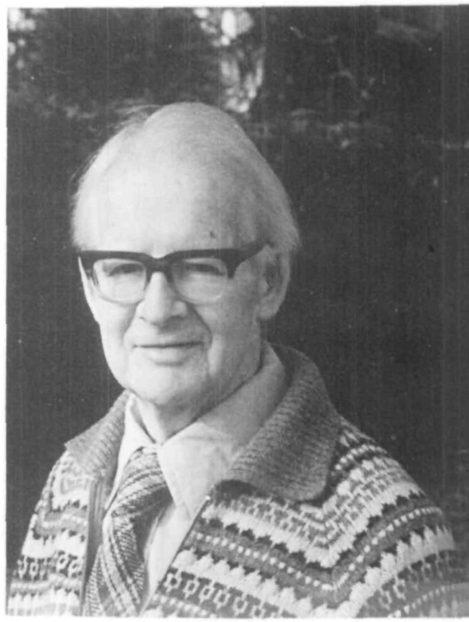
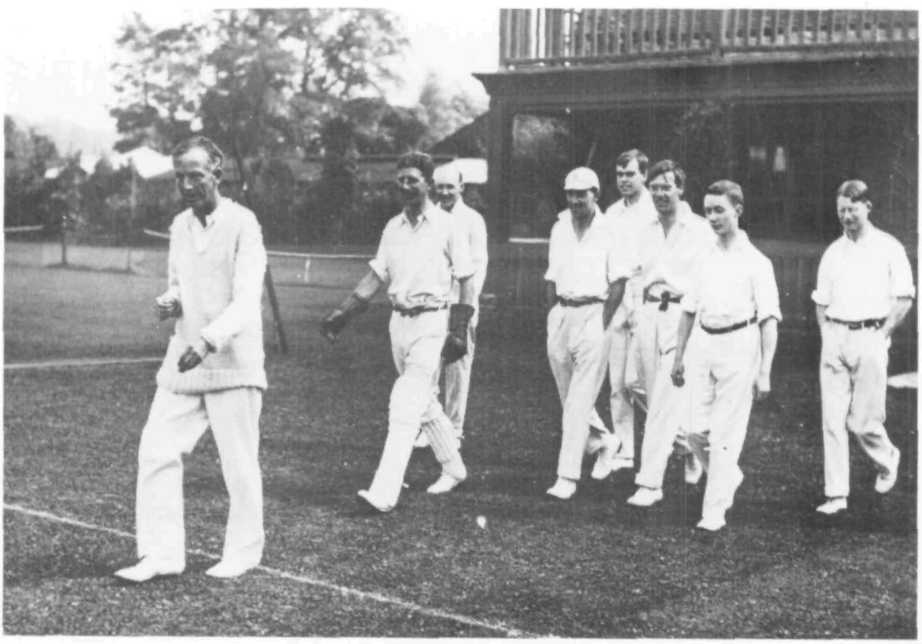
Portrait of L. S. Bosanquet,
taken about 1923.



Photograph of L. S. Bosanquet,
taken in 1980.



G. H. Hardy leading his cricket team at Oxford in August 1926. Shown are: G. H. Hardy, H. K. Salvesen, W. L. Ferrar, W. J. Langford, E. C. Titchmarsh, E. H. Neville, E. H. Linfoot, L. S. Bosanquet.

L. S. BOSANQUET 1903–1984

OBITUARY

LANCELOT STEPHEN BOSANQUET

1. *Ancestry*

Lancelot Stephen Bosanquet died at Cambridge on 10 January 1984, having just passed his 80th birthday a fortnight before. He was born on 26 December 1903 at St. Stephen's-by-Saltash in Cornwall, where his father, the Rev. Claude C. C. Bosanquet, was Vicar of the Parish. His mother Millicent was the daughter of General Percy Smith, Royal Engineers, who in the 1870s wrote five volumes on building materials and construction, and applied his expertise to good effect building bridges in India (see **[8]** for evidence that this was a non-trivial achievement).

The Rev. Claude Bosanquet came from a distinguished Huguenot family for which detailed records exist certainly back to 1500 (see **[9]**): Pierre Bosanquet of Colognac, in the Languedoc province of Southern France (he died in 1554) was Stephen's great^{10}-grandfather.

The Huguenots were Calvinists who, in the 'wars of religion' in France in the 16th century, suffered strongly for their beliefs (an infamous example being the massacre of Huguenots on St. Bartholomew's Eve 1572 by the forces of Catherine de Medici and the Duc de Guise). Finally, the victory of the Protestant Prince Henri de Navarre over his Catholic opponents, and his establishment on the French throne as Henri IV, resulted in the Edict of Nantes of 1598, guaranteeing the Huguenots their own churches, schools, legal status, and access to public office. Unfortunately this protection was gradually eroded over the next century until the Edict of Nantes was finally revoked in 1685, unleashing a wave of persecution which caused wholesale flight to Switzerland, Holland, Prussia, Scandinavia, England—or anywhere where it was safe to be Protestant and non-conformist; some of the more adventurous even made their way to North America. Those who stayed behind were imprisoned, massacred, or else converted to Catholicism.

The founder of the English branch of the family was David Bosanquet, born 31 October 1661 at Lunel, Languedoc. David fled to England on the revocation of the Edict of Nantes in 1685 and set up as a Turkey Merchant, which meant that he was a member of the Levant Company trading in the riches of the East. His arms and pedigree were recorded by the College of Heralds in 1687, and in 1695 he was joined by his brother Jean, who had been imprisoned at Dunkerque. Though Jean remained unmarried, David in 1698 married Elizabeth Hays (whose descent from King Edward III is documented) and had 14 children, of whom Stephen is descended from the second son Samuel. Elsewhere the English family tree is liberally sprinkled with knights, generals and admirals, judges and barristers, rectors and vicars, and scholars such as the philosopher Bernard Bosanquet (1848–1923) and Stephen Bosanquet himself. One member of the family in Victorian times was a knight and Privy Councillor, and another was Master of the Mint in India. Another was B. J. T. Bosanquet (1877–1936) who played cricket for England and who invented the 'googly', a style of bowling designed to be unpredictable to the batsman; the Australians at the time used to call it the 'Bosey'.

When Stephen was born in 1903, he had two older brothers—Claude, 8 years his senior, later preceded him to Balliol College, Oxford; his other brother Armytage was killed in action in the First World War.

2. *Education*

Stephen Bosanquet went to school at Lancing, where he won a Mathematical Scholarship to Balliol College, Oxford in 1923. He became Goldsmith's Senior Student in 1927 and Senior Mathematics Scholar in 1929, the year he obtained both his M.A. and D.Phil. (and he was later awarded the D.Sc. in 1935).

