From minimal non-abelian subgroups to finite non-abeian p-groups

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- Every finite non-abelian p-group contains a minimal non-abelian subgroup.
- A finite non-abelian p-group is generated by its minimal non-abelian subgroups.

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The structure of subgroups of $\mathcal{A}_t$-groups

	order
G is an $\mathcal{A}_t$-group	p^n
$\mathcal{A}_0,\ \mathcal{A}_1,\ \mathcal{A}_2,\ \cdots,\mathcal{A}_{t-2},\mathcal{A}_{t-1}$	p^{n-1}
$\mathcal{A}_0,\ \mathcal{A}_1,\ \mathcal{A}_2,\ \cdots,\mathcal{A}_{t-2}$	p^{n-2}
$\ldots\ldots\ldots\ldots\ldots$	$\ldots$
$\mathcal{A}_0,\ \mathcal{A}_1,\ \mathcal{A}_2$	$p^{n-(t-2)}$
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All possible types of $\mathcal{A}_i$-subgroups of order p^{n-j} are $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2, \cdots, \mathcal{A}_{t-2}, \mathcal{A}_{t-j}$ and G has at least one $\mathcal{A}_{t-j}$-subgroup for $j = 1, 2, \cdots, t$, $t \leq n-2$.

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- The classification of $\mathcal{A}_t$-groups for all t is hopeless. However, the classification of $\mathcal{A}_t$-groups is possible and useful for small t.

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2. H.P. Qu and X.H. Zhang, Central extension of minimal non-abelian p-groups (II), *Acta Math. Sinica*, 53:5(2010), 933–944. (in Chinese).
3. H.P. Qu and R.F. Hu, Central extension of minimal non-abelian p-groups (III), *Acta Math. Sinica*, 53:6(2010), 1051–1064. (in Chinese).
4. H.P. Qu and L.F. Zheng, Central extension of minimal non-abelian p-groups (IV), *Acta Math. Sinica*, 54:5(2011), 739–752. (in Chinese).

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2. L.J. An, L.L. Li, H.P. Qu and Q.H. Zhang, Finite p-groups with a minimal non-abelian subgroup of index p (II), *Sci China Ser A*, 57:4(2014), 737–753.
3. H.P. Qu, M.Y. Xu and L.J. An, Finite p-groups with a minimal non-abelian subgroup of index p (III), *Sci China Ser A*, 56:4(2015), 763–780.
4. L.J. An, R.F. Hu and Q.H. Zhang, Finite p-groups with a minimal non-abelian subgroup of index p (IV), *J. Algebra Appl.*, 14:2(2015), 1550020(54 pages)
5. H.P. Qu, L.P. Zhao, J. Gao and L,J. An, Finite p-groups with a minimal non-abelian subgroup of index p (V), *J. Algebra Appl.*, 13:7(2014), 1450032(35 pages).

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How is $\mathcal{A}_3$-groups classified ?

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- G is an $\mathcal{A}_3$-groups having an $\mathcal{A}_1$-subgroup of index p
  - G has an abelian subgroup of index p
    - $d(G) = 2$ → 6 types (([7]))
    - $d(G) = 3$ → 20 types ([5])
  - G has no abelian subgroup of index p
    - G has at least two $\mathcal{A}_1$-subgroups of index p
      - $d(G) = 2$ → 17 types ([7])
      - $d(G) = 3$ → 19 types ([6])
    - G has a unique $\mathcal{A}_1$-subgroup of index p → 10 types ([4,8])

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      - 8 types ([3])
      - 12 types ([7])
    - $d(G) = 3$
      - $\Phi(G) \le Z(G)$
        - 10 types ([5])
      - $\Phi(G) \not\le Z(G)$
        - 11 types
    - $d(G)=4$
      - 9 types
  - G has no abelian subgroup of index p
    - $d(M) = 2$ for all $M \lessdot G$
      - 5 types ([3])
    - $\exists\ M \lessdot G$ such that $d(M) = 3$
      - $M' \le Z(G)$ for all $M \lessdot G$ with $d(M) = 3$
        - $d(G) = 2$
          - 59 types
        - $d(G) = 3$
          - 28 types
        - $d(G)=4$
          - 6 types
      - $\exists\ M \lessdot G$ such that $d(M) = 3$ and $M' \not\le Z(G)$
        - 2 types

