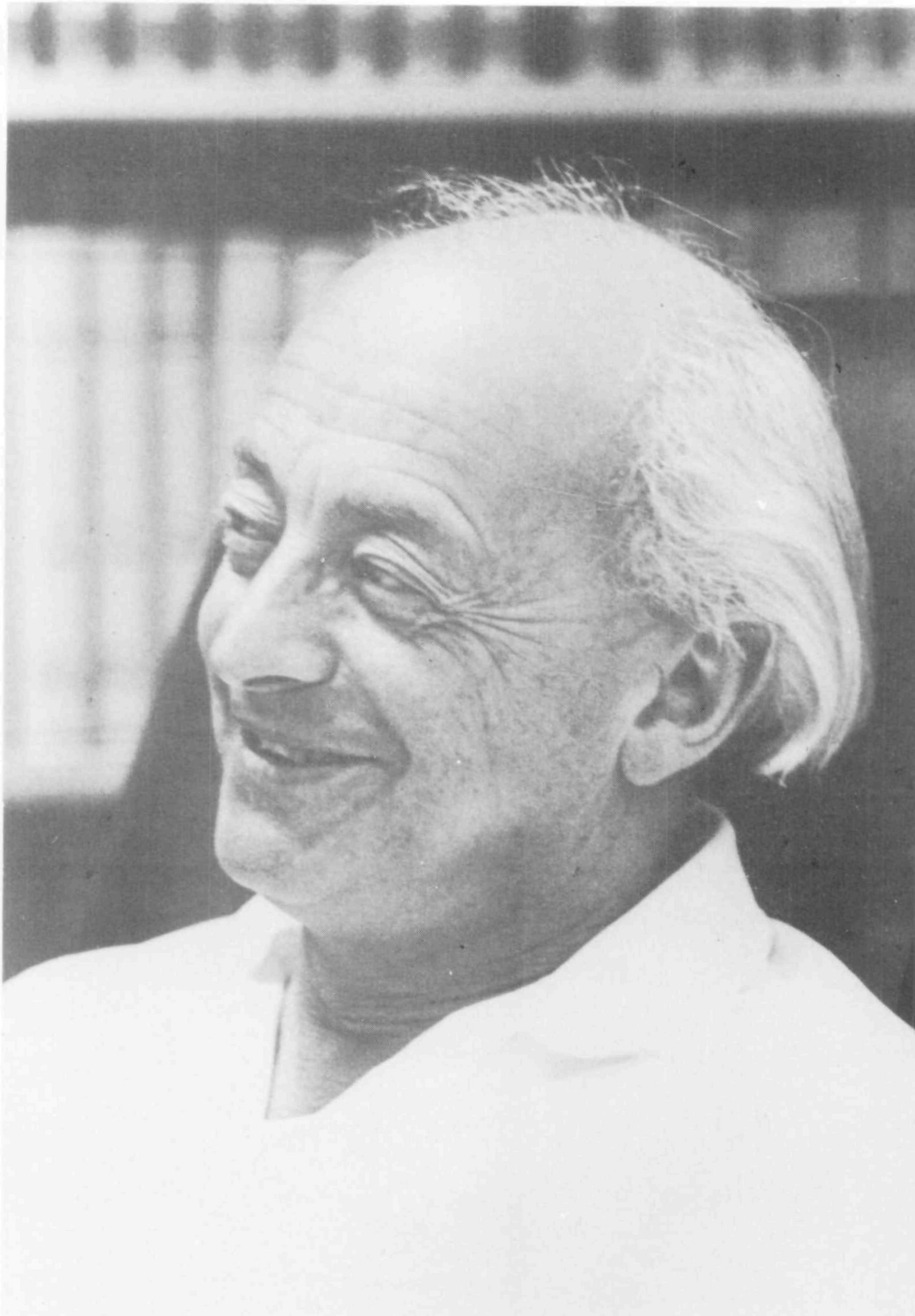

REINHOLD BAER, 1968

facing p. 339

REINHOLD BAER

Reinhold Baer, Professor Emeritus in the Johann Wolfgang Goethe-University at Frankfurt am Main, Germany, died on 22 October 1979 in Zürich, Switzerland. He had been a member of the Society since 1934.

Reinhold Baer was born in Berlin on 22 July 1902. He was the only child of Emil and Bianka Baer. Emil Baer was a successful clothing manufacturer and the family was prosperous. They lived in Charlottenburg, an elegant part of Berlin. As a small child, Reinhold travelled a good deal with his parents, and remembered staying in the best hotels and eating exotic food. But his father died when Reinhold was only twelve years old.

In 1908 he entered the Kaiser Friedrich Schule in Charlottenburg, a Humanistisches Gymnasium, and stayed there until his graduation in 1920. By this time the family fortune had largely disappeared and the future in post-war Germany looked bleak and uncertain. The young Baer dreamed of becoming an engineer, and he enrolled at the Technische Hochschule at Hannover to study mechanical engineering. It did not take him long to realise he had made a mistake. After one year and a short summer spell of practical engineering work in a factory (an obligatory part of the course) he left Hannover and began, in the autumn of 1921, to study mathematics and philosophy at Freiburg im Breisgau.

This proved to be a happy decision. He found a lively mathematical atmosphere and came into contact with F. K. Schmidt, then also a student, W. Krull, who had just been appointed Privatdozent, and above all with Alfred Loewy, who had done distinguished work in algebra (the Loewy series of a module is named after him). Baer found the town and surrounding Black Forest countryside very much to his liking and retained a great fondness for Freiburg throughout his life.

However, it was Göttingen that was the mathematical capital at the time and Baer went to study there from 1922 to 1924. He fell under the spell of Emmy Noether, whose teaching and way of looking at mathematics made a deep and lasting impression on him. He was also much attracted to the mathematics on which the young Privatdozent Hellmuth Kneser was working, the topology of surfaces from the geometric point of view initiated by Jacob Nielsen. Baer decided to write his doctoral thesis under Kneser.

In 1924 he won a year's Studentship to live and work in the "Bergmann-Haus" at Kiel, a well-known residence for gifted students. This gave him the peace and quiet needed to write his dissertation. At the University he met Ernst Steinitz, Otto Toeplitz, the theologian and philosopher Heinrich Scholz, the recently "habilitated" Helmut Hasse, and also Wolfgang Franz, newly arrived as an undergraduate. With Scholz, Baer could indulge his interest in philosophy and the foundations of mathematics; but it was with Hasse that he had his most fruitful mathematical contact that year.

Baer took his Ph.D. degree in Göttingen in 1925. His thesis was concerned with the classification of curves on surfaces and it was published later in Crelle's Journal [**3**]; cf. also [**8**], [**13**]. Baer then taught for a brief period at the Odenwaldschule, a

[BULL. LONDON MATH. SOC., 13 (1981), 339–361]

private school, famous for its unconventional teaching principles. These impressed him, but Baer was not happy at schoolmastering.

He was therefore relieved to be offered an Assistantship by Alfred Loewy in Freiburg; a position he held from 1926–28. During this period and partly through the influence of Loewy, he became an algebraist. On 1 March 1928 he was granted the Habilitation by Freiburg University and taught his first course in the summer semester 1928.

During the previous autumn, Baer had been asked by a school friend to look up the daughter of a friend of his who was coming from Liepzig to continue mathematical studies at Freiburg. According to Baer's own account he complied, but grudgingly. Thus he met Marianne Kirstein, who was later to become his wife.

In 1928 Baer received the offer of a Privatdozentur at Halle an der Saale, through Hasse, who had recently gone there as full professor. The offer must have been doubly welcome, for it brought Baer closer to Marianne's home in Leipzig, where her father was a well-known publisher of art books.

