ABRAM SAMOILOVITCH BESICOVITCH

S. J. TAYLOR

Many British mathematicians owe a great deal to A. S. Besicovitch. Several of these—H. Burkill, R. O. Davies, H. G. Eggleston, W. K. Hayman, J. M. Marstrand—have contributed to this notice, which also incorporates large sections from the biographical memoir **[B1]** written by J. C. Burkill.

Besicovitch was an exceptionally open-minded mathematician, and it was this readiness to consider all possible alternatives which made his contributions to mathematics characteristically surprising. When solving a problem most mathematicians need to make a commitment as to the nature of the solution long before the solution has been found, and this commitment interposes a psychological barrier to the consideration of other possibilities. Besicovitch never seems to have been troubled in this way. He could exercise the whole of his powerful mathematical intellect on the investigation of unlikely options and in the process illuminated the diversity of mathematical knowledge and obtained results which were astounding to his contemporaries and are surprising today. His contributions to several areas of analysis were so original that their implications are still being explored. We later discuss some of the areas in which he made substantial contributions, but it is inevitable, in a notice of finite length, that coverage will be incomplete.

Personal history and career

Abram Samoilovitch Besicovitch was born at Berdjansk, on the Sea of Azov, on 24 January, 1891, the fourth child of the family of four sons and two daughters of Samuel and Eva Besicovitch.

By descent the family belonged to the Karaim people, whose ancestors were the Khazars. The once powerful kingdom of the Khazars (from the 7th to the 11th centuries) stretched between the Volga and the Dnieper. The conversion of sects of the Khazars by the Karaite Jews led to their taking the name Karaims. Their language was originally Turkish of the Qipchaq group but is now mostly Russian.

Samuel Besicovitch was a jeweller by trade but, after losses by theft, he gave up his shop and took employment as a cashier. He married Eva when she was 15, and they had to live frugally to bring up their large family. All the children were talented. Moreover, they were united by strong bonds of affection. They all studied at the University of St. Petersburg, the older ones in turn earning money in their spare time by giving private lessons and helping to support the younger. All the children gained high qualifications, one brother of A.S. being the author of a number of mathematical books, and another a doctor of medicine. The two daughters both kept up independent careers after marriage. In later life the sons and daughters bore witness to the intelligence of their mother and agreed that, if she had not been cut off from higher education by marrying so young and having children, she would have shown conspicuous ability, notably in mathematics.

A.S. acknowledged that he owed to his father (twenty-five years older than his mother) a stern encouragement to persevere to the highest standards. From an early age the boy had shown extraordinary aptitude for solving mathematical problems. One day he brought a textbook to his father saying " I have been able to solve every problem except one in this book ". His father withheld praise for this achievement until A.S. had successfully solved the remaining problem.

Besicovitch graduated in 1912 at the University of St. Petersburg, where one of his teachers was A. A. Markov, and his first paper was on probability theory.

The last university to be opened before the revolution in Russia, in 1916, was that at Perm (later called Molotov) as a branch of the University of St. Petersburg. In 1917 it became autonomous and Besicovitch became a Professor in the School of Mathematics. After Soviet power was established in Perm the University developed rapidly. The Faculty of Physics and Mathematics had good buildings including a mechanics laboratory. At the beginning of 1918, I. M. Vinogradov, R. O. Kuz'min, A. A. Fridman and A. F. Gavrilov joined the staff. At the end of the summer of 1918 the Perm Physics and Mathematics Society was founded and started to publish a journal.

During the civil war in 1919 the University was destroyed and in part transferred to Tomsk. Besicovitch locked books away in cellars and succeeded in preserving much of the property of the Faculty. After the liberation of Perm, the University, thanks largely to the efforts of Besicovitch and Fridman, was re-established in three faculties, physics-mathematics, medicine and agriculture. In 1920 further faculties were added.

In 1920 Besicovitch returned to Leningrad as Professor in the Pedagogical Institute and Lecturer in the University from 1920 to 1924.

He had married in 1916 a mathematician, Valentina Vietalievna, rather older than himself. Because a man of the Karaims could not marry a woman of the Orthodox religion, Besicovitch had been received into the Russian Orthodox Church. During his years at Perm he became friendly with Maria Ivanovna Denisova, the widow of a mathematician, and her two young daughters (aged five and four in 1917). They relate how he kept them alive during the years of famine 1918–1919; he managed to buy half a horse which he buried in a secluded place; once a week he dug it up and cut off chunks of meat for them.

