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OBITUARY

Graham Robert Allan, 1936–2007

1. *Early life*

Graham Allan was born in Southgate, north London, on 13 August, 1936. His parents were Robert Joseph Allan, an executive civil servant, and Ellen Gwendoline, née Higgins, who had been a secretary; they were married in 1933. Both parents had left school at the age of 14, and so there was no tradition of higher education in the family; Graham was an only child, and his parents were eager that Graham should have the educational opportunities that they themselves had missed.

In 1940, during the war, the family moved to Stroud in Gloucestershire because his father worked in the Air Ministry, but they returned to London in 1943, and Graham went to Oakwood Primary School in Southgate. Then, at the age of 11, he took the then standard scholarship examination to determine the secondary school to which he would be admitted: in September, 1947, he entered Minchenden Grammar School, a state grammar school which was, somewhat unusually for that time, for both boys and girls. Of course, 1947 was the year of the brutal, cold winter, and this was a time of post-war rationing, so conditions were surely harsh.

A grammar school at that time was academically serious and challenging. Graham was particularly interested in chemistry; he was also fond of languages, taking French and German. After 'O-level' in 1952, Graham chose the scientific side for 'A-level', taking pure and applied mathematics as two separate A-levels, together with physics, in 1954; he was inspired by the head Mathematics teacher, Mr. George Bullen, and the headmaster, Dr. J. H. Walter, who taught philosophy, a subject that would be an abiding interest. Graham was very strong academically, and became head boy in 1954.

Graham was also involved with the Anglican church; he became a server at the local church, St. Thomas, Oakwood.

Minchenden often sent students to Oxford and Cambridge, and the school and his parents encouraged Graham to follow this tradition. So he stayed at school after A-levels to take the Scholarship examinations in mathematics in December, 1954, for entrance to Cambridge University, and he gained an Exhibition to Sidney Sussex College.

Military service was still compulsory in the UK in 1955; although Graham could have deferred, and thus probably have avoided this service altogether, he joined the RAF in March of 1955. Much of Graham's military service was spent at Stradishall in Suffolk, then a night fighter station with Meteors and Venoms, 11 miles from Bury St. Edmunds. His rôle was that of a radar fitter, but he also found time for reading, mostly in philosophy and religion. Graham entered Sidney Sussex to read for the Mathematics Tripos in September, 1957.

2. *A student and Fellow at Cambridge*

At that time, some students who entered Cambridge omitted part I of the Tripos, but Graham did not do this; he took Part I of the Tripos in the summer of 1958, and Part II in 1960, gaining a 1st class in each case; in Cambridge language, he was a 'Wrangler' in 1960. He then continued at Cambridge to take Part III in 1961.

In 1961, Graham started research at Cambridge as a student of Frank Smithies, and so he became a 'mathematical grandson' of G. H. Hardy; he was supported by a grant from the *Department of Scientific and Industrial Research* of that time. Smithies had been the leading figure who introduced functional analysis to Cambridge after the war, and he was the supervisor of many students who became strong professional mathematicians (see ⟨**17**⟩), but he was not an active supervisor; he proposed a topic for research and left the students to make their own way, which many did very successfully. Other lecturers in functional analysis at Cambridge at that time were John Williamson and John Ringrose (1961–1963); fellow graduate students of Smithies overlapping with Graham were Ben Garling and Trevor West (1960–1963), and Simon Bernau (1961–1964); students of Williamson included Barry Johnson and John Pym (1959–1962), Robin Harte (1960–1963), and, in the same years 1962–1965, Robert Elliott, George Reid, and Nicholas Varopoulos; students of Ringrose included Harry Dowson (1960–1963), John Erdos (1961–1964), and Chris Lance (1962–1965). One can see at once what a strong group this was; all these students went on to distinguished careers in mathematics, and most became professors of mathematics. This group was the main source of the strength in functional analysis for a generation in the UK, and they carried the subject to several other centres in the country.

There was a weekly seminar in functional analysis at that time, organized by Smithies. Graduate students spoke on their own work and were sometimes required to give a detailed exposition of a new paper, something which was probably a daunting challenge; there were occasional outside speakers. This may have been one of the first subject-specific working seminars in the country. There was then no building to form a 'Mathematics Department', but some students were squeezed into rooms in Laundress Lane; they spent a lot of time next door at the Coffee Anchor. Graham is remembered as being 'friendly, but shy'.

At the end of his first year as a research student, in 1962, Graham married Elizabeth Gemmell; they had met in Wilmslow a year earlier. Elizabeth had a degree in geography from Sheffield University, and was a town planner with Lancashire County Council.