Reinhold and Marianne's wedding took place in 1929. Their marriage was an extremely happy and close one. Their son, Klaus, was born on 22 June 1930. He has become an Egyptologist and is presently professor at the Oriental Institute and also of Near Eastern Languages, at the University of Chicago.

One of the old friends of the Kirstein family was Friedrich Levi, whom Marianne remembers from when she was a young girl. Levi was 14 years older than Baer and had started teaching at Leipzig University in 1919. Baer met Levi through Marianne and a warm friendship developed between them which lasted until Levi's death in 1966 [**158**]. They did much mathematics together during the five years Baer spent in Halle: both had a geometric background and both had become interested in group theory and its relation to topology. They wrote four papers together and these represent the only substantial research collaboration in Baer's life.

But he did collaborate on a project of a somewhat different kind with Hasse. They edited a re-publication, in book form, of the paper by Ernst Steinitz "Algebraische Theorie der Körper", which had appeared in Crelle's Journal in 1910 (vol. 87, pp. 167–309) and was already a classic twenty years later. It is now seen as one of the foundation papers of twentieth century algebra. The Baer–Hasse edition contained a commentary on the Steinitz text and an appendix, by Baer, on Galois theory.

* * *

At the beginning of 1933, Germany finally succumbed to the collective madness of Nazism. Hitler became Chancellor of Germany on 30 January 1933, the Reichstag building burned on the night of 27 February, and the elections of 5 March secured Hitler a majority in the Reichstag. The Nazis lost no time in promulgating laws removing all officials of Jewish descent from the civil service (which included University teachers) and closing the professions to all Jews. The purge of the Universities was effected by the very first of these laws, in April 1933.

Reinhold and Marianne had made arrangements to spend the whole spring and summer in Igls, in the Austrian Alps, and planned that Reinhold would return alone to Halle for the short summer semester. Baer already had a return ticket from Halle to Igls, when the University informed him that his services would not be required during the coming semester. Reinhold and Marianne left Germany just before the March elections. The return half of Baer's ticket remained unused.

The problem of where they would go, after their summer in Austria, was solved when Baer received an invitation from L. J. Mordell to come to Manchester. Hasse had been in touch with Mordell about Baer. In Britain (as in America) a committee had been formed to help German academics who had lost their jobs through the Nazis. This committee, the Academic Assistance Council, was funded by British academics who pledged part of their salaries to it. Baer was among the first refugee academics to be thus brought to Britain, and was supported by the committee at the University of Manchester for the two years 1933–35. He held the title of Honorary Research Fellow.

When the Baer family arrived in England, they stayed for the first three weeks with the Mordells, who were extremely kind and helpful to them. Marianne and Reinhold knew hardly a word of English, but they quickly learned to speak effortlessly and fluently, and were happy to be in Manchester. They met interesting non-mathematicians, something they always valued greatly wherever they lived. Among these were Leonard Palmer, then assistant lecturer in classics and later professor of comparative philology at Oxford, and the historian A. J. P. Taylor. The Taylors had a cottage in the Peak District and the Baers often spent week-ends with them there, as well as longer periods hiking in the Lake District.

During the academic year 1934/5 Hermann Weyl arrived at Oxford on a visit from America, where he had recently become a professor at the newly-created Institute for Advanced Study in Princeton. Baer had met Weyl in Göttingen in 1930 or 1931 and he now travelled down to Oxford to see him. At their meeting, Weyl invited Baer to come to Princeton for the following year and Baer accepted.

* * *

The Institute for Advanced Study was founded in 1930 and by 1935 there were 47 visiting members and six permanent professors: J. W. Alexander, A. Einstein, M. Morse, J. von Neumann, O. Veblen and H. Weyl. Princeton had replaced Göttingen as the capital of the mathematical world.

Baer was a member of the Institute from 1935 to 1937. His important work on infinite abelian groups, begun in Manchester, was continued here and led to a seminar during 1935/6. The following academic year he was invited to give a course of lectures at New York University, where Richard Courant was building up a graduate school. (Courant had been a professor at Göttingen and knew Baer from there.) Baer often claimed later that he never again had such good students at lectures as in his course at New York University.

In 1937 Baer was appointed to his first teaching job since leaving Germany. This was an assistant professorship at the University of North Carolina at Chapel Hill, where the chairman of the department was A. Henderson. At the same time N. Jacobson arrived, also coming from the Institute of Princeton. In later years Reinhold and Marianne spoke with pleasure of the beautiful rural setting of the town with its surrounding hills and woods.

But they were not to stay there for long. The University of Illinois at Urbana was anxious to strengthen its mathematics department. The head of the department, Arthur B. Coble, approached Baer, who by then had become known as a brilliant and very active algebraist but who did not, as yet, have a tenure position. The offer of an associate professorship came to Baer in 1938 and he accepted. The Baers were to live in Urbana-Champaign for 18 years, until they returned to Germany.

They quickly made friends through the weekly Friday evening Open House at the home of J. L. Doob. Here they met congenial people in a wide variety of professions, many of them also refugees from central Europe. They became part of a large and close-knit circle of friends.

But the flat countryside of central Illinois did not appeal to them. Their favourite landscape was always the mountains. During the summer of 1939 they went for the first time to the Rockies, at Estes Park, Colorado, and loved it there. They returned every year without break until around 1950. Often there would be other mathematical friends present, among them the Brauers, Weyls and Dehns.

At Urbana, Baer established himself as a very good teacher of advanced courses. But he also had to do his share of teaching elementary courses, such as calculus, where the text and content were prescribed and the students were usually uninterested. In the European context, many of these courses would be taught at school rather than university. Baer really hated this elementary teaching. The thought of escaping from it played a major part in his decision, many years later, to return to Germany.

In his research work, Baer was moving away from pure abelian group theory, although the famous paper in which he invented injective modules appeared in 1940 [52]. He turned his attention to groups appearing in geometry and to the group theoretical approach to the foundations of geometry. In a sequence of extremely influential papers [57], [72], [73], [77], [95], he showed how group theory can be used to study projective planes and how the internal structure of an abstract group can often be used to create geometrical configurations on which the group then acts as a recognisable group of motions. This work opened a new approach to combinatorics and is arguably his most distinctive contribution to mathematics.

In 1944, Baer was promoted to full professor. This was also the year that he and Marianne became American citizens.

During his time at Illinois, Baer supervised twenty students for the Ph.D. degree. Among these were R. A. Beaumont (his first), P. F. Conrad, D. G. Higman and K. G. Wolfson. His last student at Illinois was P. Dembowski, who went back to Frankfurt with Baer and actually took his degree there.

In 1948 Gerhard Hochschild joined the department and the Hochschilds and Baers became close friends. In the early fifties two other visitors arrived, with whom the Baers were to form lasting friendships: Beno Eckmann, who came only for the fall semester of 1951; and Michio Suzuki, who arrived in January 1952 on a University of Illinois Fellowship, and was to stay there permanently. Baer was instrumental in getting Suzuki to Illinois and helped him greatly when he arrived.

* * *

After the end of the war, when communication with continental Europe again became possible, Baer was quick to renew contacts with old mathematical friends in Germany. In 1950, he and Marianne travelled back briefly to Europe for the first time. Two years later they spent the whole academic year there. This was Baer's first and only sabbatical leave from Illinois. He divided his time between the Eidgenössische Technische Hochschule in Zürich and the Universities of Hamburg, Frankfurt am Main, Tübingen and Bonn. In Frankfurt he again met Wolfgang Franz, who was keen to attract Baer back to Germany, and they discussed the possibility of a position for Baer in Frankfurt.