The duties of a University teacher under the régime of the early 1920s were subject to political requirements. He had to teach classes of workers who lacked the educational background to understand the lessons; this and other duties he could not refuse. Besicovitch was offered a Rockefeller Fellowship to work abroad and applied for permission to accept this offer. Repeated efforts to obtain permission were refused and finally he made plans in 1924 to leave. In company with another mathematician, J. D. Tamarkin, he crossed the border under cover of darkness and made his way to Copenhagen, where the Rockefeller Fellowship enabled him to work for a year with Harald Bohr, who was developing the theory of almost periodic functions. Besicovitch then visited Oxford, staying for several months in New College with G. H. Hardy, who was quick to recognize his great analytical power, and secured for him a lectureship at the University of Liverpool for 1926–1927. He moved to Cambridge in 1927 as a University Lecturer becoming also a College Lecturer and in 1930 a Fellow of Trinity. This fellowship he retained until the end of his life.

Besicovitch's wife had remained in Russia when he left. There were no children and the marriage was eventually dissolved in 1926. He brought Maria Denisova and her children to England and in 1928 married the elder daughter, Valentina Alexandrovna, then aged 16. They had no family, and it was the children of his friends who, throughout his life, enjoyed his affection. Speaking Russian at home, Besicovitch's command of English remained stationary from his early days in Cambridge. For him the definite article was superfluous. A story is told that during one of his lectures an undergraduate tittered at some distortion of English idiom. "Gentlemen", said Besicovitch, "there are fifty million Englishmen speak English you speak; there are five hundred million Russians speak English I speak." There was no further tittering.

From 1927 to 1950 Besicovitch took his share of the University lecturing for the Mathematical Tripos. In most years he gave during two terms standard courses on Cours d'Analyse matter, and in another term an advanced course for graduates or for undergraduates with a flair for analysis. His advanced courses reflected his own interests in topics such as Almost Periodic Functions, Geometry of Plane Sets, Hausdorff Measure. Over many years he also maintained, for the pleasure and benefit of undergraduates, a weekly feature of "contest problems", at the level of the Advanced Problems in the *American Mathematical Monthly*. The solutions submitted were carefully read and annotated by Besicovitch and the announcement "Perfect solutions of Problem 12 were sent in by M and N" spurred several young mathematicians on to develop their analytical powers.

Besicovitch guided the early research of a number of analysts including H. D. Ursell, G. Walker, J. Gillis, D. R. Dickinson, I. J. Good, H. G. Eggleston, P. A. P. Moran, H. Mossaheb, E. R. Reifenberg, J. R. Ravetz, R. O. Davies, J. M. Marstrand and myself. In dealing with his pupils, he did not restrict his conversation to mathematics; he loved to discuss the problems of international politics as well as the ills of society and he was genuinely concerned about oppression and suffering wherever it was found. His intellectual gifts were matched by his generous sympathy which endeared him to all who were privileged to know him. A characteristic instance of his thoughtfulness was to make bequests to all the bedmakers who had looked after him during his forty years in Trinity.

In 1950, on his fifty-ninth birthday, Besicovitch was elected to the Rouse Ball

Chair of Mathematics, succeeding the first holder J. E. Littlewood. Twenty-three years earlier, on his thirty-sixth birthday, thinking that the years of greatest intensity of life were passing, he had said " I have had four-fifths of my life ". When J. C. Burkill reminded him of this in 1950, he received a postcard which read " Numerator was correct ".

After his retirement from the Rouse Ball Chair in 1958, Besicovitch remained active in teaching and research and spent eight successive years as visiting professor in various universities in the U.S.A. He then returned to live in Trinity. Towards his eightieth year his health began to fail and he died on 2 November, 1970.

He had been elected F.R.S. in 1934 and was awarded the Society's Sylvester Medal in 1952. He received in 1930 the Adams prize of the University of Cambridge for his work on almost periodic functions, and in 1950 the de Morgan medal of the London Mathematical Society.

Much of the historical material has been provided by Besicovitch's wife, and his younger sister, Mrs. Aredova, both of whom continue to live in Cambridge. Numbers refer to the bibliography at the end of this notice which was prepared by R. O. Davies using an earlier version due to Dr. Helen Alderson. References to other writers are given in the form Littlewood [**L3**].

The Kakeya problem

When Besicovitch moved to the new University of Perm in 1917, he intended to enter the field of mathematical logic, but found the library quite inadequate. He therefore continued to work on problems of analysis, including this one: if $f\colon R^2 \to R$ is Riemann integrable, must there be some choice of orthogonal axes $0xy$ for which the repeated integral $\int\left(\int f(x, y)\, dx\right)dy$ exists, in the sense that $\int f(x, y)dx$ exists for each y as a Riemann integral and its value is a Riemann integrable function of y ?