In the third year of his PhD, 1963–1964, Graham was elected a Research Fellow at his own college, Sidney Sussex. His thesis was submitted and accepted in April, 1964, and Graham was then elected a Fellow and Director of Studies in Mathematics at Churchill College. Churchill, which received its Royal Charter in 1960, was a new Cambridge college, eager to develop its reputation, especially in science, and so this was an exciting time; Graham supervised some very strong students.

Graham acted as the editor of the *Proceedings of the Cambridge Philosophical Society* (which, despite its name, published papers only on mathematics) from 1965 to 1967.

Daughter Juliet was born in October, 1966; she is now a lawyer working for *DEFRA* in London.

3. *Further moves*

Although Graham gave lectures at Cambridge from 1964 to 1967, he was not an official lecturer of the University. In 1967, he accepted an invitation from John Ringrose, by now Professor of Pure Mathematics at the University of Newcastle-upon-Tyne, to become a lecturer there. At that time Newcastle was establishing itself as a world centre in functional analysis; members of staff included Aldric Brown, Jim Ford, Barry Johnson, Chris Lance, and John Gilbert, and there were about 10 graduate students in functional analysis. Graham gave an advanced course on *Banach algebras and several complex variables* in 1968–1969. A distinguished visitor for some time was Charles Rickart of Yale.

Daughter Clare was born in Newcastle in December, 1968. She is the author of the famous novel *Poppy Shakespeare*, which became a Channel 4 drama; she is now also a journalist and teaches creative writing.

In 1969, Graham was awarded the *Junior Berwick Prize* by the London Mathematical Society for his work in [**3**, **8**].

In 1969, a position as a lecturer became available in Cambridge, and Graham was appointed; he again became a Fellow of Churchill.

However, before curtains for their new house were finished, Graham was contacted by the University of Leeds. In those days, universities such as Leeds had traditionally had just one professor of pure mathematics, who also acted as the Head of Department. In a move that was innovative for its time, Leeds decided to have three 'Research Professors', each to lead one subject area. Alfred Goldie had arrived as a *Research Professor in Algebra* in 1963, and he built a school in non-commutative ring theory; Graham became the *Research Professor in Functional Analysis* at Goldie's instigation in 1970, and began to build up a group in this subject. Towards this, Graham appointed as lecturers David Salinger in 1971 and myself in 1973, with plans for three more appointments to follow quickly. However, funding suddenly dried up, and so it took much longer than expected before the group expanded further. (The third research professor, in Mathematical Logic, arrived much later.)

The years at Leeds were a time when undergraduate student numbers in mathematics increased dramatically, and Graham played a considerable rôle in re-organizing the undergraduate syllabus to make it more realistic. (However, re-reading his first-year lecture notes on analysis of 1973 shows a striking lessening of expectations since then; one could hardly now include the difference between uniform and pointwise convergence in a course for first-year students.)

Whilst at Leeds Graham was a member of the Editorial Board of the LMS, and became Editor-in-Chief of the *Proceedings* in 1976, continuing in this rôle until 1981. Graham was a member of Council of the LMS from 1979 to 1984.

Graham and Elizabeth were very active at St. Aidan's Anglican church, which is in an inner-city parish in Leeds, at the 'Anglo-Catholic' end of the theological spectrum. Graham acted in several productions, in rôles varying from St. Paul to a pantomime dragon.

Graham became Head of Pure Mathematics at Leeds from 1975 to 1978, but he did not care for administration, and did not find it easy to work with Goldie; also, he found the mathematical level of most students at Leeds to be disappointing, and so he decided to return to Cambridge. In 1978, he became a lecturer in Pure Mathematics and, for the third time, a Fellow of Churchill; in 1980, he was promoted to *Reader in Functional Analysis* at the University, and again he directed studies in mathematics at Churchill. Graham acted as Vice-Master of Churchill from 1990 to 1993; this involved chairing committees and occasionally deputising for the Master. In the Department at Cambridge, Graham was a conscientious, but not revolutionary, member of many committees; for example, he was Chairman of the Tripos Committee and of the Mathematics Faculty Board (1997–1999), and was Part III organiser for many years. Graham always listened carefully and respectfully to suggestions for change; he had some strong views, carefully thought out, but was prepared to vary them if the evidence pointed in another direction.