The formative influence on LSB's mathematical career was G. H. Hardy, under whose direction at Oxford he obtained his D.Phil. In the early 1920s, when Stephen was a scholar of Balliol, Hardy (with the support of J. E. Littlewood in Cambridge) was intent upon creating a school of mathematics in England which would bear comparison with any in Europe or America, and he was fortunate in having an exceptionally able collection of research students. The senior of these were E. C. Titchmarsh (who was later to occupy Hardy's Savillian Chair of Geometry at Oxford 'on condition that he didn't have to lecture on geometry') and M. L. Cartwright (later Dame Mary) who was at St. Hugh's College 1919–23 and went to Cambridge in 1930, later becoming Mistress of Girton College. Among Hardy's other students during his period at Oxford (before he left for Cambridge in 1931 to join Littlewood and to occupy the Chair vacated by E. W. Hobson) were Bosanquet, Haslam-Jones, Linfoot, Oppenheim, Gertrude Stanley, Vijayaraghavan and E. M. Wright. Second only to mathematics, Hardy's absorbing passion was cricket; so one can guess that he would have been delighted to welcome to his group a kinsman of the legendary googly bowler B. J. T. Bosanquet.

Stephen Bosanquet (his fellow students used to call him 'Lance' at the time) and Hubert Linfoot became firm friends and attended Hardy's lectures regularly. Linfoot in particular kept meticulous notes, and a collection of his notebooks is presently in the Hardy Collection at Trinity College, Cambridge (for a list see [**10**, pp. 57–58]), including 11 notebooks of Hardy's lectures at Oxford during 1924–28. An intermittent attender at the lectures was John Evelyn (a descendant of the Diarist), and Bosanquet and Linfoot were frequent visitors to the Evelyn family home in Surrey. Linfoot collaborated with both Evelyn (on Number Theory) and with Bosanquet (on Fourier Series), though he later turned to Optics and Astronomy (see [**10**]). However, the three joint papers of Bosanquet and Linfoot on Fourier Series [1931b], [1931c], [1934c], typify the direction in which Bosanquet's interests were developing, namely in theory of series and integration.

Hardy had been working with Littlewood on fractional integration, and had some ideas about possible extensions of this work, which he discussed with Stephen Bosanquet. As a result, Bosanquet's first paper on this subject [1930b] was submitted in June 1929, and it became a topic to which he returned intermittently all through his career. There is no doubt that Hardy's infectious dedication and enthusiasm, and his supreme intellect and integrity, had a profound influence on young Stephen, as on all his students. It was a debt of gratitude which Bosanquet was able fully to repay when the time came to edit Hardy's last book and his Collected Papers.

3. *Family*

It was through his Balliol friend Hubert Linfoot that Stephen met Linfoot's sister Isobel, daughter of G. E. Linfoot of Sheffield, himself both a musician and a mathematician. Stephen and Isobel were married in April 1936 and have two daughters: Caroline (b. 1940) and Cordelia (b. 1946).

It was always a household full of literature and music, reflecting Isobel's degree in English Literature and the interest of both of them in music. Stephen played the piano for recreation and encouraged his daughters in musical careers: Caroline is presently Senior Lecturer in Music at the Cambridge College of Arts and Technology (when I remarked to her that 1985 marked exactly 300 years since the arrival of her ancestor David Bosanquet in England, she said it was one of four important happenings in 1685—the other three being the births of J. S. Bach, G. F. Händel and D. Scarlatti).

For Stephen, mathematics was essentially a solitary, quiet occupation, without noise or interruptions. He was happier working at home with his own books and papers, or (in the words of his daughter) 'communing with the typewriter'. He preferred to keep out of the limelight, but when cornered, he could acquit himself with vigour and wit, as when called upon for the obligatory speech at his daughter Cordelia's wedding, or at his professorial inaugural address, where his mathematical characterization of two professors approaching each other on bicycles was a rousing success.

He probably disliked large mathematical gatherings, though he did attend the International Congress of Mathematicians in Amsterdam (1954), Edinburgh (1958) and Stockholm (1962). Otherwise he preferred motoring holidays with his family to places of scenic, architectural or antiquarian interest, overwhelmingly in France or Italy. One of these trips included a visit to the old Bosanquet family seat of Château de Cardet at Lédignan, near Nîmes. Having survived the revocation of the Edict of Nantes, the château also escaped destruction during the French Revolution; this was attributed to the prudence of the steward left in charge of it at the time, who disarmed the anger of the neighbourhood revolutionaries by himself ordering the pulling down of its four turrets.

After his retirement from active teaching in 1971, he moved a few years later from London to Cambridge. But in the last six months of his life, after a major operation, he was distressed by his inability to resolve the many mathematical problems and projects which he still had outstanding.

4. *Career*

Immediately after his D.Phil. at Oxford in 1929, Bosanquet went to University College, London, as a Lecturer, and remained there for the whole of his career. For 30 years (1936–66) he was a Reader, and became a Professor for the last 5 years, retiring in 1971 as Professor Emeritus and Honorary Research Fellow. The only extended periods away from University College were the year 1964–65 when he was Visiting Professor at the University of Utah in Salt Lake City, followed by a shorter period in 1965 as Rice Visiting Professor at Washington State University at Pullman, and also the year 1969–70 when he visited the University of Western Ontario in London, Ontario. He made shorter visits at various times to the Universities of Witwatersrand and Cape Town (S. Africa), York University (Toronto, Canada), and to the Universities of Marburg, Ulm, Stuttgart (Germany). While at the University of Utah he gave an extended series of lectures [1965] on 'The history and development of the theory of divergent series and integrals' and at the University of Western Ontario a series of lectures [1970] on 'Matrix transformations and sequence spaces with applications to summability'.

He had 19 Ph.D. students stretching over 30 years, two of whom are now Fellows of the Royal Society. Over these he exercised a kind of loose supervision, standing ready with advice and ideas but allowing them to develop their own directions. His first student was Cyril Offord in 1930, when LSB himself was 26. Others include Ambrose Rogers, David Borwein, G. L. Isaacs, M. R. Mehdi, R. Mohanty, and the late Winifred Sargent (see [**12**]). Ida Busbridge and the late Lionel Cooper both started their research work with Bosanquet before transferring to D.Phil. degrees at Oxford.

He was a staunch supporter of the London Mathematical Society, joining the Society on 29 April 1926 and serving it in many non-trivial positions. He was on the Council of the Society 1944–55, Secretary 1947–50, Vice-President 1950–54, and Editor of the Journal 1947–55.