He saw that if he could construct a compact plane set F containing a unit segment in every direction, but of Lebesgue plane measure zero, then a negative solution would follow. Namely, we may suppose that F contains no vertical or horizontal segment with rational coordinate; let f be the characteristic function of the set of points of F with a rational coordinate. The discontinuity points of f are in F, so f is Riemann integrable over R^2, but in every direction there is a segment (one contained in F) along which f is not Riemann integrable.

Besicovitch constructed F in [**5**]. The key point is that one can dissect any given triangle into a large number of narrow triangles, with the original vertex and base-line, and slide them parallel to the base so that the union of the displaced triangles has arbitrarily small area; then a nesting process leads to F. Copies of the journal unfortunately did not reach the outside world, because of the disturbed conditions in Russia.

Now in a paper published in 1917 ([**F1**], also [**K1**]) the following problem was mentioned: what is the least area of a convex set in the plane within which a unit segment can be turned through two right angles, returning to its original position with the ends reversed ? The equilateral triangle of unit altitude was conjectured to be the minimal figure, and Kubota was quoted as observing that if the convexity requirement is dropped then a figure of smaller area is possible (three-cusped hypocycloid). The conjecture was proved in 1921 by Pál **[P1]**, who also emphasized the interest and difficulty of the problem without convexity (he noted that one cannot now determine *a priori* whether the lower bound of areas is attained). It was this that became known as the " Kakeya problem ".

Soon after Besicivitch left Russia, it was realized (it is not clear by whom) that his construction referred to above almost immediately yields zero as the required lower bound; this he showed in 1928 (**[28]**; see also **[P2]**). We at once get a set of triangles, with union of arbitrarily small area, together containing unit segments in all directions, and a figure in which the desired turning can take place is obtained by simple joins, also of arbitrarily small area. These were suggested by Pál, as Besicovitch acknowledged in **[112]**.

Numerous subsequent papers have been devoted to the problem and its variants. Definitive in one direction is Cunningham's result that simply connected " Kakeya sets " of arbitrarily small area can be constructed in a unit circle (**[C1]**, which also contains references to earlier work by Alphén, Schoenberg, and others). A finite number of segments can be moved " in formation " to any given new position in the plane, covering arbitrarily small area *en route*, if and only if they are parallel **[D1]**, but we do not know whether an arc of circle can be moved from one position to another (not on the same circle) in an arbitrarily small area. For movement of arcs on a sphere see **[C2]**.

Returning to the original " Besicovitch set " F of measure zero, containing unit segments in all directions, we should mention Kahane's simple construction of such a set **[K2]**, based on joining points of Cantor's set to points of a copy of it on the line $y = 1$. However, it was Besicovitch himself who in 1964 transformed the subject by connecting it with geometric measure theory. He showed **[118]** that if E is a linearly measurable plane set of finite positive linear measure, and E^* denotes the union of the polar lines of the points of E, with respect to a fixed circle, then (a) if E is regular, then E^* has infinite Lebesgue plane measure, provided multiplicity of covering is counted (not necessarily otherwise **[D2]**), and (b) if E is irregular, then E^* has zero plane measure: this is essentially the dual of his result that E projects into a set of linear measure zero in almost all directions. If we take for E an irregular set having a point on each line through the centre of the fixed circle, then E^* will serve as F. Thus a rather subtle *ad hoc* construction is replaced by a trivial one in combination with an appeal to important general theory.

It is surprising that Besicovitch did not notice that the same approach gives an easy solution **[D3]** to the problem of constructing a plane set of measure zero containing circumferences of all radii, which was solved by an *ad hoc* construction in his last paper **[127]**, with Rado (see also **[K3]**). We do not know whether there exists a null set containing a translate, or a congruent copy, of every ellipse—or of every rectifiable arc! There is a null set containing a translate of every polygon, but this, like every Besicovitch set, must have Hausdorff dimension two; however, there is a set of dimension unity containing a congruent copy of every polygon (see **[W1, 2]**, **[D1]**, **[M1]**). In three-space we do not know whether there is a null set containing a translate of every plane.

In 1927, Nikodým **[N1]** solved a problem of Banach by constructing an F_σ set of full measure in the unit square, each point of which is accessible, i.e. on a straight line not meeting the set again; one consequence is the impossibility of extending Saks' strong density theorem in a certain direction. This was completely independent of Besicovitch's work, but in fact Nikodým's construction can profitably be understood and exploited when regarded as a kind of dual of Besicovitch's. This was the basis for **[D4]** (written under the supervision of Besicovitch), where a set of full measure in the whole plane was constructed, with continuum-many lines of accessibility through each point. It was also proved that any plane set can be extended to a union of lines without increasing its measure, and a variant of this result has just been used by Putnam **[P3]** to establish a characterization of the spectra of hyponormal operators—thus showing that Besicovitch's construction can yield positive results as well as counter-examples. Larman **[L1]** used **[D4]** in constructing a compact set of positive measure in three-space, whose points are accessible by continuously-varying disjoint unit segments with union of measure zero. Finally, the most well known and important application of Besicovitch's construction has been by Fefferman **[F2]** in his negative solution of the multiplier problem for the ball; and there has also been a recent related application by Mityagin and Nikishin **[M2]**.