After his return to Cambridge, Graham's abiding interest in theology and religion deepened; he made personal contact with some leading theologians in Cambridge, and was influenced by the writings of the Dominican, Father Herbert McCabe; he took instruction from the Roman Catholic Chaplain to the University, The Most Rev. Maurice Couve de Murville (subsequently Archbishop of Birmingham), and was received into the Catholic Church in 1979; he was active on the Council of Fisher House, the home of the Catholic Chaplaincy to the University. Graham enjoyed the Tridentine Mass, and grew to love the sacred music of the church.

Graham was diagnosed with a brain tumour in October, 2006. Treatment by operations and therapies reduced the tumour, but after some months there was a resurgence, and Graham died on 9 August 2007, just short of his 71st birthday. A Requiem Mass was held in the Church of Our Lady and the English Martyrs, Cambridge.

4. *Conferences and foreign travel*

Graham was interested in foreign travel, and keen to talk to other mathematicians. He made several trips to countries where he sought to assist in the development of mathematics. For example, he visited École Normale Supérieure, Takkadoun, in Rabat, Morocco, several times in the 1970s and 1980s, and in particular he was a plenary lecturer at the *First Pan-African Congress of Mathematicians* in 1976; he visited Nigeria for some time in 1977; he lectured at the annual meeting of the *Iranian Mathematical Society* in Isfahan in 1978.

A productive sabbatical of three months in 1979 was spent with Charles Rickart at Yale. Then, in 1988 he spent three further months as a Visiting Professor at UCLA, at the invitation of Phil Curtis, and he certainly enjoyed his time in California, visiting the major mathematics departments there. In 1996, he spent three months at Université Laval, Québec, at the invitation of his former student, Tom Ransford, also working there with Jaroslav Zemánek. Graham enjoyed several visits to Bordeaux, where he had many discussions with Jean Esterle.

In the early part of Graham's career contact with eastern block countries was almost impossible; nevertheless, he communicated with Alexander Helemskii in Moscow, and was eventually able to visit Moscow in 1992. Graham and Elizabeth were subsequently the hosts of Helemskii in Cambridge and in Wales; this was a wonderful opportunity for highly educated people with very wide cultural, historical, and philosophical interests to converse gently and to exchange their different views of the world. Helemskii writes:

Graham was an extremely educated and cultured man.... a guest in his hospitable home used to be astonished by the abundance of books, dedicated to science and literature, fine *arts and music. Most of all, ... he was fond of philosophy and theology, and his knowledge of these subjects was tremendous.... Sometimes great erudition makes a person somewhat dry and aloof, but people who knew Graham and listened to him were much delighted by his impeccable sense of humour, so English and so universal .... Graham was very kind, very human, very tolerant, very modest.*

Graham also made strong contacts with Wiesław Żelazko and Jaroslav Zemánek in Warsaw; he first visited the Banach Center in Warsaw in the 1980s, and returned about 10 times over the years, usually giving lectures; he was a main speaker in Będlewo for a meeting in honour of Żelazko's 70th birthday in 2003; a special seminar in Warsaw in September, 2008, was devoted to his memory.

A great feature of our mathematical life over nearly 40 years has been a series of conferences on *Banach algebras*; they originated with a meeting in Los Angeles in 1974 sponsored by Bill Bade of Berkeley and Phil Curtis of UCLA. The series has now achieved its 20th meeting in Waterloo, Canada, in 2011. Graham was an enthusiastic participant at many of these meetings: for example, he attended the 3rd conference in Long Beach in 1981, the 4th conference in Copenhagen in 1985, the 5th, 7th and 9th conferences in Berkeley in 1986, 1988, and 1990, and the 8th conference in Canberra in 1989. Then Graham himself, with Tom Ransford, became the organiser of the 10th conference, held in Cambridge from 1 to 12 July, 1991. His last conference was the 17th in Bordeaux in 2005; at the 18th, in Laval, we could only remember Graham with affection and respect.

5. *Graduate students*

It was a particular strength of Graham's mathematical career that he was the supervisor of a substantial number of graduate students who obtained PhDs under his care.

It now seems remarkable that, immediately after his own PhD, Graham accepted a research student; this was John F. Rennison (1964–1967), who became a Lecturer in Mathematics at the University of Kent; John studied *Arens products on the second duals of Banach algebras*. Then Ian G. Craw became Graham's second student in 1966, obtaining his PhD in 1969 for a thesis on *Continuity problems for Fréchet algebras*. Ian became a Senior Lecturer in Mathematics at the University of Aberdeen. At that time there was great pressure on people at Cambridge to accept research students in functional analysis; for example, nine such students started in 1966, and these had to be shared out among available supervisors.