During that sabbatical year Baer made a new contact, which was to be of importance later: W. Süss invited Baer to his international mathematical institute at the former hunting lodge of a Hessian aristocrat, located near the tiny hamlet of

Oberwolfach-Walke in the Black Forest. Baer was to spend many happy weeks here in the years to come, talking and listening to mathematics.

The Baers returned to America. But they had been impressed to find cultural life renewing itself in Germany and felt it possible to consider living there again.

Then Baer received the invitation to a professorship at Frankfurt University. The chair he was offered had once been occupied by Max Dehn until he too had been driven out by the Nazis. An extremely difficult choice faced Baer and his wife. There were obvious and strong reasons for not returning. Moreover, their son was firmly launched on his academic career in Chicago and entertained no thought whatever of leaving America. On the other hand, Baer felt that his position in Illinois did not give him sufficient independence to teach and order things in his own way. German professors still had this independence. He finally accepted the chair; and with it, the challenge of creating his own teaching unit. His great success at Frankfurt is proof of the wisdom of that decision.

* * *

Baer was fifty-four years old when he returned to Germany in 1956. The mathematical community that he joined at Frankfurt was very small: Franz and Ruth Moufang held the only senior positions and E. A. Behrens, A. Burger and L. Sauer the junior ones. Within an astonishingly short time Baer had built up at Frankfurt one of the liveliest schools of algebra in Europe. Between 1956 and his retirement in 1967 he supervised nearly thirty students for their Ph.D. degree. Many of these now teach at Universities across Germany and North America. Among his students were H. Bender, B. Fischer, H. Heineken, C. Hering, O. H. Kegel, H. Lüneburg, and G. Michler, all of whom now hold full professorships in Germany.

The German teaching system suited him. Students attended lecture courses in the usual way but they also had to participate in seminars. Professors had assistants, whose task it was to prepare students for the seminars and to advise them with their thesis work. Theses were required not only for the Ph.D., but also for the first examination (the Diplom or Staatsexamen). Baer was equally involved with both types of theses, discussing them at length and returning the drafts with copious annotations. Yet he never imposed his own mathematical style on his students. Although always ready to provide help and criticism, he encouraged them to develop their own mathematical personalities.

His flexible teaching methods also involved his assistants. They were given a free hand to direct students, each in his own way; and thus became personally involved in the teaching process. Baer's first assistant was H. Goetz, who was succeeded by Kegel in 1960. A second assistantship went to H. Salzmann. After Salzmann "habilitated", he stayed on as Diätendozent (lecturer). Among Baer's assistants, at various times, were Heineken, Fischer, Michler and W. Kappe. At one stage there were three assistants. But Baer managed, by means of fellowships and research grants, to have always at least double this number of young mathematicians in his entourage. He was by no means isolated from the rest of the faculty: he readily helped Behrens' students; and his own loved pupil, Peter Dembowski, whom he had brought back from Urbana, became an assistant to Ruth Moufang. Kegel recalls that everyone, be they assistants, fellowshipholders or lecturers, helped with the general work as and when necessary.

Baer had created a real community of working mathematicians. But it evolved further. He and Marianne made it into a social community, with wives and husbands an integral part. It became an extended family for Reinhold and Marianne. They were hospitable, generous in help and very protective of their "children". Baer took enormous care and trouble to help his pupils into suitable jobs. He showed them off at international meetings at Oberwolfach, he sent them abroad and he exposed them to visitors at Frankfurt.

There were plenty of such visitors. Baer was very clever at discovering sources of money, but of course was lucky to be operating during a decade of expanding resources. He tapped NATO funds long before it became common mathematical practice. "Uncle Nato will pay", he wrote to me once. All of us who visited Frankfurt were charmed by the relaxed and generous welcome provided by Reinhold and Marianne and enjoyed wide ranging and entertaining conversations with them. There is no denying that Reinhold's wit could be biting and his scorn devastating. But neither was ever directed at young people, only at well-established and, on the whole, well-deserving targets.

At the University, Baer controlled his time tightly. He usually arrived in the early afternoon. His secretary and his assistants were primed and ready to help him deal rapidly with the paper work and interviewing of students; he attended his Seminar and he gave his lectures. By early evening he was on the train back to his home in the Taunus mountains.

Baer did a great deal of mathematical travelling, both during his Frankfurt years and later in his retirement. He held Visiting Professorships at Chicago (1958), the University of California at Berkeley (1963) and the Universities of Florence, Naples and Padua (1965); and he was Distinguished Professor at the New Mexico State University at Las Cruces for 1967/8. He attended two international group theory meetings in Canberra, Australia, lectured in New Zealand, South Africa, Japan and widely in North America. He often came to England, and he was warmly appreciative of the excellent working atmosphere at Warwick, where he attended Symposia in 1966, 1973 and 1977.

His most frequent trips were to the Mathematical Research Institute at Oberwolfach. Nowadays, the Institute functions practically continuously and meetings need to be booked two years in advance. This was not so in the fifties. Süss desperately wanted to keep his Institute going and was inviting everyone possible to use it for meetings. Baer needed little persuasion: he could see the tremendous potentialities of the place and became responsible over the years for many of the regular international meetings on group theory and geometry. At least once a year he would bring his Frankfurt group there in order to have the leisure to listen to and discuss everyone's mathematics. After the sudden death of Süss in 1958, a legal basis for the Institute had to be created and the "Gesellschaft für Mathematische Forschung" came into being. Baer was one of the 15 founding members. He served on the scientific advisory council (Wissenschaftlicher Beirat) until 1968 and continued as a member of the association until his death. During the seventies, he and Marianne would often spend 4 or 5 weeks at the Institute, usually in the spring, where there would be a sequence of meetings of interest to him. It was rare indeed for him to miss a single lecture.

* * *

Baer retired at Frankfurt in the summer of 1967 and planned to leave the Frankfurt area. He felt that no university professor should be burdened by having his predecessor around; but he had no intention of retiring from mathematics. He and Marianne chose to live in Zürich. Here there were old friends such as Hopf, van der Waerden and Eckmann. Some high level help secured them the status of permanent residents (not a triviality in a Swiss canton!). They liked the city and the many cultural activities it offered; they were still close to Oberwolfach and closer to Pontresina, their favourite spot in the Alps. It was in all respects a good choice.

Beno Eckmann ensured that Baer became an honorary member of the Mathematisches Forschungsinstitut. Eckmann had founded this in 1964 and it has become a flourishing research centre with a constant and large stream of visitors. Baer probably attended more lectures and colloquia here than any other member.

He continued to be sought out by young mathematicians and to give help freely. He never ceased to do mathematics. Indeed, it is a remarkable fact that the flow of his mathematical invention continued unabated until his death.

He carried on his editorial work for the Archiv der Mathematik, begun in 1959. Throughout his career he had been associated in an editorial capacity with the publishing of mathematics, and always regarded this as an important part of his work. Apart from the Archiv, he served as an editor for Compositio Mathematica (1934–1966), the American Journal of Mathematics (1949–1955), the Illinois Journal of Mathematics, which he helped to found (1956–1963) and the Springer–Verlag series, Ergebnisse der Mathematik (from 1952).

Three Universities conferred Honorary Degrees on Baer: Giessen in 1974, Kiel in 1976 and Birmingham in 1978.