In 1958 the Mathematical Association of America made the experiment of constructing motion films illustrating mathematical processes which lent themselves to this form of illustration. The Kakeya construction was eminently suitable, accompanied by the exposition of Besicovitch in his inimitable style. The Kakeya film has been widely seen, and any reader of this notice who ever has the chance to see it should not let it slip.

Almost periodic functions

In 1924 Bohr published his first paper on almost periodic functions. He defined a continuous complex-valued function f on the real line to be almost periodic if, roughly speaking, it repeated its values at reasonably regular intervals. More precisely, Bohr called a subset E of the real line *relatively dense* if every sufficiently long interval contains a member of E; and he called a continuous function f *almost periodic* (a.p.) if, to every $\varepsilon > 0$, there corresponds a relatively dense set E_ε such that

$$\sup_{x \in R^1} |f(x+\tau)-f(x)| < \varepsilon \tag{1}$$

for all $\tau \in E_\varepsilon$. Bohr proved the remarkable result that this structural property is equivalent to f being uniformly approximable by trigonometric polynomials of the form

$$\sum_{k=1}^{m} a_k e^{i\lambda_k x} \quad (\lambda_k \text{ real});$$

in other words, f is almost periodic if and only if there exists a sequence (P_n) of trigonometric polynomials such that

$$\sup_{x \in R^1} |f(x) - P_n(x)| \to 0 \quad \text{as} \quad n \to \infty. \tag{2}$$

Another striking feature of a.p. functions is that they possess generalized Fourier series. For

$$a_\lambda = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(x) e^{-i\lambda x}\, dx$$

exists for all real λ and differs from 0 only when λ belongs to a countable set $\{\lambda_1, \lambda_2, \ldots\}$; and the series

$$\sum_n a_{\lambda_n} e^{i\lambda_n x}$$

behaves very much like the Fourier series of a purely periodic function.

Subsequently Bohr defined the property of almost periodicity for complex functions analytic in a vertical strip; and Stepanoff obtained two generalizations of Bohr's original concept by replacing the left side of (1) with

$$\sup_{x \in R^1} \int_x^{x+1} |f(x+\tau) - f(x)|\, dx \quad \text{or} \quad \sup_{x \in R^1} \int_x^{x+1} |f(x+\tau) - f(x)|^2\, dx.$$

Besicovitch's interest in almost periodic functions was aroused when he visited Bohr during 1925–26. His first results were announced in **[17]** and detailed proofs appeared in **[21]**. He began by showing that the analogue of the Riesz–Fischer theorem does not hold for Bohr's or even Stepanoff's a.p. functions; and then proceeded to define a type of almost periodicity (later designated B^2 almost periodicity) for which this analogue does hold. Thus, given complex numbers $a_1, a_2, \ldots$ such that $\sum |a_n|^2 < \infty$, and arbitrary real numbers $\lambda_1, \lambda_2, \ldots$, there exists a B^2 a.p. function with Fourier series $\sum a_n e^{i\lambda_n x}$. The definition of Besicovitch's almost periodicity is far less straightforward than Bohr's, and the argument used to prove the Riesz–Fischer theorem demands the kind of ingenuity which is the hallmark of Besicovitch's work. The second problem tackled in this paper concerns sums of everywhere convergent trigonometric series: is such a sum almost periodic in some sense and is the original series its Fourier series? A typically Besicovitch type of construction showed that functions satisfying quite mild conditions and with no conceivable almost periodic character have infinitely many trigonometric series converging to them. So the Riemann theory of periodic trigonometric series has no immediate analogue in the more general situation considered by Besicovitch.

The paper [19] is partly concerned with complex functions which are B^2 a.p. on vertical lines. It is curious that here Besicovitch came very near to defining analytic B^2 a.p. functions, but never actually did so. In fact, the concept has only quite recently been introduced by Bauermeister [B2].