Stephen Bosanquet belonged to a generation of mathematicians which did its work on the backs of envelopes and old examination scripts, not because they were the only things handy when an idea struck, but as a conscious policy of thrift and conservation. In this respect he resembled his mentors Hardy and Littlewood: Hardy always recycled his manuscripts and lecture notes (once all through on one side, and then on the other), while Littlewood reached the age of 92 without apparently ever buying any paper for mathematical purposes (and Mary Cartwright tells me that Littlewood's draft manuscripts often contain notes of appointments and grocery lists in the margins). I still treasure one particular mathematical letter from Bosanquet, of several pages, which resembles a set of paper doilies—he had cut away all the margins, paragraph indentations and other blank space, to save airmail postage.

One of LSB's most characteristic attributes was his kindness to younger mathematicians and his concern for the integrity of the published work. Ambrose Rogers remarks, in the *Times* Obituary [**3**]: 'He was an extremely generous mathematician, always striving to help and encourage his students to sharpen their ideas. Indeed, he would help anyone who sent him a mathematical manuscript, in this way'. His colleague Milne Anderson has also remarked to me: 'He was always extremely kind to me as a young man, and always very generous with his time and advice'. B. Choudhary (I.I.T. Delhi) says: 'He was generous and kind and helped his research scholars to the utmost'. One can also find evidence of Stephen Bosanquet's generosity in many well-known works, for example:

(a) Dienes, *The Taylor series* (Oxford, 1931).
'I can hardly overestimate the value of Dr. Bosanquet's help. He went through all the heavy proofs of fundamental results, clearing up the intricate reasoning involved.'

(b) Hardy, Littlewood, Polya, *Inequalities* (Cambridge, 1934).
'Dr. Bosanquet, Dr. Jessen and Professor Zygmund have read the proofs and corrected many inaccuracies.'

(c) Hardy, *Divergent series* (Oxford, 1949).
'Dedicated by the author to L. S. Bosanquet, without whose help this book would never have been finished.' Hardy died in 1947, and LSB checked everything, adding annotations, and piloted it through the press.

(d) Hardy, *Collected papers* (7 volumes, Oxford, 1967–79).
This project was planned in the early 1960s under the chairmanship of H. Davenport, and LSB was on the committee from the outset. He was originally in charge of Volume 6 on the Theory of Series, and added at the end of each of the 52 papers in the volume the same kind of meticulous and informative notes and references which he had put at the end of the chapters in Hardy's *Divergent series*. On the death of Davenport midway through the publication process, Bosanquet became General Editor of the whole set.

(e) Littlewood, *Collected papers* (2 volumes, Oxford, 1982).
Here LSB took charge of the section 2(d) on Theory of Series and Tauberian Theorems and performed a similar service of annotations as he had done with Hardy's *Collected papers*.

When Bosanquet refereed a paper or commented on a manuscript which someone had sent him, he always tried to put himself in the position of a young mathematician in labour, struggling to give birth. And LSB behaved like a sympathetic midwife, never putting himself in an adversarial position, but rather striving to help the author bring forth a strong healthy child at the proper time. Although I was not one of his students (though he was later my D.Sc. examiner) I went regularly to his postgraduate lectures and to his analysis seminar at University College. At last my first paper was ready, honed (as I thought) to perfection, and with diffidence I showed it to Dr. Bosanquet. He took it home and came back a few days later with 3 foolscap pages of comments and suggestions for avenues to explore and further references to check, followed a little later by a further 10 pages. Six months later the paper was again ready, but I had learned in the meantime a great deal about the philosophy of mathematical research. Later, when we collaborated in a paper for the *Proceedings* of the London Mathematical Society, the manuscript went through seven quite different editions before it finally passed his eagle-eyed scrutiny.

But if his papers show a meticulous observance to detail and an obsession with accuracy down to the last comma, he often cast caution aside in his letters and his lectures, and one could obtain from these the flavour of heuristic which showed his mind at work. He drew his audience along with him, inviting them to contradict him or to suggest a different direction. And my mathematical correspondence with him was voluminous because he just sat down at his typewriter and typed out arguments and proofs as they came to him, using the typewriter quite informally. For example (after he had gone back to put in Greek symbols and forgot to change the type-ball) I find the phrase: '→*ηισ ρθο* × *εσ τηε ωιμιτατιον* Sorry! I was speaking in tongues. In translation "this proves the limitation theorem".' It was also not unknown for me carefully to check a long page of a letter only to find typed at the end of the page 'I don't think this argument is going to lead anywhere; let's try a different way'. Of course, I suppose his idea was that whatever he wrote, even if not immediately effective, might trigger some response in the observer which could lead to a method he had not thought of himself. In this respect, he was the most generous of mathematicians: he shared his ideas as soon as they came to him, as though they were God's property and he had no right to keep them for himself.

As might be supposed from someone so punctilious, it took a long time to bring a paper to gestation and final delivery, so that in 50 years of writing he published about 65 papers; but many of them are of substantial length and all of them are models of scholarly exposition. Besides the three papers with E. H. Linfoot, he collaborated with M. L. Cartwright, A. C. Offord, J. M. Hyslop, H. Kestelman, J. B. Tatchell, H. C. Chow, G. Das, and myself.

5. *Mathematical work*

While Bosanquet's very first paper was 'Generalizations of Minkowski's inequality' [1928], some 20 of his papers, overwhelmingly in the first 10 years after his doctorate, concern the *convergence and summability of Fourier series*. These began with his second paper, 'On the summability of Fourier series' [1930a], and essentially ended with [1941b], except for a paper [1945b] (actually written in 1941) in which he tidied up a number of difficult loose ends. His second contribution was in the *theory of fractional integrals* (or various types): unlike his work on Fourier series, which was more or less consolidated into a decade, he returned to this theme at various times through his career, ranging from his third paper [1930b] through his massive pair of papers on the Laplace–Stieltjes integral [1953], [1961], to his very delicate analysis of 'Liouville's extension of Abel's integral equation' [1969b]. A third main area to which he turned his attention was that of *convergence and summability factors*: while some of these results were for specific kinds of series (for example, Dirichlet series [1947], [1948b]), others were for specific kinds of summability (for example, Nörlund means [1979], and particularly summability factors for Cesàro means which, in a long series of papers from [1942] to [1983], he effectively cleaned up). *Inequalities*, *mean-value theorems*, *Tauberian theorems*, *convexity theorems* are other topics to which he made interesting, and in some cases substantial, contributions; a number of these turn out to provide fundamental examples of important principles, particularly in the functional analytic aspects of sequence spaces and function spaces. There remains in much of this work a mine of untapped information which we may still expect to draw upon.