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For $p = 2$, the problem was solved by Janko, see [2, Theorem 90.1]. In particular, finite non-abelian 2-group all of whose $\mathcal{A}_1$-subgroups are isomorphic to Q_8 or D_8 were classified by Janko, respectively, see [1, Theorem 10.33] and [2, App.17. Cor.17.3].

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The results of classification

Theorem(Q.H. Zhang). Assume G is a finite nonabelian p-group with $d(G) = n$, p an odd prime. Then all $\mathcal{A}_1$-subgroups of G are isomorphic to $\mathrm{M}_p(1,1,1)$ if and only if G is one of the following groups:

(1) nonabelian groups with $\exp(G) = p$;

(2) $G = H_p \rtimes \langle a \rangle$, a semidirect product of H_p and $\langle a \rangle$, where $H_p = B_1 \times B_2 \times \cdots \times B_{n-1}$ is an abelian Hughes subgroup of index p, $a^p = 1$. Moreover, $\langle B_i, a \rangle$ is a groups of maximal class with an abelian subgroup of index p and whose union elements are of order p, or an elementary abelian group of order p^2, where $i = 1, 2, \ldots, n-1$.

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$\ldots\ldots\ldots\ldots\ldots$	$\cdots$
$\mathcal{A}_2, \cdots, \mathcal{A}_2, \mathcal{A}_0, \cdots, \mathcal{A}_0$	p^4
$\mathcal{A}_1, \cdots, \mathcal{A}_1, \mathcal{A}_0, \cdots, \mathcal{A}_0$	p^3

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$\mathcal{A}_2, \cdots, \mathcal{A}_2, \mathcal{A}_0, \cdots, \mathcal{A}_0$	p^4
$\mathcal{A}_1, \cdots, \mathcal{A}_1, \mathcal{A}_0, \cdots, \mathcal{A}_0$	p^3

All possible types of $\mathcal{A}_i$-subgroups of order p^{n-j} are $\mathcal{A}_0$ and $\mathcal{A}_{t-j}$ and G has at least one $\mathcal{A}_{t-j}$-subgroup for $j = 1, 2, \cdots, t-1$, $t \leq n-2$.

The structure of subgroups of $\mathcal{A}_t$-groups and more

In addition, my colleagues have also classified finite p-groups with the structure of subgroups showed as follows.

	order
G is an $\mathcal{A}_t$-group	p^n
$\mathcal{A}_{t-1}, \cdots, \mathcal{A}_{t-1}, \mathcal{A}_0(\leq p)$	p^{n-1}
$\mathcal{A}_{t-2}, \cdots, \mathcal{A}_{t-2}, \mathcal{A}_0(\leq p)$	p^{n-2}
$\dots\dots\dots\dots\dots$	$\dots$
$\mathcal{A}_2, \cdots, \mathcal{A}_2, \mathcal{A}_0(\leq p)$	$p^{n-(t-2)}$
$\mathcal{A}_1, \cdots, \mathcal{A}_1, \mathcal{A}_0(\leq p)$	$p^{n-(t-1)}$
$\mathcal{A}_0, \cdots, \mathcal{A}_0, \mathcal{A}_0$	p^{n-t}

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$\mathcal{A}_{t-2}, \cdots, \mathcal{A}_{t-2}, \mathcal{A}_0(\leq p)$	p^{n-2}
$\ldots\ldots\ldots\ldots\ldots$	$\ldots$
$\mathcal{A}_2, \cdots, \mathcal{A}_2, \mathcal{A}_0(\leq p)$	$p^{n-(t-2)}$
$\mathcal{A}_1, \cdots, \mathcal{A}_1, \mathcal{A}_0(\leq p)$	$p^{n-(t-1)}$
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The structure of subgroups of ordinary metacyclic p-groups