My own story is as follows. I attended Graham's Part III lectures at Cambridge on Banach algebras in the year 1966–1967. I found the subject to be an exciting blend of algebra and analysis, and was attracted by the clarity and beauty of his exposition; lovely theorems were carefully delineated. We progressed through the Gel'fand theory of commutative Banach algebras, touched upon C^*-algebras, and developed the single-variable holomorphic functional calculus. Some of the style and content of the notes of those days still shines through in Part II of the text [**41**]. I remember in particular the application of Gel'fand theory to show very elegantly the well-known theorem of Wiener that the reciprocal of a non-vanishing function on the circle $\mathbb{T}$ with absolutely convergent Fourier series also has an absolutely convergent Fourier series. The lectures also discussed the question of the 'uniqueness of topology for all semi-simple Banach algebras', which was then an open question; it was resolved positively very soon afterwards by Barry Johnson ⟨**13**⟩. Another topic of great interest was the 'Michael problem': Is every character on a Fréchet algebra automatically continuous? Remarkably, this latter question, much discussed over the years, has still defeated us and many other people.

During that year I arranged with Graham to be his research student at Cambridge; in the event, Graham's move to Newcastle in 1967 meant that I was his student at Newcastle. Graham asked me to read the lectures of Ken Hoffman ⟨**11**⟩ given at a Summer School in Bruges that

Graham had attended. The lectures related Banach algebras to the holomorphic functional calculus in several variables, and these topics became of lasting interest to Graham. There are two fascinating questions in these lectures: (i) Let A be a uniform algebra with character space $\bar{\mathbb{D}}$, the closed unit disc in $\mathbb{C}$. Must the Shilov boundary of A meet, or even contain, the unit circle $\mathbb{T}$? (ii) Let A be a uniform algebra with character space $\mathbb{I}$, the closed unit interval in $\mathbb{C}$. Is it true that $A = C(\mathbb{I})$? Graham suggested that I work on these two problems. He was very kind and helpful, but somewhat diffident. Somehow I never understood at that time that these were not 'simple exercises'; Graham was very tactful in pointing out the errors in my initial approaches. I failed to make useful progress on either of these two questions which, embarrassingly, are still open. Fortunately, some other related work did progress: I am grateful indeed to Graham for recognizing the potential of some work, and for leading me to a thesis in 1970.

Another student at Newcastle was J. Peter McClure (1967–1970), with a Commonwealth Scholarship from Canada; McClure became a Professor of Mathematics, and eventually Chairman, at the University of Manitoba in Winnipeg. Also in 1970, Peter G. Dixon received a Cambridge PhD under Graham; Dixon became a Reader in Pure Mathematics at the University of Sheffield.

Graham had two students from Iran at Leeds: these were Ghodsieh Vakily (1971–1974) and Y. Nejad-Degan (1974–1977); both returned to Iran and became Professors of Mathematics, and Graham visited them there.

After his final return to Cambridge, Graham's students were Amir Khosravi (1977–1980), Thomas J. Ransford (1981–1984), Frédéric Gourdeau (1984–1989), Michael C. White (1986–1989), Christopher E. J. Kilgour (1990–1993), Simon E. Morris (1990–1993), Thomas Vils Pedersen (1991–1994), Timothy J. D. Wilkins (1992–1996), Michael K. Kopp (1999–2002), and Daniel L. Neale (2000–2004). Ransford is now a *Canada Research Chair Professor* at the Université Laval, Québec, Gourdeau is Director of the Department of Mathematics and Statistics at Laval, White is a Senior Lecturer at the University of Newcastle-upon-Tyne, and Pedersen is an Assistant Professor at the University of Copenhagen.

For many years at Cambridge, Graham gave Part III lectures (to mostly 4th-year students) on functional analysis, Banach algebras, and several complex variables. These lectures were always very clear and well-prepared; some of the students attending these lectures were encouraged to be 'students of students', and so 'mathematical grandchildren' of Graham include Joel Feinstein, now an Associate Professor at Nottingham, Matthew Daws, now a Lecturer at Leeds, and Yemon Choi, now an Assistant Professor at the University of Saskatchewan. In all, the *Mathematics Genealogy Project* lists over 50 mathematical descendants of Graham.

Graham went a long way towards preparing a book based on various of his Part III courses; sadly he was not able to complete this. His notes were prepared for eventual publication in [**41**].

It is clear that Graham had great influence on future generations of mathematicians working in his subject.