By the late 1920's, several isolated generalizations of Bohr almost periodicity had been made. Stepanoff's and Besicovitch's have already been mentioned; another was due to H. Weyl. Besicovitch and Bohr now set about unifying and systematizing the theory of almost periodicity. Their principal idea in [34] was to make each type of almost periodicity correspond to a suitable distance D in a functional space. For instance, they put

$$D_U(f, g) = \sup_{x \in R^1} |f(x) - g(x)|$$

and, for $p \geqslant 1$,

$$D_{S^p}(f, g) = \sup_{x \in R^1} \left(\int_x^{x+1} |f-g|^p \right)^{1/p}, \quad D_{B^p}(f, g) = \limsup_{T \to \infty} \left(\frac{1}{2T} \int_{-T}^{T} |f-g|^p \right)^{1/p}.$$

They defined almost periodic functions, other than Besicovitch's, simply by substituting the inequality $D(f(x+\tau), f(x)) < \varepsilon$ for (1); and they proved that in all cases the mode of approximation by trigonometric polynomials was obtained by using the expression $D(f, P_n)$ in (2). This paper became the middle portion of Besicovitch's book [35] on a.p. functions. The first part is a very clear exposition of Bohr's a.p. functions, with many of Bohr's original and very complicated proofs replaced by later simplifications; and the last part gives a concise but eminently readable account of analytic a.p. functions. The book has remained the standard work on the classical theory of a.p. functions.

The relative complexity of the structural definition of B^p almost periodicity appears to have worried Besicovitch. Both [34] and [35] included an appendix which showed that a seemingly more natural definition would not do. But Besicovitch returned to the attack and in [36] he obtained a much simpler characterization in the case $p = 1$. Although he retained his interest in almost periodic functions, this paper, dating from 1932, was his last published work on the subject.

Translation numbers and almost periodicity can be defined for a function $f(x+iy)$ analytic in a strip $a < x < b$. There is then a Dirichlet series. The relations between a function and its Dirichlet series are worked out in [35].

Hausdorff measure

Measure of fractional order in the sense of Hausdorff was a topic to which Besicovitch returned repeatedly. For any subset E of a metric space, and positive numbers δ, s, $\bigwedge^s{}_\delta E$ denotes the infimum of

$$\sum_{j=1}^{\infty} (dA_j)^s$$

where dA denotes the diameter of A and the infimum is taken over all possible sequences $\{A_j\}$ of sets of diameter less than δ whose union covers E. It is easy to show that $\Lambda^s{}_\delta$ is an outer measure. Most of the theory involves the more useful outer measure Λ^s formed by taking the limit, as $\delta \to 0$ of $\Lambda^s{}_\delta$. The Hausdorff dimension of a set E is the infimum of the set of s for which $\Lambda^s E = 0$. For example, Cantor's ternary set C on the line has dimension $\alpha = \log 2/\log 3$ and in fact $\Lambda^\alpha C = 1$. For a recent account of Hausdorff measures, see Rogers [**R1**]. Besicovitch's contribution to the subject is vast, and we shall describe part of it in this section and the more geometric results in the following section.

In [**85**] Besicovitch gives an example of a linear Borel set which differs from any F_σ subset by a set of dimension 1. This example illustrates the fundamental difference between Lebesgue measure on the one hand and Hausdorff measure on the other—Hausdorff measure is not merely a refinement of Lebesgue measure but it enables one to construct examples " in the opposite direction ".

There is a series of papers [**39, 41, 42, 43, 49**] which investigate the dimension of certain exceptional sets which arise naturally in simple problems of analysis. For example, in (41) he considers the set E of real x for which infinitely many rationals m/n exist which differ from x by less than $n^{-q}(q > 2)$ and proves that dim $E = 2/q$.

The papers [**39**], [**43**] are concerned with the distribution of digits in the decimal expansion of a real number: results are obtained about the size of the exceptional sets in Borel's normal number theorem. In [**49**] he considers plane curves $(x, f(x))$ where f satisfies a Lipschitz condition of order δ

$$|f(x+h)-f(x)| < A|h|^\delta,$$

and shows that the dimension d of such a graph satisfies $1 \leqslant d \leqslant 2-\delta$. The inequalities are easily proved, but the construction of examples to show that they are best possible is more difficult.

Bescicovitch applied properties of Hausdorff measure to define the s-derivatives of a real function (where $s < 1$) as the upper and lower limits of $\Delta f/h^s$ where $\Delta f = f(x+h)-f(x)$. He showed in (**31**) that a function can have a non-zero s-derivative only on a set of zero Λ^s-measure. A set function version of this analysis was later explored by Rogers and Taylor (**RT1**) to give results which have proved useful in analysing the structure of Borel measures in Euclidean space. In [**63**], written jointly with Moran, it was proved that if A is a linear s-set and B a linear t-set, then

$$\Lambda^{s+t}(A \times B) \geqslant k \Lambda^s A . \Lambda^s B$$

for an absolute constant k. As with much of Besicovitch's work, a thorough investigation of the product sets problem is carried out, inequalities in the apposite direction are obtained under suitable conditions, and specific examples illustrate the extreme cases.