Qu et al. in [J. Algebra Appl. **13:4**(2014)] classified finite p-groups with the structure of subgroups showed as follows.

	order
G is an $\mathcal{A}_t$-group	p^n
$\mathcal{A}_{t-1}$	p^{n-1}
$\mathcal{A}_{t-2}$	p^{n-2}
....	...
$\mathcal{A}_2$	$p^{n-(t-2)}$
$\mathcal{A}_1$	$p^{n-(t-1)}$
$\mathcal{A}_0$	p^{n-t}

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$\mathcal{A}_{t-2}$	p^{n-2}
....	...
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$\mathcal{A}_1$	$p^{n-(t-1)}$
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It turns out that such p-groups are exactly ordinary metacyclic p-groups.

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The structure of subgroups of $\mathcal{A}_t$-groups and more

My colleagues Zhang et al. have classified finite p-groups with the structure of subgroups showed as follows.

	order
G is an $\mathcal{A}_t$-group	p^n
$\mathcal{A}_0,\ \mathcal{A}_1,\ \mathcal{A}_2,\ \cdots,\mathcal{A}_{t-2},\mathcal{A}_{t-1}$	p^{n-1}
$\mathcal{A}_0,\ \mathcal{A}_1,\ \mathcal{A}_2,\ \cdots,\mathcal{A}_{t-2}$	p^{n-2}
....	...
$\mathcal{A}_0,\ \mathcal{A}_1,\ \mathcal{A}_2$	$p^{n-(t-2)}$
$\mathcal{A}_0,\ \mathcal{A}_1$	$p^{n-(t-1)}$
$\mathcal{A}_0$	p^{n-t}

The structure of subgroups of $\mathcal{A}_t$-groups and more

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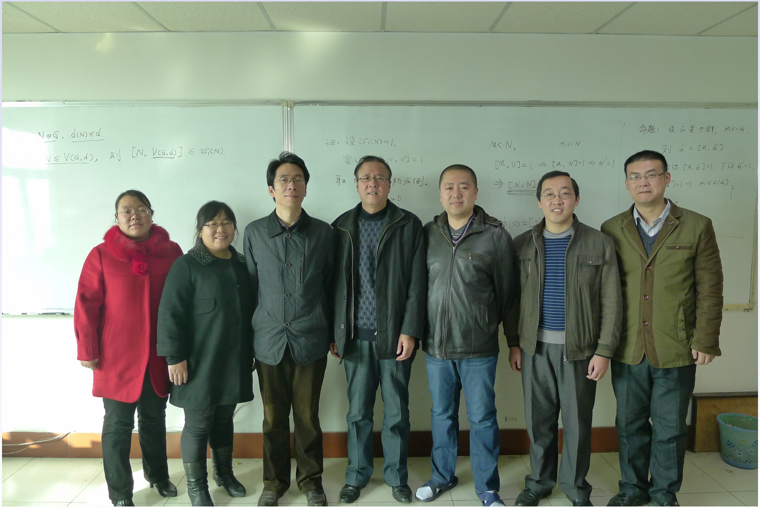
	order
G is an $\mathcal{A}_t$-group	p^n
$\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_{t-2}, \mathcal{A}_{t-1}$	p^{n-1}
$\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_{t-2}$	p^{n-2}
$\ldots\ldots\ldots\ldots\ldots$	$\ldots$
$\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$	$p^{n-(t-2)}$
$\mathcal{A}_0, \mathcal{A}_1$	$p^{n-(t-1)}$
$\mathcal{A}_0$	p^{n-t}

Such p-groups can be regarded as the p-groups "with most possible types of $\mathcal{A}_i$-subgroups".

Members of p-group team of Shanxi Normal University



Thank you!