6. *Mathematical papers*

Graham's thesis was written in the years 1961–1964, under the supervision of Frank Smithies. At that time, the subject of Banach algebras was well developed and the books of Naimark ⟨**15**⟩ and Rickart ⟨**16**⟩, often referenced by Graham, had established themselves as the definitive texts. The theory had been extended to cover locally m-convex algebras by Michael ⟨**14**⟩.

In his thesis, Graham took the then existing theory of Banach algebras, and especially of Banach $*$-algebras, and married this to the more general theory of locally convex spaces; this work was expounded in [**3**, **4**]. Let A be a locally convex space which is also a unital algebra, with identity e, in which multiplication is separately continuous. An element $a \in A$ is *bounded* if the set $\{(\varepsilon a)^n : n \in \mathbb{N}\}$ is bounded for some $\varepsilon > 0$; the subset of bounded elements is A_0.

Let $a \in A$. The *resolvent set* $\rho(a)$ is the set of numbers $z \in \mathbb{C}$ such that $ze - a$ has an inverse in A_0, together with ∞ if $a \notin A_0$; the *spectrum* $\sigma(a)$ of a is the complement of $\rho(a)$ in the extended complex plane, $\mathbb{C}^*$. Suppose that A has a continuous involution. Then A is *symmetric* if $-1 \in \rho(aa^*)$ $(a \in A)$. The algebra A is a *GB^*-algebra* if it is symmetric, pseudo-complete, and the class of bounded, closed, absolutely convex subsets B with $e \in B$, $B^2 \subset B$, and $B^* = B$ has greatest member. Graham developed a representation theory for commutative GB^*-algebras involving continuous functions taking values in $\mathbb{C}^*$ on the space of maximal ideals of A_0; an analogous theory for non-commutative GB^*-algebras was given by Dixon ⟨**7**⟩.

These GB^*-algebras have flourished because of their relevance to unbounded operators in quantum field theory, and more generally in mathematical physics; they occur now among the 'unbounded Hilbert algebras'. Papers are still being written about these algebras; for a recent survey, with many references, see ⟨**10**⟩.

The above work was related to the major study of the analytic functional calculus for Banach algebras and more general topological algebras of Waelbroeck ⟨**21**⟩, a memoir studied by Graham. This was the beginning of Graham's interest in several complex variable theory, and he became perhaps the only person in England to master and lecture on the work of Cartan, Oka, and Weil, etc.

As a Fellow at Churchill, Graham studied functions taking values in a Banach algebra. The following theorem is proved in [**6**]. Let A be a Banach algebra with identity e. For an open subset D of $\mathbb{C}^n$, denote by $A(D)$ the algebra of all holomorphic A-valued functions on D.

Theorem 6.1. *Let A be a Banach algebra with identity e, and suppose that D is a domain of holomorphy in $\mathbb{C}^n$. Let $f_1, \ldots, f_m \in A(D)$, and suppose that, for each $z \in D$, there exist $\alpha_1, \ldots, \alpha_m \in A$ with $\sum_{i=1}^m \alpha_i f_i(z) = e$. Then there are $g_1, \ldots, g_m \in A(D)$ such that $\sum_{i=1}^m g_i(z) f_i(z) = e$ $(z \in D)$.*

In particular, for each open $D \subset \mathbb{C}$ and $f \in A(D)$ such that $f(z)$ is left invertible in A for each $z \in D$, there is a holomorphic function $g : D \to A$ with $g(z)f(z) = e$ $(z \in D)$, a result partially proved earlier in [**5**].

These results were presented in a more abstract Banach-algebra setting in [**8**]. In particular, this enabled Graham to establish the above theorem in the case where D is an arbitrary Stein manifold. These theorems have subsequently been much generalized; for example, see [**23**].

The connection between the theories of several complex variables and commutative Banach algebras was developed in an important sequence of papers on the holomorphic (or analytic) functional calculus in several variables.