In [**81**], Besicovitch shows that every closed set of infinite Λ^s-measure in Euclidean space has a subset of positive finite Λ^s-measure. This result is trivial when s is the dimension of the whole space (for then Λ^s is a constant times Lebesgue measure); and the difficulty for general s results from the fact that a set of infinite Λ^s-measure cannot be expressed as the union of countably many sets of finite Λ^s-measure. In the proof, Besicovitch introduced a special "net-measure" in which the class of covering sets is severely restricted: this technique has since been exploited by many writers including Larman **[L2]** who introduced a new theory of dimension. Another interesting sequel to **[81]** is the example, due to Davies and Rogers **[DR1]** which shows that the result is false for subsets of a "sufficiently large" metric space.

Geometric measure theory

The case $s = 1$ of Hausdorff measure is called linear measure: it was introduced by Carathéodory in 1914 for subsets of the plane. The linear measure of a rectifiable curve is its length, but the class of linearly measurable sets is much wider than the class of sets obtained as subsets of rectifiable curves. Federer in his comprehensive book **[F3]** remarks (page 2) that "Much of geometric measure theory during the first half of this century consisted of detailed studies of certain peculiar sets...From this analysis of pathology there gradually evolved, thanks largely to the pioneering genius of A. S. Besicovitch, a pattern of structure." The general structure theorems, obtained in general form by Federer, were first systematically obtained by Besicovitch for linear measurable subsets of the plane in a series of papers, though some of the results had been previously obtained independently by Wazewski **[W3]**. The results about the structure of linearly measurable subsets of the plane are perhaps the most beautiful of Besicovitch's work. The apparent simplicity of this case is deceptive —and there are still today fundamental and difficult unsolved problems.

Let a be any point of the plane, $c(a, r)$ the circle with centre a and radius r. Then $D^*(a, E)$ and $D_*(a, E)$, the upper and lower circular densities of E at a, are defined as the upper and lower limits, as $r \to 0$ of $\Lambda\{E \cap c(a, r)\}/2r$. If $D^*(a, E) = D_*(a, E)$, the common value is called the density $D(a, E)$.

If $D(a, E) = 1$, then a is called a *regular* point of E. Any other point of E is called *irregular*. A set E for which almost all its points are regular (irregular) is called regular (irregular). The fundamental structure theorems show that regular sets have tangential properties similar to those of rectifiable curves so that they have a tangent almost everywhere. On the other hand, at almost all irregular points of E, neighbouring points of E are dense in every sector however small. Any linearly measurable set decomposes into the union of two sets, one regular and the other irregular. Besicovitch gave two proofs **[26, 50]** of the fact that a regular set is a subset of the union of a countable collection of rectifiable curves. Each of these proofs is intricate and interesting, but the second is particularly elegant and makes use of plane continua of finite linear measure. In **[50]** he only needs the weaker hypothesis that $D_*(a, E) > \frac{3}{4}$ at almost all points, so it emerges as a corollary that $D_*(a, E) \leqslant \frac{3}{4}$ at almost all points of an irregular set. Besicovitch conjectured that $\frac{3}{4}$ can be improved to $\frac{1}{2}$ but this problem is still unsolved. The other density bounds

$$\tfrac{1}{2} \leqslant D^*(a, E) \leqslant 1 \quad \text{and} \quad 0 \leqslant D_*(a, E) \leqslant 1$$

at almost all points of any linearly measurable E are known to be best possible.

In **[26]** he gave an example of an irregular linearly measurable set whose projection in all directions has zero Lebesgue measure, exhibiting another difference in behaviour to that of regular sets. This example has features in common with his construction which solved the Kakeya problem. He later proved, in **[52]**, that every irregular linearly measurable set projects onto a set of zero linear measure in almost all directions. A surprising consequence of the techniques used in the proof is that every linearly measurable set has this projection property or else it projects onto a set of measure zero in at most one direction. Both the proofs and the examples constructed are difficult and ingenious. The other references in the bibliography are **[18, 25, 33, 37, 38, 47, 118]**.

The geometrical properties of sets of non-integer dimension are much less satisfactory. Besicovitch considered these for $0 < s < 1$ and subsets of the line, and he proved in **[30, 31]** that at almost all points of a fractional dimensional set the one-sided upper densities are unity and lower densities zero. A corresponding more detailed investigation of s-sets in higher dimension was carried out by Marstrand **[M3]** and again, most of the results about structure are negative. The extension of Besicovitch's work to k-dimensional subsets of Euclidean n-space was successfully carried by Federer **[F4]** for all integers k, n.