It was characteristic of Bosanquet's work that he never creamed off the most obvious cases, leaving the more messy details to others. If he was cooking the dinner, then cleaning up was part of the process, and indeed he took pride in dealing systematically with the more difficult side-issues, for he regarded these as the most likely bridges to neighbouring problems. A man of considerable patience, he did not mind leaving a paper lying around for several years while he gradually filled in all the details. He was also careful never to give the impression that he had really done something which he had not thoroughly checked. He once cautioned me against claiming in a paper that a particular result was 'obtainable by suitable modifications', because he said that it left the priority in limbo and could prevent others from working on the problem. On another occasion, when we were working on a joint paper and needed a result from a previous paper of mine where I had said 'it is not difficult to show that...', he insisted on putting in (despite production of a proof from my files) 'Russell has stated without proof that...' (see [1976, p. 575])!

One of the tools which he used frequently was the Cesàro transform (of series or integrals). The Cesàro (C, α)-means ($\alpha \in \mathbb{R}$, $\alpha > -1$) of a series $\sum_0^\infty a_k$ (or of its sequence of partial sums (s_n)) form the sequence

$$s_n^\alpha = \frac{1}{E_n^\alpha} \sum_{k=0}^{n} E_{n-k}^{\alpha-1} s_k = \frac{1}{E_n^\alpha} \sum_{k=0}^{n} E_{n-k}^{\alpha} a_k \quad (n = 0, 1, \ldots), \tag{1}$$

where

$$E_0^\alpha = 1, \quad E_n^\alpha = \binom{n+\alpha}{n} = \frac{1}{n!}(\alpha+1)\ldots(\alpha+n) \quad (n \geqslant 1);$$

$\sum a_k$ (or (s_n)) is said to be (C, α)-summable to the value s if $\lim_{n\to\infty} s_n = s$ exists, and $|C, \alpha|$-summable if $\sum |s_n^\alpha - s_{n-1}^\alpha| < \infty$. Cesàro summability increases in power as the parameter increases, namely: if $-1 < \alpha < \beta$ and $s_n^\alpha \to s$ then $s_n^\beta \to s$ [that is, $(C, \alpha) \subset (C, \beta)$].

For a function $\phi(\cdot) \in L(0, t)$, $t > 0$, the Riemann–Liouville αth fractional integral of ϕ ($\alpha \geqslant 0$) is

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \phi(u)\, du\, (\alpha > 0), \quad \Phi_0(t) = \phi(t), \tag{2}$$

and the corresponding Cesàro (C, α)-integral mean of ϕ is

$$\phi_\alpha(t) = \Gamma(\alpha+1)\, t^{-\alpha}\, \Phi_\alpha(t) \quad (\alpha \geqslant 0). \tag{3}$$

Then $\phi(t) \to s$ (C, α) means $\phi_\alpha(t) \to s$; the limiting value of t can be either $t \to +\infty$ or $t \to 0+$, and must therefore be specified.

5.1. *Fourier series*

The average mathematician knows the highlights about Fourier series. If he is globally-minded he is satisfied to know Carleson's Theorem that the Fourier series of an L^2 (and hence of a continuous) function is convergent almost everywhere. If he is locally-minded, he then objects that, given some particular point, how do we know whether it belongs to some divergence set, even if this set is of measure zero? But he is somewhat mollified by Jordan's Theorem that the Fourier series of a function of bounded variation converges everywhere, or that, in any case, if we have a continuous function then its Fourier series must be $(C, 1)$-summable everywhere, by Fejér's Theorem (a standard reference is [15]). Stephen Bosanquet belonged to the latter school; while he certainly appreciated the power of (non-constructive) existence theorems, particularly in functional analysis, he was essentially a constructivist who liked to work with something concrete which he could use as his building blocks.

Let us suppose throughout this section that f is $L(-\pi, \pi)$ and 2π-periodic, and for a fixed x denote $\phi(t) = \frac{1}{2}\{f(x+t)+f(x-t)\}$; write $\mathscr{F}[f(x)]$ for the Fourier series of f at the point x.

An improvement on Fejér's Theorem was that found by S. Chapman (1910), who replaced $(C, 1)$-summability by Cesàro summability of any positive order. Bosanquet [1930a, Theorem 1] says:

If $\beta > \alpha \geqslant 0$ and if $\phi(t) \to s$ (C, α) as $t \to 0+$, then $\mathscr{F}[f(x)] \in (C, \beta)$.

The case $\alpha = 0$ gives Chapman's Theorem, and if also $\beta = 1$ we get Fejér's Theorem. It is noteworthy here (and typical of Bosanquet's future work) that his theorem is simply and elegantly expressed, the surprise being that the two Cesàro summabilities are of essentially different types, since the (C, α) is an integral transform and the (C, β) a series transform.

The Abel-transform of a series $\sum_0^\infty a_k$ is the function $a(x) = \sum_{k=0}^\infty a_k x^k$. If $a(x)$ exists (that is, the power series converges) for $0 \leqslant x < 1$, then Σa_k is said to be A-summable to s when $\lim_{x\to 1-0} a(x) = s$ exists, and $|A|$-summable when $\int_0^1 |da(x)| < \infty$. While we have the inclusion relations

$$(C, \alpha) \subset (C, \beta) \subset A, \quad |C, \alpha| \subset |C, \beta| \subset |A| \quad (-1 < \alpha < \beta),$$

it turns out that convergence, that is $(C, 0)$, is not comparable with $|A|$. Now Jordan's

Theorem says that if f is locally of bounded variation near a point x, then its Fourier series converges at x. More precisely:

If $\phi(\cdot) \in \mathrm{BV}(0, \eta)$ for some $\eta > 0$, then $\mathscr{F}[f(x)] \in (C, 0)$.

In [1934b], Bosanquet proved a generalized analogue for $|A|$-summability:

If $\alpha \geqslant 0$, and if $\phi_\alpha(\cdot) \in \mathrm{BV}(0, \eta)$ for some $\eta > 0$, then $\mathscr{F}[f(x)] \in |A|$.

This theorem includes several earlier results, notably for the cases $\alpha = 0, 1$.

In [1936c] and [1936d] Bosanquet considers the consequences of replacing local BV by global BV in his hypothesis:

If $0 \leqslant \alpha < \beta$ and if $\phi_\alpha(\cdot) \in \mathrm{BV}(0, \pi)$, then $\mathscr{F}[f(x)] \in |C, \beta|$.