He became ill in 1976. Cancer of the stomach was suspected in early summer (but not confirmed until a year later). He was put on a severely restricted diet but his mathematical life continued almost normally. In July 1977 he even managed a lecturing trip to East Germany. This gave him a chance to return for the first time to the scenes of his youth in Halle. Early in 1978 a deterioration in his condition made an operation necessary. It was a success. In July 1978 he could travel to Birmingham to receive his honorary doctorate. His last Oberwolfach appearance was in May 1979 when he was judged to be alert as in the years before his illness. But in August his condition worsened rapidly; he entered hospital and died, suddenly of heart failure, on October 22nd.

* * *

The joy that Baer had in creating mathematics never left him. It was one of his remarkable gifts that he could communicate this excitement and fun to his pupils. His own research during the last two decades of his life was certainly overshadowed by his extraordinary success as a teacher. The memorial meeting in February 1980 at Frankfurt, organised by Professor H. Behr (who presently holds Baer's chair), drew well over two hundred people from all over Germany, a large proportion of them former students or associates.

Many of us will always remember Reinhold Baer as he was at Oberwolfach. A trim and fit figure, dressed in an open-neck white shirt, grey flannel trousers and tennis shoes. A happy smile on his face, Marianne at his side, and surrounded by young people.

Acknowledgments. The preparation of this biographical notice would not have been possible without the generous assistance of many people. I am particularly indebted to O. Kegel; and I wish to thank him and also all the following, who provided me with much useful information: M. Barner, E. A. Behrens, N. Blackburn, A. Clifford, J. L. Doob, B. Eckmann, W. Franz (who kindly allowed me to see the manuscript of his lecture at the Baer Memorial Meeting in Frankfurt), P. Halmos, D. G. Higman,

K. A. Hirsch, G. Hochschild, N. Jacobson, I. Kaplansky, W. Magnus, B. H. Neumann, E. Shiels, M. Suzuki, A. J. P. Taylor, C. Underwood and H. Wielandt.

Above all, I am deeply grateful to Marianne Baer for giving me many hours of patient help and constructive criticism, and I thank her most warmly.

THE MATHEMATICAL WORK OF REINHOLD BAER

This survey of Baer's work is necessarily incomplete and does not even mention some of the areas to which he contributed, such as field theory and ring theory. The work that I have selected for detailed discussion is arranged under the following headings:

1. Topology
2. Abelian groups
3. Geometry
4. Extension theory
5. Other group theory

The section on geometry is largely based on a manuscript very kindly made available to me by H. Salzmann. I am also most grateful for help by D. B. A. Epstein, T. O. Hawkes, D. R. Hughes, I. Kaplansky, U. Stammbach, D. J. S. Robinson and H. Wielandt.

Topology

Baer's most influential papers on topology were his first two, [**3**], [**8**], on surfaces, and his last one, [**32**], on the Kuroš subgroup theorem, which was joint work with F. Levi.

The problem solved in [**3**] (1927) is the classification of the homotopy classes of closed curves on a closed orientable surface of genus > 1. Baer's idea is to choose a particular family Σ of standard simple closed curves on such a surface S and to code an arbitrary closed curve C by specifying a sequence in its intersections with the members of Σ. When C is a geodesic, these intersections can be taken as the actual geometric meeting points, but in the general case one must use "intersection classes", an idea of J. Nielsen: Let U be the universal cover of S and choose a fixed open covering $\tilde{C}$ of C (assumed not null-homotopic). If $K \in \Sigma$ and $\tilde{K}$ is one of its open covering curves, then the image on C of the point-set $\tilde{C} \cap \tilde{K}$ is the intersection class "generated" by $\tilde{K}$. These intersection classes are therefore effectively determined by the way each $\tilde{K}$ separates (or fails to separate) the end points of $\tilde{C}$ on the boundary of the hyperbolic plane U. The intersection sequence of C with Σ, together with a simple additional condition, provides a complete system of invariants for the homotopy class of C. These ideas of Baer foreshadow the fundamental work of Marston Morse on symbolic dynamics.

The 1928 paper [**8**] is a continuation of [**3**]. Here Baer proves that two simple closed curves on an orientable closed surface S of genus > 1 are homotopic if, and only if, they are isotopic. He then uses this theorem to investigate when two self-homeomorphisms of S can be deformed one into the other. Let μ be the natural

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homomorphism of Aut S, the group of all self-homeomorphisms of S, into Out $\pi_1(S)$, the group of all automorphism classes (the outer automorphism group) of $\pi_1(S)$. He shows that a self-homeomorphism ϕ belongs to the kernel D of μ if, and only if, ϕ is a deformation (*i.e.*, is isotopic to the identity). The image of μ was found by Nielsen in the first of his three long papers on surfaces (Nielsen (1927)): every automorphism class is the image of some self-homeomorphism of S. Hence Aut S/D, which is often called the mapping class group of S, is isomorphic to Out $\pi_1(S)$. This is a basic result in the theory of Teichmüller spaces. Baer's work was later generalised by various authors, most notably and most recently by D. B. A. Epstein (1966). (Cf. also Joan Birman (1977).)

Baer's involvement in topology ended with **[32]** (1936), the last of the four papers that he wrote with Levi. This contains a beautifully transparent proof, by covering space arguments, of the Kuroš theorem on the structure of subgroups of free products of groups, as well as some new consequences of the theorem. It was thirty years before the Baer–Levi proof found its rightful way into the textbook literature, when it appeared in the book by W. S. Massey (1967).

Abelian groups

The theory of infinitely generated abelian groups was a young subject when Baer began working in it during the early thirties. Levi's thesis in 1917 had been followed a few years later by the basic contributions of H. Prüfer and in 1933 the subject had its first spectacular success with H. Ulm's classification of the countable torsion groups. Baer moved into the area on a broad front. He studied torsion groups **[29]**, **[27]**, torsion-free groups **[36]**, mixed groups **[33]**, **[122]**, extensions **[52]**, **[86]**, **[88]** and automorphism groups and endomorphism rings **[38]**, **[60]**. Many of his results and ideas have become indispensable ingredients in any modern exposition of abelian group theory. Some of the problems he raised in the early papers required decades of work by many people before they were solved.

A recurring theme in Baer's work is the extent to which the structure of an abelian group A is determined by the ideal structure of its endomorphism ring End A and by the normal structure of its automorphism group Aut A. His first major paper on these matters is **[29]** (1935), where A is assumed to be a direct sum of cyclic p-groups and groups of type p^∞. He determines the orbits in A of Aut A; enumerates the characteristic subgroups of A and establishes a correspondence between them and the normal subgroups of Aut A. A generalised version of these results is given in §18 of Kaplansky's book (1954).

Similar questions are studied in chapter 3 of the influential paper **[36]** (1937), but for torsion-free groups. Here there appears for the first time the standard classification of torsion-free groups of rank 1 in terms of Steinitz's "supernatural numbers", and the notion of separability: the torsion-free group A is separable if every finite subset is contained in a completely decomposable direct summand of A. Complete decomposability is the main theme of the paper: a torsion-free group is called completely decomposable (Baer says "completely reducible") if it is the direct sum of groups of rank 1. Baer proves that two such direct decompositions are isomorphic; that countable separable groups are completely decomposable; that a direct power R^n of a group R of rank 1 is completely decomposable if, and only if, the cardinal n is finite or $R = \mathbb{Q}$ (so that, for example, the group of all sequences of integers is not free). He also establishes a special case of the celebrated Baer–Kulikov–Kaplansky theorem: direct summands of completely decomposable groups are completely decomposable. Baer proved this when the genus set (the set of "types" in modern terminology) of the direct summand satisfies the maximal condition; Kulikov (1952) when the group is countable; and Kaplansky (1958) showed how countability can be removed. A good account of this material and later developments can be found in L. Fuchs' book (1973). Baer's paper was the first attempt at setting up a structure theory for torsion-free groups.