Surface Area

Charles Burkill recalls that in 1942 Besicovitch asked him "Will you lend me book of Saks for some time?". He was embarking on a study of the area of parametric surfaces

$$\text{(A)} \qquad x = x(u, v), \quad y = y(n, v), \quad z = z(v, v)$$

Saks **[S1]** had given an account of the area of a surface $z = f(x, y)$, according to Lebesgue's concept of area as the lower limit of approximating polyhedra. This account, making use of Tonelli's definitions of bounded variation and absolute continuity, satisfied all requirements, such as semi-continuity.

Besicovitch set himself the natural problem of proving the equality of the Lebesgue–Fréchet area of a surface (A) and its Carathéodory–Hausdorff two-dimensional measure. As he was to write in **[75]**, "I came to results very different from ones I was hoping for".

He constructed **[62]** an example of a topological disk (or a topological sphere) having arbitrarily small Lebesgue–Fréchet area but, nevertheless, the Lebesgue three-dimensional measure could be an assigned positive number. He summed up the implications of this discovery by saying that the Lebesgue–Fréchet area, though it had held the field for thirty years, has these defects

(a) it cannot be fitted into any scale of measures,

(b) it has no property of additivity—a similar construction would yield a surface of arbitrarily small area including all points of a unit square,

(c) it cannot be extended to surfaces defined as point-sets.

He concluded that the only satisfactory concept of area is two-dimensional Hausdorff measure, and undertook a programme of solving anew problems such as the expression of area as an integral, Plateau's problem and large parts of the calculus of variations. These problems are treated in the papers [**68**—which contains an error corrected in **86**—, **71**, **72**, **73**]. Besicovitch's pupil, Reifenberg, made notable advances in the problems arising from parametric surfaces, in particular the Plateau problem. He had, just before his death at the age of 35 in a mountaineering accident, written penetrating papers on minimal surfaces. (See the list of references in [**R2**].)

General Real Analysis

Throughout his mathematical life, Besicovitch kept up an interest in problems which arose from properties of real functions. His most interesting contributions arose from the construction of sets or functions designed to disprove plausible but false conjectures. For example, in [**55**] he constructs a linear set X that has property C (that is, it can be covered by a sequence of intervals with any prescribed lengths) but is such that there is no enumerable set Y such that any G_δ-set containing Y contains all of X except for a countable subset. In this paper he introduces the idea (still largely unexploited) of a fundamental sequence of functions which he used later in [**84**] to discuss the behaviour of series $\sum u_n(x)$ which are either conditionally convergent for x in [0, 1] or can be rearranged to be conditionally convergent. He showed, for example, that by choosing $u_n(x)$ appropriately one can ensure, with at most enumerably many exceptions that the series converges to 0.

In some of his early papers [**11**, **12**, **15**, **16**] he obtained the Denjoy properties of derived numbers by methods which are simpler than those of Denjoy or W. H. and G. C. Young. [**16**] contains the first example of a continuous function having at no point a one-sided derivative—see Pepper [**P4**]. In [**83**] Besicovitch produced an example of a function $\phi(x, y)$, jointly measurable in x and y such that no absolutely continuous $y(x)$ exists satisfying $dy/dx = \phi(x, y)$; in [**102**] he produces a function $f(x)$ continuous on [0, 1] with $f(x)+f(1-x) \neq 0$ such that, for every integer n,

$$\sum_{m=0}^{n} f\left(\frac{m}{n}\right) = 0;$$

and in [**113**], [**117**] he obtains further relations between continuous functions and power series with non-negative coefficients.

In [**7**, **20**] Besicovitch established the existence for almost all values of t of the integral

$$\int_0^1 \frac{f(x)dx}{x-t}.$$

This important integral, which first attracted the attention of Kellogg and of Hilbert in integral equations, had been discussed by a variety of methods including Fourier series and transforms (Young, Plessner, Titchmarsh) and complex variables (M. Riesz). We find in [7] an elementary proof of the existence p.p. of the principal-value integral if f is L^2-integrable. Titchmarsh extended this proof to L^p-integrable f, for any p greater than 1, but no argument for $p = 1$ other than M. Riesz's complex-variable was devised until Besicovitch's proof in [**20**] which depended only on sets of points. The beautiful proofs in [7, **20**] which Besicovitch, staying in Oxford, showed to Hardy made an impression on him which he often recalled in later years.

Vitali stated his well-known covering principle as a property of sets in R_n having zero n-dimensional Lebesgue measure. In [**50**] Besicovitch made the extension which he needed to $\bigwedge^m$-measure for integers $m < n$. Later, in [**61, 64**] he made the complete extension to an arbitrary measure. If $F(X)$ is a non-additive function of sets X of an additive class, and $F(X) = 0$ for any X outside a set G, then a set Γ of circles covering G in the Vitali sense contains a subset Γ_1, of non-overlapping circles for which $F(\Gamma_1) = F(G)$. This result enabled him to prove final results about differentiating one additive function with respect to another, and the differentiation of indefinite integrals. Other papers on general measure theory are [**37, 47, 55, 81, 85**].