He also gives a theorem in the converse direction:

If $\beta > \alpha + 1 \geqslant 1$ and if $\mathscr{F}[f(x)] \in |C, \alpha|$, then $\phi_\beta(\cdot) \in \mathrm{BV}(0, \pi)$.

A question raised in these papers was whether $|C, 1|$-summability of a Fourier series at a point was a local property of the generating function, and this was answered (in the negative) in a joint paper with H. Kestelman [1938a].

The typical feature which strikes one on reading LSB's papers on Fourier series is the beautiful simplicity of many of his results. That this is deceptive comes from a closer scrutiny of his methods, where hard work and supreme analytic skill has allowed him to express his result in a form free of all artificial restrictions (such as restriction of a parameter to an integer because of the method of proof chosen); the final form of his result thus appears entirely natural.

5.2. *Fractional integration*

On first glance, the results in Bosanquet's first paper on fractional integration [1930b] look essentially similar to known consistency theorems, but on closer inspection they give a foretaste of his ability, refined and developed all through his career, to pare down hypotheses to a minimum. Thus [1930b, Theorems 4 and 5] give:

If, for some fixed $b \geqslant -\infty$, and for $\alpha > 0$, $\beta > 0$,

$$\Phi_{\alpha+\beta}(t) = \frac{1}{\Gamma(\alpha+\beta)} \int_b^t (t-u)^{\alpha+\beta-1} \phi(u)\, du$$

exists as a general Denjoy integral then $\Phi_\alpha(u)$ exists in the Denjoy sense for almost all u in (b, t), and

$$\Phi_{\alpha+\beta}(t) = \frac{1}{\Gamma(\beta)} \int_b^t (t-u)^{\beta-1} \Phi_\alpha(u)\, du.$$

For example, the case $b = -\infty$ gives a consistency theorem for the Weyl fractional integral.

In the much more comprehensive paper [1969b], Bosanquet takes as his definition of fractional integral and derivative, one due to Cossar (1941), which includes both the Liouville and Weyl definitions; the mode of integration is now that of Lebesgue. One object of the paper is to show [1969b, Theorem 1], with Cossar's integral and differential operators I^α, D^α ($\alpha > 0$), that:

If $f(x)$ is representable in the form $f(x) = I^{\alpha}g(x)$ almost everywhere for $x > A$ (where $A \geqslant -\infty$), then $g(x) = D^{\alpha}f(x)$ a.e. for $x > A$.

Further theorems [1969b: Theorems 2, 4, 5 for $0 < \alpha < 1$, Theorem 3 for $\alpha \geqslant 1$] then proceed to examine in comprehensive detail the necessary and sufficient conditions under which a function f is in fact representable in the required form $f(x) = I^{\alpha}g(x)$.

Two of his most interesting papers are 'The summability of Laplace–Stieltjes integrals, I, II' [1953 and 1961]—a total of 75 pages in which none of the irksome details are avoided. They represent a natural outcome of his work on fractional integration. The formal integral is of the form $\int_0^{\infty} k(u)\,dA(u)$ with k continuous and A locally BV; thus $k(u) = e^{-su}$, $s = \sigma + i\tau$, gives the Laplace–Stieltjes transform of $A(\cdot)$, while if we extend to $\int_{-\infty}^{\infty}$ and take $k(u) = e^{i\tau u}$, we get the Fourier–Stieltjes transform of $A(\cdot)$. Since A is only locally BV, the integrals may not converge in the usual sense. However, if we apply the Cesàro (C, α)-transform to $\phi(t) = \int_0^t k(u)\,dA(u)$, then the corresponding fractional integral is

$$\Phi_{\alpha}(t) = \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-u)^{\alpha} k(u)\,dA(u) \quad (\alpha \geqslant 0,\ t > 0). \tag{4}$$

By Fubini's Theorem for Lebesgue–Stieltjes integrals, this $\Phi_{\alpha}(\cdot)$ is the same as that defined in (2). Defining $\phi_{\alpha}(t) = \Gamma(\alpha+1)\,t^{-\alpha}\,\Phi_{\alpha}(t)$ as before in (3), $\int_0^{\infty} k(u)\,dA(u) = s(C, \alpha)$ then means $\phi_{\alpha}(t) \to s\ (t \to +\infty)$, while $\int_0^{\infty} k(u)\,dA(u) \in |C, \alpha|$ means $\int_b^{\infty} |d\phi_{\alpha}(t)| < \infty$ for some $b > 0$. Bosanquet examines in great detail, mainly for $k(u) = e^{-su}$, the properties of these integrals, which could well prove useful to anyone working on unusual cases of the Laplace–Stieltjes transform. He relates his results to earlier work of M. Riesz, Hardy and Littlewood, H. Bohr, M. Fekete, N. Obrechkoff, as well as to contemporary work of G. L. Isaacs and D. Borwein.

When $k(u) \equiv 1$ in (4), we write $A_{\alpha}(t)$ for the corresponding $\Phi_{\alpha}(t)$:

$$A_{\alpha}(t) = \frac{1}{\Gamma(\alpha+1)} \int_0^t (t-u)^{\alpha}\,dA(u) \quad (\alpha \geqslant 0,\ t > 0),$$

where we assume without loss of generality that $A(0) = 0$.

Bosanquet [1953, Theorem 2] shows that:

If $\alpha \geqslant 0$ and if $\int_0^{\infty} e^{-s_0 u}\,dA(u) \in |C, \alpha|$, then $\int_0^{\infty} e^{-su}\,dA(u) \in |C, \alpha|$ for $\mathrm{Re}\,s > \mathrm{Re}\,s_0$.

The analogous result for ordinary (C, α) summability had been given by M. Riesz (1924); his result shows the existence of an abscissa of (C, α)-summability γ_{α}, namely γ_{α} is the least ξ such that $\int_0^{\infty} e^{-su}\,dA(u) \in (C, \alpha)$ for $\mathrm{Re}\,s > \xi$; if $\gamma_{\alpha} \geqslant 0$, then its value is given by

$$\gamma_{\alpha} = \limsup_{t \to +\infty} \frac{1}{t} |A_{\alpha}(t)|.$$

Bosanquet's result just quoted likewise establishes an abscissa of $|C, \alpha|$-summability $\bar{\gamma}_{\alpha}$, and in [1953, Theorem 20] he shows that if $\bar{\gamma}_{\alpha} \geqslant 0$, then its value is given by

$$\bar{\gamma}_{\alpha} = \limsup_{t \to +\infty} \frac{1}{t} \log \int_0^t |dA_{\alpha}(t)|.$$

Several of Bosanquet's other results establish relations between different orders of Cesàro summability for different forms of the Laplace integrals. As an example, we select a converse-type result [1953, Theorem 4]:

If $\alpha \geqslant 1$ *and if* $\int_0^\infty e^{-s_0 u}\, dA(u) \in (C, \alpha)$ [*or* $|C, \alpha|$] *then there is a number A (arbitrary if* $\operatorname{Re} s_0 \geqslant 0$) *such that* $\int_0^\infty e^{-su}\{A(u)-A\}\, du \in (C, \alpha-1)$ [*or* $|C, \alpha-1|$] *for* $\operatorname{Re} s > \operatorname{Re} s_0$.