Let A be a commutative Banach algebra with identity e, let $n \in \mathbb{N}$, and let $a_1, \ldots, a_n \in A$. The joint spectrum $\sigma(a)$ of $a = (a_1, \ldots, a_n)$ is the complement of the set of elements $(z_1, \ldots, z_n)$ of $\mathbb{C}^n$ such that there exist $b_1, \ldots, b_n \in A$ with $\sum_{i=1}^n b_i(z_i e - a_i) = e$; in fact, $\sigma(a) = \{(\varphi(a_1), \ldots, \varphi(a_n)) : \varphi \in \Phi_A\}$, where Φ_A is the character space of A, and so $\sigma(a)$ is a non-empty, compact set in $\mathbb{C}^n$. Let K be a non-empty, compact subset of $\mathbb{C}^n$. The algebra of germs of functions holomorphic on a neighbourhood of K is denoted by $\mathcal{O}_K$. The *weak form* of the Shilov–Arens–Calderón (SAC) theorem states that, for each $F \in \mathcal{O}_{\sigma(a)}$, there exists $b \in A$ with $\varphi(b) = F(\varphi(a_1), \ldots, \varphi(a_n))$ for all $\varphi \in \Phi_A$; the *strong form* says that there is a continuous, unital homomorphism $\Theta_a : \mathcal{O}_\sigma(a) \to A$ such that $\Theta_a(Z_i) = a_i$ $(i = 1, \ldots, n)$ and $b = \Theta_a(F)$ satisfies the above condition for each $F \in \mathcal{O}_{\sigma(a)}$. (This functional calculus satisfies various other properties, and conditions for its uniqueness are known; see ⟨**3**, **19**⟩.)

First, in [**7**], Graham gave a neat characterization of Φ_A as a subset of A' that leads to a different proof of SAC, described in ⟨**2**, § 21⟩. Next, in [**9**], Graham extended the weak form of SAC: a special case of his theorem shows that, if $\sigma(a)$ is contained in a closed, analytic subvariety V of $\mathbb{C}^n$ and F is analytic on a neighbourhood of $\sigma(a)$ in V, then again there exists

$b \in A$ with $\varphi(b) = F(\varphi(a_1), \ldots, \varphi(a_n))$ $(\varphi \in \Phi_A)$. It is natural to wonder if there is a version of the strong form of SAC that holds in the above situation; Graham proved in [**11**] that this is so when V is a complex submanifold of an open neighbourhood of $\sigma(a)$ in $\mathbb{C}^n$, but he gave an example to show that, surprisingly, the strong form need not hold when V is a variety that is not a submanifold.

The modestly entitled [**10**] is significant. An attractive approach, originating with Waelbroeck, to the functional calculus was given in the book ⟨**12**⟩ of Hörmander. However this approach had the deficiency that it gave only the weak form of SAC. This deficiency had been remedied by Waelbroeck by using deep results in several complex variable theory; Graham gave in [**10**] an elementary proof of a key lemma that led to a simple deduction of the strong form. This is a now-standard approach.

A delightful consequence of SAC, using a solution of a Cousin problem, is *Rossi's local maximum modulus theorem* ⟨**18**⟩: roughly, 'a local peak set of a Banach function algebra is a peak set'. Here is Graham's extension, from [**12**]. The element F of the following theorem is *analytically approximable on* U if there are $a_1, \ldots, a_n \in A$ and a function h holomorphic on a neighbourhood of $\{(\varphi(a_1), \ldots, \varphi(a_n)) : \varphi \in U\}$ such that $F(\varphi) = h(\varphi(a_1), \ldots, \varphi(a_n))$ $(\varphi \in U)$.

THEOREM 6.2. *Let A be a commutative, unital Banach algebra, and let L be a compact subspace of Φ_A. Suppose that there exists a neighbourhood U of L in Φ_A and an analytically approximable function F on U such that $f(L) = \{1\}$ and $|F(\varphi)| < 1$ $(\varphi \in U \setminus L)$. Then there exists $a \in A$ such that $\varphi(a) = 1$ $(\varphi \in L)$ and $|\varphi(a)| < 1$ $(\varphi \in \Phi_A \setminus L)$.*

The proof required the solution of two different Cousin problems; a nice application is a 'local minimum modulus theorem'.

Some of Graham's work on the functional calculus was summarized in [**13**]; there is a detailed account of his results in ⟨**19**, § 8⟩. An important sequel to Graham's work on the functional calculus was that of Zame ⟨**22**⟩, who established a strong uniqueness result; see ⟨**5**, p. 228⟩.

The several-variable functional calculus for commutative Banach algebras was extended to the wider class of *pseudo-Banach algebras* in [**14**]: these algebras have a bound structure as described above for locally convex algebras, and are algebraically an inductive limit of Banach algebras connected by continuous monomorphisms; various examples were given. Indeed, pseudo-Banach algebras are characterized among commutative, unital algebras with compact character spaces by the existence of a functional calculus.