Complex Variable

The papers on complex function theory include [**9, 13, 23, 27, 32, 59, 120**]. We give two examples.

In [**32**] the author proves that if f is regular in the complement of a closed set E then f can be analytically continued onto E provided that either

(i) E has linear measure zero and f is bounded outside E *or*

(ii) E has at most countably infinite linear measure and f is continuous in the plane.

The first result had been discovered about 30 years earlier by Painlevé (see Collingwood and Lohwater [**CL1**], page 5, for a discussion of the history of this result).

Papers [**9**] and [**23**] are concerned with the so-called cos $\pi\rho$-Theorem. If $f(z)$ is an entire function of order less than one and $0 < \rho < \rho' < 1$, then this theorem, first proved by Wiman and Valiron in 1914, asserts that

$$m(r) > M(r)^{\cos \pi\rho'} \tag{1}$$

on an unbounded set E, where $m(r)$, $M(r)$ are the minimum and maximum modulus of $|f(z)|$ on $|z| = r$. In [**9**] Besicovitch gave a new proof of (1); in [**23**] he showed that (1) holds on a set E of upper linear density at least $1-\rho/\rho'$.

Twenty-five years later it was shown by Barry [**B3**] that upper density can here be replaced by lower logarithmic density. Kjellberg [**K4**] and Hayman [**A1**] showed that the corresponding result is then sharp. It is still not known whether Besicovitch's original theorem is sharp although several mathematicians have recently worked on this.

Geometrical problems

One of Besicovitch's interests in later life was in convexity and if he did not achieve such brilliant discoveries in this as in other fields, it must be remembered that he was already over fifty years old when his first paper **[65]** on convexity was published. He constructed an infinite sequence of disjoint open convex sets each two of which had frontiers that met in a set of positive area. In this he had been anticipated by H. Tietze. His second paper [**67**] and subsequent paper [**77**] introduced the idea of a measure of asymmetry of a convex set. He showed in [**67**] that a triangle is the most asymmetrical and in [**77**] that amongst sets of constant width a Reuleaux triangle is the most asymmetrical. The first of them is easy to prove, but Besicovitch's proof depended on the interesting observation that it is possible to inscribe an affine transform of a regular hexagon in any closed bounded convex curve. The second paper is difficult and in it Besicovitch uses a lemma concerning the lengths of intercepts on four concurrent circles of equal radius which may well have other applications in the theory of sets of constant width. In papers [**74**] and [**79**] Besicovitch considered extensions of the plane isoperimetric problem; his most significant result being to describe those plane figures which are of given length of perimeter, enclose the largest area subject to the condition of not enclosing a circle of radius larger than 1. Surprisingly enough some of these figures are not convex.

The papers [**92**] and [**95**] deal with the problem of sets which contain a given sphere and one wants to minimize some function of the set. Paper [**101**] proves (by an unnecessarily complicated argument] that a convex curve with the property that if three vertices of a square lie on the curve, then so does the fourth, is necessarily a circle.

In [**60**], Besicovitch shows that the answer to Zarankiewicz's problem " Does a tree always contain a homeomorphic proper subtree? " is negative by constructing a continuum of such peculiar construction that no neighbourhood of any arc point of it is homeomorphic to any neighbourhood of any other point of it. In one of his last papers [**123**] he constructs a plane arc of length less than 1 which cannot be covered by an equilateral triangle of side unity. Here the construction is simple and the difficulties lie largely in believing that such an arc exists.

A final amusing example of Besicovitch's power is his solution of the " Lion and man " problem proposed by R. Rado about 1930, and solved by Besicovitch in 1950. " A lion and a man in a closed circular arena have equal maximum speeds. What tactics should the lion employ to be sure of his meal? " (see Littlewood [**L3**] and Croft [**C3**]). No one seems to have doubted that if L gets on to the radius joining O, the centre of the arena, to M and keeps on OM however M moves, then L would catch M. Besicovitch constructed a path for M such that L never catches him (though he gets arbitrarily close). Let M start at M_0 and follow at full speed the polygon $M_0 M_1 M_2 \ldots$ where (i) $M_n M_{n+1}$ is perpendicular to $L_n M_n$, (ii) $\sum M_n M_{n+1}$ is infinite, (iii) the path stays in the arena. We have only to take $M_n M_{n+1} = cn^{-\frac{3}{4}}$ for a suitable constant c. Since $M_0 M_1$ is perpendicular to $L_0 M_0$, L cannot catch M

while he is on $M_0 M_1$. By induction, he never catches him as it takes infinite time to cover the polygon.

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