A year earlier, in [**33**] (1936), Baer attacked the problem of mixed groups. Levi (1917) had given the first example of an abelian group that does not split over its torsion subgroup. Baer wanted general theorems. Let T be a torsion group and J a torsion-free group. Baer produces necessary and sufficient conditions for every extension of T by J to split, provided only that J satisfies a certain set-theoretic restriction (satisfied, for example, if J is countable). These conditions are interlocking structural properties of T and J. They reduce to properties of the groups separately if one asks for universal splitting conditions: (1) T is a direct summand of every abelian group of which it is the torsion group if, and only if, T is a direct sum of a group of bounded order and a divisible group. (2) Every abelian group, whose torsion-free factor is J, splits over its torsion group if, and only if, J is free abelian, provided that J satisfies the above set-theoretic restriction. What happens without the restriction was left unresolved. There the matter stood for more than 20 years. By then, homological algebra had revived interest in abelian group theory and the problem could be attacked with this new tool. The final solution was found by P. Griffith (1969): J is free, irrespective of any set-theoretic restriction (cf. Fuchs, §101).

In [**52**] (1940), Baer considered a problem related to (1) above. It was well known that an abelian group G is a direct summand of every containing abelian group if, and only if, G is divisible (the first published proof apparently appears in [**33**]). Baer generalized this to modules over an arbitrary ring R. The splitting property is now called R-injectivity. Baer proved that a left R-module is injective if, and only if, every module homomorphism of a left ideal of R can be extended to one of R; and that, if every left ideal of R is principal, then injectivity is equivalent to divisibility. He also established the existence of injective envelopes. The rise of homological algebra pulled this little article into prominence and it has become one of Baer's most widely quoted papers.

Geometry

The study of abelian groups led Baer to the foundations of projective geometry. The rapid development of lattice theory in the mid-thirties suggested that a projective geometry should be viewed as a special kind of lattice, the lattice of all subspaces of a vector space. It was reasonable then to think of the subgroup-lattice of an abelian group as a generalized projective geometry and of an isomorphism of subgroup-lattices as a projectivity of the underlying groups. Baer proved many results about such projectivities ([**44**] 1939), and also about dualities, which means order-reversing bijections of subgroup-lattices ([**48**] 1939). For a concise account of some of these results, I refer to Suzuki's book (1956) (particularly Theorem 2, p. 35 and Theorem 1, p. 87). In an impressive paper in 1942 [**58**], Baer partially solved the problem of treating abelian groups and projective spaces in a unified way by dealing with a class of modules over a type of non-commutative discrete valuation ring. (Cf. also the later, but independent study by E. Inaba (1948), and Baer's 1968 papers [**166**], [**167**].)

Baer's work on these matters culminated in his book [**96**] (1952). This is an account of the representation of projective geometries by vector spaces over division rings, of projectivities by semi-linear transformations and of dualities by semi-bilinear forms. The treatment is restricted throughout to vector spaces: [**58**] is only mentioned in passing, though many ideas of that paper appear here in specialised form. No finiteness restrictions are imposed, except where they are known to be necessary. Baer's book remains the only full and general account of the foundations of projective geometry.

It does not touch at all, however, on the subject of projective planes, except those that are desarguesian. Until the nineteen-forties, non-desarguesian planes occupied little more than a footnote to projective geometry. This changed with the basic work of Marshall Hall (1943) and Baer [**57**], [**72**], [**73**], [**77**]. Projective planes began to develop into a subject in its own right. Baer's ideas proved to be extremely fruitful and much of what he did then has become standard material, as can be seen, for example, in the books of Dembowski (1968) and Hughes–Piper (1970).

Until the appearance of [**57**] (1942), projective planes were always approached via coordinization procedures and the significance of Desargues' Theorem was always expressed in the following form: Desargues' Theorem is true in a projective plane if, and only if, the plane can be realised as the lattice of subspaces of a 3-dimensional vector space over some division ring. Baer introduced an entirely different way of recognising the desarguesian property, by expressing it in group-theoretic terms. Let P be a point and l a line in a projective plane $\mathscr{P}$. Then $\mathscr{P}$ is called (P, l)-*transitive* if to any distinct points A, B on a line through P (where A, B are distinct from P and not on l) there exists a perspectivity (central collineation) σ (necessarily unique) with centre P, axis l and $A^\sigma = B$. The connexion with Desargues' Theorem is expressed in the following result: the plane is (P, l)-transitive if, and only if, every pair of triangles in perspective from P has an axis of perspective and this is l (or, as one usually says, $\mathscr{P}$ is (P, l)-*desarguesian*). The possible configurations formed by the set of all points P and all lines l for which a plane can be (P, l)-transitive were found by Lenz (1954) and Barlotti (1957) (cf. Dembowski pp. 123–127). They provide a very useful classification of projective planes.

In [**73**] (1946) Baer turned his attention to the geometric structure determined by a single collineation. Let $\mathscr{P}$ be a projective plane and σ a collineation of it. If $\mathscr{F}(\sigma)$ denotes the fixed configuration for σ (the set of all points and all lines fixed by σ), then $\mathscr{F}(\sigma)$ is closed (in the sense that the join of two points in $\mathscr{F}(\sigma)$ is a line in $\mathscr{F}(\sigma)$ and also dually). Baer discovered that if $\mathscr{F}(\sigma)$ is dense in $\mathscr{P}$ (meaning that every element in $\mathscr{P}$ is incident with an element of $\mathscr{F}(\sigma)$), then the configuration $\mathscr{F}(\sigma)$ already yields important information on σ. Dense subsets are now called *Baer subsets*. A proper closed Baer subset is a maximal closed configuration and is actually a proper subplane of $\mathscr{P}$ or else it consists of a line l and all points on it and a point P and all lines through it. A *quasiperspectivity* is a collineation σ for which $\mathscr{F}(\sigma)$ is a Baer set. All involutions (collineations of order 2) and all perspectivities are quasiperspectivities. The result above on closed Baer sets implies that every quasiperspectivity is planar or is a perspectivity. Baer also shows that if $\mathscr{P}$ has finite order n, then a non-identity collineation is a quasiperspectivity if, and only if, it has at least n fixed points. He gives a classification of quasiperspectivities according to their numbers of fixed points: $n+1$ for elations, $n+2$ for homologies, or $n+\sqrt{n}+1$ for planar quasiperspectivities.

In [72] (1946), Baer subjects the polarities of a finite projective plane to a similar investigation. A polarity is an auto-duality (or correlation) of order 2. Let π be a polarity in the plane $\mathscr{P}$ of order n. The configuration of interest is the set of absolute points and absolute lines for π. (A point P is absolute for π if P lies on the line P^π; and dually for absolute lines.) Baer shows that the number of absolute points of π is always $\geqslant n+1$ and that it is precisely $n+1$ if n is non-square. In the case of $n+1$ absolute points, these form an oval if n is odd and they are collinear if n is even. More delicate results concern what Baer calls regular polarities: the polarity π is regular if the number of absolute points on a nonabsolute line is $s+1$ for some fixed integer $s = s(\pi) \geqslant -1$. (In a desarguesian plane, all polarities are, of course, regular and s is 1 or $\sqrt{n}$.) If π is regular, then $1 \leqslant s^2 \leqslant n$ (it is still unknown if $1 < s^2 < n$ is possible); and if $s^2 = n$, then the absolute points and nonabsolute lines form a unital. The kind of argument used in this paper (and in [77]) has become the prototype for many subsequent investigations, where pure counting techniques lead to far-reaching structural consequences for finite incidence geometries.