Probably the most important and original of Graham's papers is [**15**], written in 1971, soon after his move to Leeds. In the background is the *problem of Kaplansky*. Let X be a non-empty, compact, Hausdorff space, and denote by $C(X)$ the uniform algebra of all complex-valued, continuous functions on X, so that $C(X)$ is a commutative, unital Banach algebra with respect to the uniform norm $|\cdot|_X$. Now let $\|\cdot\|$ be an arbitrary algebra norm (not necessarily complete) on $C(X)$. Kaplansky had proved that $\|f\| \geqslant |f|_X$ $(f \in C(X))$; his problem was to determine whether $\|\cdot\|$ and $|\cdot|_X$ are necessarily equivalent on $C(X)$. One can also phrase the questions as: Is every homomorphism from $C(X)$ into each Banach algebra automatically continuous? It was known from the work of Badé and Curtis ⟨**1**⟩ that such a homomorphism is necessarily continuous on a dense subalgebra of $C(X)$.

Graham's brilliant new idea in [**15**] was to seek to embed the Fréchet algebra $\mathfrak{F} = \mathbb{C}[[X]]$ of all formal power series in one variable into a Banach algebra; at first sight it seems implausible that this might be possible. Graham introduced in this paper the notion of an element of *finite closed descent* (fcd). Let A be a commutative Banach algebra, and take $a \in A$. Then a has fcd if there exists $n \in \mathbb{N}$ such that $a^n \in \overline{a^{n+1}A} \setminus a^{n+1}A$; see ⟨**5**, Definition 2.2.10⟩. Let $\theta : \mathfrak{F} \to A$ be a unital embedding, where A is a commutative, unital Banach algebra, and set $a = \theta(X)$. Then it is rather easy to show that $a \in \operatorname{rad} A$, the Jacobson radical of A, and that a has fcd. Graham's theorem is the converse: given such an element a, there is a unital embedding $\theta : \mathfrak{F} \to A$ with $\theta(X) = a$. The construction of θ is necessarily quite long and algebraic; it considers separately elements of $\mathfrak{F}$ that are, respectively, algebraic and transcendental with respect to a subalgebra. The main tools introduced were the Mittag-Leffler theorem on inverse limits and a theorem of Arens and Calderón on the solution of certain equations with coefficients in a Banach algebra. Of course Graham exhibited an algebra that has elements of fcd: his example is the Volterra algebra with identity adjoined.

All these ideas have been subsequently much used. For example, elements that have fcd are used in Thomas's solution of the Singer–Wermer problem on the range of derivations on commutative Banach algebras ⟨**20**⟩; see ⟨**5**, Theorem 5.2.36⟩.

In fact, Kaplansky's problem was solved independently by Jean Esterle ⟨**9**⟩ and myself ⟨**4**⟩. Thus, with the continuum hypothesis (CH), there is a discontinuous homomorphism from $C(X)$ for every infinite, compact space X. Curiously, the solution of Esterle was much influenced by the techniques of [**15**], but my own approach was different. One would like to know which Banach algebras A are such that there is a discontinuous homomorphism from some $C(X)$, and hence, with CH, from $C(X)$ for all infinite X, into A. The answer is remarkable: they are exactly the algebras A that contain an element $a \in \operatorname{rad} A$ of fcd, and hence exactly the algebras that contain images of $\mathfrak{F}$, as identified by Graham; further, each unital integral domain with a character and of cardinality $\mathfrak{c}$ can be embedded in each of these algebras. These are theorems of ZFC + CH; they are not true as theorems of ZFC itself ⟨**6**⟩. Several other characterizations of these algebras are known; full details are given in ⟨**5**, § 5.7⟩.

Graham extended his ideas on elements of finite closed descent in [**16**, **20**, **28**]; the latter paper develops the theory for such elements in Fréchet algebras.

An important result in Banach algebra theory is *Cohen's factorization theorem*. Graham and Allan Sinclair have a fine extension of this result in [**17**]. Let A be a Banach algebra with a bounded left approximate identity, let X be a Banach left A-module, and take $x \in \overline{A \cdot X}$. For each sequence (α_n) which increases to infinity, there exist $a \in A$ and (y_n) in X with $x = a^n y_n$ and $\|y_n\| \leqslant \alpha_n^n \|x\|$ for all $n \in \mathbb{N}$. Thus a radical Banach algebra with a bounded approximate identity cannot have $\|x^n\|^{1/n}$ tending to zero uniformly in the unit ball of A.