In [1961] results similar to this last one (where the two orders of summability differ by 1) are extended to cases where the orders of summability differ by an arbitrary (not necessarily integer) value.

Many of the results in the papers [1953] and [1961] are motivated by, and have applications to, the theory of Dirichlet series. Thus if $0 < \lambda_0 < \lambda_1 < \ldots < \lambda_n \to \infty$ and if $A(t) = \sum_{\lambda_n \leqslant t} a_n$, then $\int_0^\infty e^{-su}\, dA(u)$ reduces to the Dirichlet series $\sum_{n=0}^\infty e^{-\lambda_n s} a_n$, while the Cesàro (C, α) mean of the integral reduces to

$$\phi_\alpha(t) = \sum_{\lambda_n \leqslant t} \left(1 - \frac{\lambda_n}{t}\right)^\alpha e^{-\lambda_n s} a_n$$

which is the typical Riesz (R, λ_n, α)-mean of the Dirichlet series. See also §§5.4 and 5.6 below, and for general properties of Riesz means see [**4**].

5.3. *Inequalities and mean value theorems*

Bosanquet's first paper on the subject of mean values [1931a] was the completion of a partially-proved result of Hardy, Ingham and Polya (1927) on mean values of a function of a complex variable, holomorphic in a strip. But while Bosanquet was prepared to draw on the full power of complex function theory when applicable, the intricacies and delicacies of real variable theory appealed more to him.

Thus it is natural that he should be interested in Riesz's Inequality, which estimates a section of a fractional integral in terms of values of the full fractional integral:

If $0 < \alpha < 1$, $0 \leqslant y \leqslant x$ *and* $\phi(\cdot) \in L(0, y)$, *then*

$$\left| \int_0^y (x-u)^{\alpha-1} \phi(u)\, du \right| \leqslant \operatorname*{ess\,sup}_{0 \leqslant t \leqslant y} \left| \int_0^t (t-u)^{\alpha-1} \phi(u)\, du \right|. \tag{5}$$

If ϕ is real-valued and the integral on the right is continuous in t, this result can be written in the form of a mean-value theorem. A sharper inequality is obtained by inserting, on the right of (5), a factor

$$W(\alpha, x, y) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^y (x-u)^{\alpha-1} u^{-\alpha}\, du; \tag{6}$$

here $W(\alpha, x, y) \leqslant 1$ and the factor is best-possible in the sense that equality is then obtained for $\phi(u) = u^{-\alpha}$.

In [1967b] Bosanquet replaces the fractional integral by a more general convolution integral (in Stieltjes form) $F(t) = \int_b^t G(t-u)\, d\phi(u)$, where $G(\cdot)$ is positive. He considers a related positive function H such that $\int_0^t G(t-u) H(u)\, du = 1$ $(t > 0)$, $= 0$ $(t = 0)$, and a kernel

$$h_y(x, t) = -\int_y^x G(x-u)\, d_u H(u-t).$$

Several theorems consider various hypotheses on G, H, ϕ under which we can assert that

$$\int_b^y G(x-u)\, d\phi(u) = \int_b^y h_y(x, t) F(t)\, dt$$

and Bosanquet then deduces generalizations of (5) in both mean-value and inequality form; the choice $G(t) = t^{\alpha-1}$, $H(t) = t^{-\alpha}/\{\Gamma(\alpha)\Gamma(1-\alpha)\}$ leads us back to the original Riesz form (with the factor W).

As a sequel, Bosanquet in [1969a] asks what functions G, H other than these last examples, satisfy the functional equation $\int_0^t G(t-u)H(u)\,du = 1$ for almost all $t > 0$.

The analogues for sequences, of many of these results on integrals, were also considered by Bosanquet. Indeed (using the notation in (1)), the direct analogue of (5) is [1941c, Theorem 1]:

If $0 \leqslant \alpha \leqslant 1$ *and* $0 \leqslant m \leqslant n$ *then, for any* (s_k),

$$\left|\sum_{k=0}^{m} E_{n-k}^{\alpha-1} s_k\right| \leqslant \max_{0 \leqslant r \leqslant m} \left|\sum_{k=0}^{r} E_{r-k}^{\alpha-1} s_k\right|. \qquad (7)$$

As remarked in §5.4 below, this inequality shows directly that the space of (C, α)-summable sequences, $0 \leqslant \alpha \leqslant 1$, has the AB (abschnittsbeschränkt = section-bounded) property, which is equivalent to a basis property. In fact, a variant of (7), namely [1941c, Theorem 2] is readily generalized (explicitly in [1949]) to give a criterion for the AB property for summability fields of more general triangular matrixes (a_{nk}). See [**14**, p. 43].

Inequalities such as (7), but with more general matrix elements in place of $E_t^{\alpha-1}$, were considered in detail by Bosanquet [1966b]; these include the insertion of best-possible factors, and other inequalities and mean-value theorems for matrices and sequences, analogous to those for integrals which appear subsequently in [1967b].

5.4. *Convergence and summability factors*

The basic problem here is:

Given two linear summability methods $(A, B$ *say) for an infinite series, what are necessary and sufficient conditions on the sequence* (ε_n) *in order that we always have* $\Sigma a_n \in A \Rightarrow \Sigma a_n \varepsilon_n \in B$?

The sequence (ε_n) is an '(A, B)-summability factor'.

A simplified question is to take B as ordinary convergence, and then the problem reduces to asking:

When does $\Sigma a_n \in A$ *imply* $\Sigma a_n \varepsilon_n$ *convergent?*

Now (ε_n) is a *convergence-factor for A*, and in this case much can be done from a more modern viewpoint by treating (a_n) as an element of a sequence space (λ, say), and defining

$$\lambda^\beta = \{(\varepsilon_n) \mid \Sigma a_n \varepsilon_n \text{ converges for every } (a_n) \in \lambda\} = \beta\text{-dual space of } \lambda.$$

If the original sequence space λ has a particular type of Schauder basis, a great deal of information about λ^β comes out by methods of functional analysis; LSB anticipated this for the space λ of (C, α)-summable sequences, $0 \leqslant \alpha \leqslant 1$, for his paper 'A mean value theorem' [1941c] is precisely a proof that this λ has the Schauder basis required. More details of this approach can be found in [7]. However, in the more general case of summability factors, the approach via sequence spaces has hardly been touched, and LSB's results still await incorporation into a unified theory.