There is another type of problem connected with polarities that fascinated Baer. Can the group of a given polarity (not now necessarily a plane) be characterised by the way it acts on the incidence structure? An example is in [**92**], where Baer effectively solves the Helmholtz space problem for geometries over arbitrary ordered division rings. Another example is in the interesting paper [**95**] (1951). Here Baer associates an incidence structure $D(G)$ with an abstract group G by defining the elements of G to be both the points and the hyperplanes of $D(G)$ and "a incident with b" to mean that ab is an involution. He proves the surprising result that $D(G)$ is a projective geometry of dimension > 1 if, and only if, G is isomorphic to the special orthogonal group of a symmetric form, with no absolute points, on a 3-dimensional vector space over a field of characteristic not 2.

The concept of incidence group developed from this paper. The idea is to have a group whose elements are the points of a geometric structure so that multiplication by each group element is a geometric automorphism.

A good example is the case of a group G admitting a non-trivial partition: this is a family of proper subgroups $\mathscr{U} = (U_i\,; i \in I)$ so that every $g \neq 1$ lies in exactly one U_i. The points of the resulting incidence structure are the elements of G, the blocks are the cosets $U_i x$ and incidence is defined by containment. [**143**] (1963) begins a possible programme for classifying the partitions of an abelian group. This is of geometric importance because of André's discovery (1954) of a canonical correspondence between the congruence partitions of an abelian group and (affine) translation planes. (A partition $\mathscr{U}$ is a congruence partition or "plane partition" as Baer calls it, if $U_i U_j = G$ for all $i \neq j$.) A little earlier, in [**136**], [**137**], [**138**], Baer studied normal partitions of a finite group (meaning that every U_i is normal). If G has such a partition, then G is a Frobenius group or else each U_i is nilpotent (Kegel 1961), and if G is non-soluble, the structure of G is known explicitly (Suzuki 1961). These normal partitions also have geometric significance (cf. Dembowski pp. 164–166).

The 1963 paper [**143**] is the last one of primarily geometric interest that Baer wrote. But his influence on combinatorial geometry, and in particular his insistence that the subject needs strong links with group theory, continued to be felt through his teaching and mathematical contacts. It remains very much alive today.

Extension theory

The first paper that Baer published on extension theory [**24**] (1934) is a remarkably original piece of work. Eight years earlier, in 1926, Schreier's classic paper on group extensions had appeared. These two papers together established the modern theory of group extensions. It is worth taking a detailed look at Baer's paper.

He begins by considering a group extension $\mathfrak{N} \rightarrowtail \mathfrak{G} \twoheadrightarrow C$ and the associated coupling $\chi : C \to \operatorname{Out} \mathfrak{N}$, which arises from the conjugation action of $\mathfrak{G}$ on $\mathfrak{N}$ (he calls χ a "Kollektivcharakter von C in $\mathfrak{N}$"). If $\mathfrak{Z}$ is the centre of $\mathfrak{N}$ and $G = \mathfrak{G}/\mathfrak{Z}$, then conjugation by $\mathfrak{G}$ yields a homomorphism $\chi_G : G \to \operatorname{Aut} \mathfrak{N}$ (a "character of G in $\mathfrak{N}$") so that

$$\begin{array}{ccc} G & \twoheadrightarrow & C \\ {\scriptstyle \chi_G}\downarrow & & \downarrow{\scriptstyle \chi} \\ \operatorname{Aut} \mathfrak{N} & \twoheadrightarrow & \operatorname{Out} \mathfrak{N} \end{array}$$

is a commutative square. Conversely, given two groups $\mathfrak{N}$, C and a homomorphism $\chi : C \to \operatorname{Out} \mathfrak{N}$, there always exists a square as above and the resulting group extension $N \rightarrowtail G \twoheadrightarrow C$, where $N = \operatorname{Inn} \mathfrak{N}$, is uniquely determined. Baer proves this (§1, Satz 2) by constructing the pull-back and verifying its uniqueness.

Let $\operatorname{Ext}(C, \mathfrak{N}; \chi)$ denote the set of all equivalence classes of extensions of $\mathfrak{N}$ by C with coupling χ. Assuming this set to be non-empty, Baer proves (§7, Satz 1 and p. 415) that it is bijective with a coset in $\operatorname{Ext}(G, \mathfrak{Z}; \chi)$ of a subgroup E defined as follows: E consists of all extensions which are split when restricted to N and which act by conjugation on N exactly as does G. (E is bijective with $\operatorname{Ext}(C, \mathfrak{Z}; \chi)$, but this is not stated explicitly.) An example is given to show that not every triple $\mathfrak{N}, C, \chi$ gives rise to an extension. However, the fact that there exists a whole theory of obstructions to group extensions had to await discovery until 1947, by S. Eilenberg and S. Mac Lane.

The case of extensions with an abelian kernel is discussed in §§2, 3. To obtain the structure of $\operatorname{Ext}(G, \mathfrak{A}; \chi)$, where $\mathfrak{A}$ is now abelian (so that χ makes $\mathfrak{A}$ into a G-module), Baer takes a free presentation of $G : R \rightarrowtail F \twoheadrightarrow G$, where F is a free group. He shows (§2, Satz 1) that every extension of $\mathfrak{A}$ by G determines a group homomorphism of R into $\mathfrak{A}$ with the property that conjugation by F in R translates to the module operation on $\mathfrak{A}$; and conversely (here a push-out construction is used). This result therefore establishes a surjection:

$$\operatorname{Hom}_F(R, \mathfrak{A}) \twoheadrightarrow \operatorname{Ext}(G, \mathfrak{A}).$$

Suppose the homomorphisms ϕ_1, ϕ_2 determine equivalent extensions. This happens (§2, Folgerung 2 zu Satz 2) if, and only if, there exists a derivation (or crossed homomorphism) δ of F into $\mathfrak{A}$ so that δ restricted to R is $\phi_1 - \phi_2$. (Baer does not use the term derivation but gives the appropriate definition.) Thus, if $\operatorname{Der}(F, \mathfrak{A})$ is the group of all derivations of F in $\mathfrak{A}$, then the cokernel of the restriction map

$$\operatorname{Der}(F, \mathfrak{A}) \twoheadrightarrow \operatorname{Hom}_F(R, \mathfrak{A})$$

is bijective with Ext $(G, \mathfrak{A})$. This theorem reappeared, from a different point of view, in Mac Lane (1949). Baer also discusses his results in terms of Schreier's factor-sets, but emphasises the advantages of his multiplicative approach. He was right to do so: it is now known that factor-sets are actually a special case of Baer's multiplicative theory. In §3, he determines explicitly the group structure of Ext $(G, \mathfrak{A})$ as induced by $\mathrm{Hom}_F(R, \mathfrak{A})$. This group operation has come to be called the *Baer sum* of extensions. The same construction is valid for many other algebraic systems.

Baer returned to problems involving extension theory in many later papers. A particularly interesting one is [**91**], which contains an unexpected generalization of Artin's Zerfällungsgruppensatz (Zassenhaus 1937, p. 98).

The very long three part paper [**69**] was surely inspired by H. Hopf's discovery (1942) that, for any extension $R \rightarrowtail F \twoheadrightarrow G$, where F is free, the groups $[F, F]/[R, F]$ and $R \cap [F, F]/[R, F]$ depend only on G and not at all on the free extension. Baer's programme was to classify expressions of a given group G in the form H/M, where H belongs to some sensible family of groups, and to define invariants of G relative to this family. The general classification processes occupy Parts I and III of the paper. Baer was aware that portions of his theory might be expressible in the framework of the then infant homological algebra (cf. [**67**], p. 158). This was shown to be the case by A. Fröhlich in his 1963 paper on *Baer-invariants* of algebras (cf. also J. Stallings (1965)). Baer-invariants have found application in associative algebras, Lie algebras and varieties of groups.