Let ω be a continuous function on $\mathbb{R}^+$. Then ω is a *weight* if $\omega(s+t) \leqslant \omega(s)\omega(t)\ (s,t \in \mathbb{R}^+)$, and in this case $L^1(\mathbb{R}^+, \omega)$ is a commutative Banach algebra with respect to the convolution product; it is radical in the case where ω is *radical*, that is, $\lim_{t\to\infty} \omega(t)^{1/t} = 0$. For each $a \in \mathbb{R}^+$, the set $\{f \in L^1(\mathbb{R}^+, \omega) : \inf \operatorname{supp} f \geqslant a\}$ is a closed ideal; these are the *standard* ideals. In the now-inaccessible paper [**19**], Graham initiated an attempt to prove that, for 'reasonable' radical weights ω, such as $\omega : t \mapsto \exp(-t^2)$, every non-zero, closed ideal is standard, and proved some attractive and significant partial results. A general theorem was eventually proved by Yngve Domar ⟨**8**⟩, using some of Graham's ideas; see ⟨**5**, Theorem 4.7.72⟩. The case where $\omega(t) = (1+t)^{-1/t}\ (t \in \mathbb{R}^+)$ is still open.

An attractive paper, with Anthony O'Farrell and Thomas Ransford, is [**24**]. A result of Katznelson and Tzafriri shows that, for a bounded operator T on a Banach space such that $(\|T^n\|)$ is bounded and the spectrum of T meets the unit circle in at most one point, necessarily $\lim_{n\to\infty} \|T^n - T^{n+1}\| = 0$. The authors generalize this result, essentially by replacing the single point in the circle by a set of linear measure 0; the proof involves only complex-variable theory. These results, with some additions, were summarized in [**25**] and extended in [**31**]. In [**27**], the authors provide several characteristically elegant extensions, involving harmonic and complex analysis, to various classical theorems, in particular considering elements x of a Banach algebra such that $\|x^n\| \leqslant \mu(n)\ (n \in \mathbb{N})$ for suitable sequences $(\mu(n))$.

After a gap from about 1989 to 1996, Graham became active again in research. First, with co-authors, he studied the question of the approximation of functions by polynomials in smooth functions. Let $f, \psi_1, \ldots, \psi_r$ be infinitely-differentiable functions on $\mathbb{R}$ such that the r-tuple

$(\psi_1, \ldots, \psi_r) : \mathbb{R} \to \mathbb{R}^r$ is injective. Then it is proved in [**34**] that f is the limit of a sequence of polynomials in the functions ψ_i precisely when f satisfies the obvious necessary condition that its Taylor series at each $a \in \mathbb{R}$ is a formal power series in the Taylor series of the functions $\psi_1 - \psi_1(a), \ldots, \psi_r - \psi_r(a)$. One feels that this should have been known long ago, but, as the reviewer states, 'the proof is more than substantial enough to show why it might not have been'. The main ideas are described in [**35**], and related results, when the functions ψ_i are defined on $\mathbb{R}^d$, are given in [**37**], but it seems that the underlying general question is still quite open.

In later years, Graham returned to his earlier ideas of using the Mittag-Leffler theorem to establish the existence of certain elements. The setting is as follows. An *inverse-limit sequence* has the form

$$\mathcal{V} : V_1 \xleftarrow{T_1} V_2 \xleftarrow{T_2} V_3 \xleftarrow{T_3} \cdots ,$$

where each V_n is a linear space and each T_n is a linear map. The sequence is said to be *stable* if, for each sequence (v_n), where $v_n \in V_n$ $(n \in \mathbb{N})$, there are elements $x_n \in V_n$ with $x_n = T_n(x_{n+1}) + v_n$ $(n \in \mathbb{N})$. A form of the Mittag-Leffler theorem shows that, if each V_n is a complete metrizable space and each T_n is a continuous map with dense range, then the above sequence is stable. There are obvious close connections between this notion and that of an element of fcd.

In [**30**], Graham studied when limits of connected sequences 'fit-together', proving a surprisingly strong and general principle that implies several classical theorems, including the Mittag-Leffler theorem. The paper [**32**] places his earlier work of [**15**] in a more abstract and general setting. In [**36**], Graham considered two inverse-limit sequences of complete, metrizable, abelian groups and continuous group homomorphism, with connecting homomorphism commuting with the sequence maps, and showed that the 'separating subspaces' of classical automatic continuity theory form a stable inverse-limit theorem; this leads to yet more general results involving fcd properties. There is a partial summary of this work in [**38**]. In fact, Graham produced in 2006 a draft of a substantial account of 'stable elements' in a variety of settings, initially very general; maybe one day this will be polished for publication.

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