His first excursion into this area dealt with the case of 'Bohr–Hardy summability factors', namely the case where the summability methods A and B are the same. In [1942, Theorem B] he proves:

If $\alpha \geqslant 0$, then in order that (ε_n) should be a $((C, \alpha), (C, \alpha))$-summability factor, it is necessary and sufficient that both (ε_n) and $\left(\binom{n+\alpha}{n}\Delta^\alpha \varepsilon_n\right)$ should be of bounded variation.

This is given as a corollary of the more general [1942, Theorem A], which contains an additional parameter. The fractional difference Δ^α is defined in (15) below.

In a subsequent long series of papers Bosanquet (sometimes with collaborators) dealt with the characterizations of (A, B)-summability factors for a large variety of combinations of ordinary or absolute summability methods A and B, mainly chosen from Cesàro methods of different orders, Abel methods, or Nörlund methods. For example [1945c, Theorem 2]:

If $0 \leqslant \rho \leqslant \alpha$, then (ε_n) is a $(|C, \alpha|, |C, \rho|)$-summability factor if and only if $\varepsilon_n = O(n^{\rho-\alpha})$ and $\Delta^\alpha \varepsilon_n = O(n^{-\alpha})$. If $\rho > \alpha \geqslant 0$, the conditions are the same as in the case $\rho = \alpha$.

For an excellent survey of the earlier work on Cesàro summability factors see [**1**].

As another example, Bosanquet and Tatchell [1957b, Theorem 2] prove:

If $\alpha \geqslant -1$, then (ε_n) is a $((C, \alpha), |A|)$-summability factor if and only if $\Sigma n^{-1} |\varepsilon_n| < \infty$ and $\Sigma n^\alpha |\Delta^{\alpha+1} \varepsilon_n| < \infty$.

Here $|A|$ denotes absolute Abel summability, and it is worth noting that in this last paper, many of the arguments are simplified by making use of functional analytic techniques.

In a comprehensive paper, Bosanquet and Das [1979] consider (A, B)-summability factors for which A and B are ordinary or absolute Nörlund methods; in general these methods have different defining sequences, but specialization yields a number of earlier results for Cesàro and harmonic summability.

For Dirichlet series, the appropriate related summability method is that of the Riesz typical means (see the last paragraph of §5.2 above); we may write $l_n = e^{\lambda_n}$ in order to obtain our series in the form $\Sigma l_n^{-s} a_n$ instead of $\Sigma e^{-\lambda_n s} a_n$, so that $1 < l_0 < l_1 < \ldots < l_n \to \infty$. In considering the relation between Σa_n and $\Sigma l_n^{-s} a_n$, the sequence (l_n^{-s}) takes the role of a convergence or summability factor. Thus [1947, Theorem A] gives:

If α is a positive integer and $D = \limsup_{n\to\infty} (\log n/\log l_n)$, and if Σa_n is bounded (R, l_n, α), then $\Sigma l_n^{-\sigma} a_n$ is summable $|R, l_n, \alpha|$ for $\sigma > D$.

Bosanquet remarked at the time [1947, p. 191] that the truth of this result for *all positive* α remained unsettled, and I am not aware that an answer to this question has been given, though he does give a result later in which $|R, l_n, \alpha|$ is replaced by $|R, l_n, \alpha+1|$, namely [1948b, p. 35]:

If $\alpha \geqslant 0$ and $\Sigma a_n \in (R, l_n, \alpha)$, then $\Sigma l_n^{-\sigma} a_n \in |R, l_n, \alpha+1|$ for $\sigma > 0$.

Summability factors remained one of Stephen Bosanquet's favourite topics, and it is perhaps fitting that his last paper [1983] was on this subject. As already remarked, his results in this area still await incorporation into a unified general theory.

5.5. *Tauberian theorems*

The classical Tauberian theorems are of the following type: X is a set of functions (or sequences) with subsets F, T ($F \cap T \neq \varnothing$), and $L: F \cup T \to X$ is a linear operator such that

$$f \in F \Rightarrow Lf \in F; \tag{8}$$

$$Lf \in F \text{ and } f \in T \Rightarrow f \in F. \tag{9}$$

The condition $f \in T$ is the 'Tauberian condition' for the operator L and the set F. The prototype was Tauber's Theorem (1897) that

$$\lim_{x \to 1-0} \sum_{k=0}^{\infty} a_k x^k = s \text{ and } ka_k = o(1) \Rightarrow \sum_{k=0}^{\infty} a_k \text{ converges to } s; \tag{10}$$

this corresponds to (9) with F as the set of convergent series and L as the Abel-summability operator (the consistency part of the theorem, (8), is Abel's continuity theorem for power series). The term 'Tauberian theorem' was first used by Hardy and Littlewood in 1913. However, the famous Littlewood Tauberian Theorem (1911) replaced the Tauberian condition $ka_k = o(1)$ in (10) by $ka_k = O(1)$; and, in general, 'o-Tauberian theorems' are usually easy to prove, while the 'O-Tauberian theorems' lie much deeper (Littlewood, in his 1911 paper, proves (10) from elementary first principles in 8 lines). The O-type conditions have been substantially generalized, such as to functions or sequences which are 'slowly oscillating' or 'regularly varying' (in this regard see recent papers of N. H. Bingham and C. M. Goldie [**2**], which contain extensive bibliographies). In the powerful theory of N. Wiener [**13**] (see also [**5**, Chapter 12]), L is a convolution operator and the Tauberian condition requires the non-vanishing of a Fourier transform. A short survey of the initial results in classical Tauberian theory is given by Bosanquet [1982].

In [1944] Bosanquet proves the following Tauberian theorem:

Let $0 \leqslant \lambda_0 < \lambda_1 < \ldots < \lambda_n \to \infty$. *If* $\lim_{x \to 1-0} \sum_{k=0}^{\infty} a_k x^{\lambda_k} = s$, *and if*

$$\lim_{\delta \to 0+} \limsup_{n \to \infty} \max_{\lambda_n < \lambda_m < (1+\delta)\lambda_n} |a_{n+1} + \ldots + a_m| = 0 \tag{11}$$

then $\sum_{k=0}^{\infty} a_k$ *converges to* s.