Part II of [**69**] deals with some concrete group-theoretic consequences of the general theory. If M is a normal subgroup of H, define ${}_iM$ inductively by ${}_0M = M$, ${}_iM = [H, {}_{i-1}M]$. When $M = H$, this becomes the lower central series $({}^iH)$ of H. Baer proves (p. 369) that

(*) if H/M is finite, then so is ${}^iH/{}_iM$, for all i.

(The case $i = 1$ is due to I. Schur.) If H is a free group, then (by Part I) the groups ${}^iH/{}_iM$ and $({}_iM \cap {}^{i+1}H)/{}_{i+1}M$ are invariants of $G = H/M$. Let $W/{}_{i+1}M$ be a complement to $({}_iM \cap {}^{i+1}H)/{}_{i+1}M$ in ${}_iM/{}_{i+1}M$. The set of all groups ${}^iH/W$, for fixed i and varying W, is classified by means of suitable groups of central extensions. (When $i = 0$, the groups ${}^iH/W$ are the Darstellungsgruppen of Schur.)

The result (*) has given rise to interesting group-theoretic developments. In [**97**] (1952) Baer generalizes the case $i = 1$ in the following characteristic way: If A is a group of automorphisms of H so that the set H^{-1+A} (of all elements $h^{-1}h^a$) is finite (for example, if A and $(H : H^A)$ are finite), then the subgroup $[H, A]$ is finite. Philip Hall in his Edmonton Lectures (1957, chapter 8) suggested a general context in which to view these types of problems. There is a full discussion of all this in D. Robinson (1972, §§4.1, 4.2).

Other group theory

Baer's principal contributions to group theory from 1950 onwards are in soluble and nilpotent groups and in finiteness conditions on groups.

The study of infinite groups "which are in one sense or another close to abelian groups, under restrictions which are in one sense or another close to the finiteness of the groups" had become by 1950 a new branch of group theory. This was largely due to the work of Russian group theorists and especially to the efforts of A. G. Kuroš and his school. The quotation above comes from the introduction to the survey article by Kuroš and S. N. Černikov published in 1947 (but not made available in English until 1953). This survey and the book of Kuroš (the first edition appeared in 1944 and the English translation of the second edition in 1955) were both immensely influential in the development of the subject, first in Russia and later in the West. The present state of the theory is excellently described in Derek Robinson's two books (1972). In what follows I shall use Robinson's terminology.

When Baer began working in this area he was quite alone in America. His first substantial study of generalized nilpotent groups is **[53]** in 1940. He returned to this theme on a massive scale in 1953 with a series of papers on the hypercentre **[103]**, **[104]**, **[105]** (also **[106]**, **[113]** and **[100]**). His aim, as so often in his group theoretic work, was to characterise the positioning of a subgroup (in this case its containment in the hypercentre) by purely lattice theoretic and factorial properties. He proved in **[103]** that the normal subgroup N of G is contained in $\zeta(G)$, the hypercentre of G, if, and only if, N satisfies the following two conditions: (a) for every normal subgroup M of G properly contained in N there exists a normal subgroup V of G so that $M < V \leqslant N$ and V/M, as a G-operator group is residually finite and simple; (b) for every M as in (a) and x of order p^s in N/M, g in G/M, there exists m so that x and g^{p^m} commute. In **[104]** he introduced a hypercentrality condition of an extremely general type: N is "hypercentral" in G, or *Baer-hypercentral* as one says nowadays, if the image of N in every finite section of G is hypercentral in that section. He showed that, in the above characterisation of normal subgroups in $\zeta(G)$, condition (b) can be replaced by Baer-hypercentrality. He also proved that N is Baer-hypercentral in G if, and only if, for each $x \in N$, $g \in G$, $\langle x, g\rangle$ is Baer-nilpotent. Here, a group G is *Baer-nilpotent* if G itself is Baer-hypercentral in G. This is the first appearance of a type of criterion that was to become important later in the work of many people in connexion with Engel conditions.

Relatively little is still known about Baer-nilpotent groups. Baer himself showed **[105]** that a soluble Baer-nilpotent group is locally nilpotent; and he pointed out in **[152]** that this fact and results of Mal'cev and Zassenhaus together imply that a Baer-nilpotent linear group is hypercentral (cf. Robinson II, p. 35).

Many years later, in **[183]** (1974), Baer again studied the problem of hypercentral containment. Here he also generalized Černikov's criterion for hypercentrality (cf. Robinson I, p. 50) in the following way. In a group G, a sequence $(c_1, c_2, \ldots)$ is called a G-commutator sequence if $c_{i+1} = [c_i, x_i]$ for some x_i in G and all $i \geqslant 1$. Baer proves that the set of all elements g, so that every G-commutator sequence containing g contains only finitely many distinct elements, is precisely the characteristic subgroup Q with the property that $Q/P = \zeta(G/P)$, where P is the product of all finite normal subgroups of G. (There is much more on these matters in **[188]**.)

Perhaps the most influential single paper that Baer wrote on generalized nilpotent and soluble groups is **[111]** (1955). In this, he introduced a very useful class of generalized nilpotent groups and showed how a radical can be associated with the class. More important, he demonstrated that Wielandt's notion of subnormality also has a significant part to play in this area of infinite group theory. The objects of study in **[111]** are groups in which every cyclic subgroup is subnormal. Such groups are now called *Baer groups*. He proved that they are locally nilpotent and that, in every group G, the product $B(G)$ of all normal Baer subgroups is again a Baer group. In fact, $B(G)$ is exactly the set of all elements in G that generate a subnormal subgroup. One calls $B(G)$ the *Baer-radical* of G. It generalizes the Fitting radical (which is the product of all normal nilpotent subgroups) and it is always contained in the Hirsch Plotkin radical.

Engel conditions make their first appearance in Baer's work in [**111**]. They reappear frequently, notably in [**116**], [**144**], [**152**]. The 1957 paper [**116**] contains his famous theorem that in a group G with Max, the maximal condition on subgroups, the set of all left Engel elements is precisely the Fitting radical of G and the set of all right Engel elements is the hypercentre of G. The proof is extremely ingenious. The theorem is still almost the only general and non-trivial fact known about the class Max. It is also one of those rare results on infinite groups that specialises to a striking new theorem on finite groups: A p-element g of a finite group generates a normal p-subgroup if, and only if, any two conjugates of g generate a p-subgroup. This result (rediscovered by Suzuki (1965)) has turned out to be very useful in the theory of finite simple groups. Baer returned to this theorem in [**189**] (1977). Of the many papers written by others on these and related questions, the relatively recent one by Wielandt (1974) is perhaps one of the most striking. The work done on Engel theory before 1972 is fully discussed in Robinson, volume II.

Baer extended his theorem on the left and right Engel elements to groups with Max-*ab*, the maximal condition on all abelian subgroups, in [**152**] (1965). In the same work he replaces the Engel conditions by abstract properties of subsets that encapsulate what is needed to make the Engel arguments work. He was fond of this kind of experimentation. It stems partly from his attempts to understand proofs by isolating exactly what is used, precisely where. These analyses led to constructions of new group-theoretic properties and functions from given ones. The papers [**117**], [**123**], [**140**], [**141**], [**146**], [**149**], [**153**], [**168**], [**191**] are good examples of this approach. But it is present to some degree in almost all his later work and it influenced that of many others.

One of his favourite constructions was the following (*e.g.*, [**142**], §3): If $\mathfrak{A}, \mathfrak{B}$ are classes of groups (or group theoretic properties), then $(\mathfrak{A}, \mathfrak{B})$ is to denote the class of all groups G so that if H is a non-trivial homomorphic image of G, then H contains a non-trivial normal subgroup N with $N \in \mathfrak{A}$ and $\mathrm{Aut}_H N \in \mathfrak{B}$ (where $\mathrm{Aut}_H N$ is the group of automorphisms induced by H on N). If $\mathfrak{D}$ is the class of all groups and $\mathfrak{C}$ is an arbitrary class, let $(\mathfrak{C}, \mathfrak{D}) = \mathfrak{C}_*$ and $(\mathfrak{D}, \mathfrak{C}) = \mathfrak{C}^*$ [**141**]. Thus, provided $\mathfrak{C}$ is inherited by homomorphic images, $\mathfrak{C}_*$ is just the class of groups having an ascending normal $\mathfrak{C}$-series. These are now called hyper-$\mathfrak{C}$ groups. The class $\mathfrak{C}^*$ is often smaller than $\mathfrak{C}_*$ because, as is simple to see, if $\mathfrak{C}$ is subgroup closed and contains all abelian groups, then $\mathfrak{C}^* \leqslant \mathfrak{C}_*$. If $\mathfrak{C}$ is image closed and $G \in \mathfrak{C}^*$, then the $\mathfrak{C}$-residual of G is hypercentral. The construction $\mathfrak{C} \to \mathfrak{C}^*$ first appears in [**123**] and is studied further in [**141**].

[**140**] is devoted to an investigation of "countably recognisable" classes. The class $\mathfrak{C}$ is countably recognisable if every group in which all countable subgroups are $\mathfrak{C}$-groups is itself an $\mathfrak{C}$-group. Baer proves that if $\mathfrak{A}, \mathfrak{B}$ are subgroup closed, image closed and countably recognisable, then so is $(\mathfrak{A}, \mathfrak{B})$. In particular, all hyper-$\mathfrak{A}$ groups are countably recognisable. Among the many classes shown to be countably recognisable is the class of Baer groups.

A property that naturally fascinated Baer is that of finiteness itself: how can the borderline between the finite and infinite in group theory be recognised? [**134**] is an essay on this theme. So are his numerous papers on various types of chain conditions. [**85**] (1949) is the first systematic study in the literature of Min-n, the minimal condition on normal subgroups. Here he proves, for example, that a hyperfinite group with Min-n is a Černikov group (a finite extension of an abelian group with Min). In [**148**] (1964) he shows that a soluble group with Min-n is locally finite. This remains to this day one of the very few results known about soluble groups with Min-n. The proof is a simple consequence of a module theoretic fact: If G is a locally finite group and A is a faithful irreducible G-module, then A is an elementary abelian p-group. Baer shows further that the centralizer C of G in End A is an absolutely algebraic field of characteristic p; and that if G is also abelian-by-finite, then A is finite dimensional over C. (Cf. Robinson I, §5.2.)

Baer wrote a number of papers on generalized soluble groups that satisfy finiteness conditions on their abelian subgroups. [**179**], with H. Heineken, contains the strongest results to date on radical groups (meaning hyper-(locally nilpotent)-groups). For example, if G is a radical group whose abelian subgroups have finite p-rank for all $p \geqslant 0$, then every abelian section of G has the same property and G is hyperabelian; if G has all abelian subgroups of finite rank, then G has finite rank and is soluble modulo the torsion subgroup of its Hirsch–Plotkin radical. The paper is also a storehouse of useful results on modules. A concise, recent survey of this area of soluble group theory is Robinson (1976).

One of the finiteness conditions introduced by Baer in his 1948 paper [**83**] is FC, the property of having all conjugacy classes finite. Here and in B. H. Neumann's 1951 paper, the basic properties of FC-groups were established. During the fifties and sixties, FC groups were widely studied. Neumann (1954) proved that a group G has boundedly finite conjugacy classes if, and only if, G' is finite. Twenty years later Baer discovered a beautiful generalization of this, [**181**] (1973): If N is a normal subgroup of G, then the elements of N have boundedly finite conjugacy classes in G if, and only if, there exists a finite normal subgroup E of G with $E \leqslant N$, N/E abelian and $\mathrm{Aut}_G(N/E)$ finite. Like many results in group theory, this one depends on a theorem about modules: If A is a G-module in which the G-orbits are boundedly finite, then there exists a finite submodule E of A so that $\mathrm{Aut}_G(A/E)$ is finite.

Baer also wrote extensively on finite groups. His work here has not had the same impact as his researches on groups in general. Yet it is precisely in the area of finite groups that he had pupils who produced work of quite exceptional importance and who were surely inspired and stimulated in no small measure by Baer himself.

The first of the series of papers listed above on group-theoretical properties, [**117**] (1956), deals exclusively with finite groups. Two types of properties introduced here were to play a special part in later work. The first is that of dispersion. Let σ be a partially ordered set of primes and call a subset τ a segment of σ if $p \in \tau$ always implies that $q \in \tau$ for every q preceding p. Also, if π is an arbitrary set of primes, then a group G is π-closed if the set of all π-elements in G forms a subgroup. Baer defines a group G to be σ-dispersed if G is τ-closed for every segment τ in σ. For example, if σ is the set of all primes with the trivial partial order, then σ-dispersion is nilpotency; if σ is all primes with some complete ordering then σ-dispersion is the Sylowtower property (B. Huppert (1967), p. 695). The term "dispersed" goes back to O. Ore (1939), who used it to mean σ-dispersed where σ is all primes with the inverted natural order. Baer shows that dispersion can be considered a generalization of nilpotence which is restricted enough for many features of nilpotence to persist. For instance, if σ is the set of all primes with some partial ordering and if every proper subgroup of G is σ-dispersed, then G is soluble ([**117**], p. 172). In [**150**] (1965) he determines the minimal non σ-dispersed groups and introduces the notion of σ-radical and σ-hypercentre.

The second group-theoretic property introduced in the 1956 paper [**117**] is the following: If J is a finite simple group and $\mathfrak{C}$ is some class of finite groups, let $(J \mid \mathfrak{C})$ be all finite groups G so that every homomorphic image H of G induces an $\mathfrak{C}$-automorphism group on every minimal normal subgroup of H that is a direct product of copies of J. This construction is clearly related to that of $(\mathfrak{A}, \mathfrak{B})$ discussed above. It also contains the seeds of formation theory. But this is only hindsight. Formation theory was really conceived by W. Gaschütz (1963) and quickly developed to provide a new approach to research in finite soluble groups (cf. Huppert, chapter 6, §7). Baer contributed to this theory in [**177**] (1972) and began to study how one might go beyond soluble groups with the formation concept. He was working on a book about these matters in the last years of his life.

* * *

When one inspects the totality of Baer's work in group theory one cannot fail to be greatly impressed by the catholicity of his interests and the ceaseless experimentation with new concepts and structures.

His attitude to mathematics throughout his long and productive life is well expressed in the following words, which he wrote about his old friend Friedrich Levi [**158**], but which apply, I believe, with even greater force to Baer himself:

"Er war ein typischer Vertreter der mathematischen Zeitenwende der zwanziger Jahre, die zu einer völlig anderen Auffassung der Mathematik führte: die Ablösung der Formel durch den Begriff. Entsprechend war ihm das Verständnis eines Sachverhalts wichtiger als die Entdeckung eines neuen Sachverhalts. Dass die verständnisvolle Durchdringung dann doch zu neuen Ergebnissen führte, mag ebenso unbeabsichtigt wie zwangsläufig gewesen sein."

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December 1980.
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