The Tauberian condition (11) is of slowly oscillating type, and is implied by

$$a_k = O\{(\lambda_k - \lambda_{k-1})/\lambda_k\}. \tag{12}$$

The corollary obtained by replacing (11) by (12) is stated in [**5**, Theorem 104]; the case $\lambda_k = k$ of this is exactly Littlewood's Theorem. However, Bosanquet's more general result just stated also includes a 'high indices theorem' when (λ_k) increases sufficiently rapidly; see [**5**, p. 177].

Bosanquet and Cartwright [1933b] had considered generalizations of Littlewood's Theorem in several directions, including the change to a complex variable and approach to the limiting value 1 from within an angle inside the unit disc. Among their results is [1933b, Theorem 1]:

Let $f(z) = f_0(z) = \sum_{k=0}^{\infty} a_k z^k$ *be holomorphic for* $|z| < 1$, *and*

$$f_n(z) = \frac{1}{1-z} \int_z^1 f_{n-1}(u)\, du \quad (n = 1, 2, \ldots). \tag{13}$$

If, for some positive integer r, $f_r(z) \to A$ as $z \to 1$ along $\arg(1-z) = \gamma$, $0 \leqslant |\gamma| < \frac{1}{2}\pi$, *and if* Σa_k *is bounded*, (C, α), $\alpha \geqslant -1$, *then* Σa_k *is summable* $(C, \alpha+\delta)$ *for every* $\delta > 0$.

This is actually a four-fold generalization of Littlewood's Theorem (which we get when $\alpha = -1$, $\gamma = 0$, $\delta = 1$, $r = 0$).

When, in the characterization of a Tauberian theorem in (8), (9) above, no Tauberian set T is required, that is when L is a bijective operator on F, we obtain a Mercerian theorem, the prototype here being Mercer's Theorem (1907) for $(C, 1)$-summability:

Let $q > -1$. *Then* $s_n + (q/n)(s_1 + \ldots + s_n) \to s \Leftrightarrow s_n \to s$.

In [1938b] Bosanquet gives an analogue of this theorem in which convergence of the sequence is replaced by absolute convergence (that is, bounded variation), namely:

Let $t_n = s_n + (q/n)(s_1 + \ldots + s_n)$. *If* $\operatorname{Re} q \neq -1$ *and if* $\Sigma |\Delta t_n| < \infty$, *then there is a constant C such that* $\Sigma |\Delta\{s_n - C\Gamma(n)/\Gamma(n+1+q)\}| < \infty$. *If* $\operatorname{Re} q > -1$ *then* $C = 0$.

5.6. *Other topics*

It is often a difficult task to divide a mathematician's work into separate compartments, since much of it will have implications in several areas. A striking example is the result of Bosanquet and Kestelman [1938a, Theorem 1]:

For $n = 1, 2, \ldots$, *let* $f_n(\cdot)$ *be measurable on* (a, b), $0 < b-a \leqslant \infty$. *In order that for every* $\phi(\cdot) \in L(a, b)$, *we should have*

$$f_n(\cdot)\,\phi(\cdot) \in L(a, b) \quad \text{and} \quad \sum_{n=1}^{\infty} \left| \int_a^b f_n(x)\,\phi(x)\,dx \right| < \infty,$$

it is necessary and sufficient that

$$\operatorname*{ess\,sup}_{x \in (a,\, b)} \sum_{n=1}^{\infty} |f_n(x)| < \infty.$$

Not only does this contain an important principle for the theory of Lebesgue integration, but it has an implication in the theory of Fourier series (already mentioned at the end of §5.1), and there is also a sense in which the $f_n(\cdot)$ in this theorem can be regarded as convergence factors.

Three papers, namely [1932b], [1933a] and [1946], deal mainly with *Hölder means*, though [1932b] could fairly be regarded as a paper on fractional integration, and [1933a] contains much of Tauberian character. The Hölder transform (of integer order r) is a $(C, 1)$-transform iterated r times and bears a natural relation (without being entirely equivalent) to the (C, r)-transform. In [1946] Bosanquet gives two counter-examples for Hölder summability $(H, 2)$ of a series, of which the most interesting [1946, Theorem B] is the existence of a series whose $(H, 2)$-means tend to $+\infty$, but which has an Abel transform with $-\infty$ as a limit point. In [1933a] Bosanquet and Cartwright consider the (H, r)-transform of a function $f(z)$ of a complex variable (namely $f_r(z)$ as defined by (13) above), and also the (C, α)-transform of $f(z)$ defined analogously to (3) but using complex variables throughout. They give a number of Tauberian-type and convexity-type theorems for both Hölder and Cesàro means, many of which have no analogues for transforms of series or of functions of a real variable.

Two papers [1943, 1959a] explicitly concern *convexity-type theorems*, in which we have a set of functions (or sequences) indexed by a real parameter σ, say $\{f_\sigma(\cdot)\}$, and from specified behaviour of two fixed members $f_{\sigma_1}(\cdot), f_{\sigma_2}(\cdot)$, we wish to deduce the behaviour of $f_\sigma(\cdot)$ for some σ between σ_1 and σ_2.

For a series Σa_k and a fixed sequence $(\lambda_n): 0 \leqslant \lambda_0 < \lambda_1 < \ldots \lambda_n \to \infty$, Bosanquet [1943] considers the convexity of the Riesz sums

$$A_\sigma(t) = \sum_{\lambda_k \leqslant t} (t-\lambda_k)^\sigma a_k \quad (t > 0); \tag{14}$$

for example [1943, Theorem 4]:

If $\rho > 0$, $\beta > -1$, *and* (i) $\liminf_{n\to\infty} (\lambda_{n+1}/\lambda_n) > 1$, (ii) $A_\rho(t) = o(t^\beta)$ $(t \to +\infty)$, *then* $A_\sigma(t) = o(t^{\beta-\rho+\sigma})$ $(t \to +\infty)$ *for* $0 \leqslant \sigma < \rho$.

This is a convexity theorem because hypothesis (i) implies an order condition on the behaviour of $A_0(t)$. Other results in [1943] concern analogues for Cesàro transforms.

The fractional differences of order σ ($\sigma \in \mathbb{R}$) of a bi-infinite sequence $(u_n)_{-\infty}^{\infty}$ are defined by

$$\Delta^\sigma u_n = \sum_{k=n}^{\infty} E_{k-n}^{-\sigma-1} u_k \quad (-\infty < n < \infty) \tag{15}$$

whenever this series converges; as usual,

$$E_n^\alpha = \binom{n+\alpha}{n}.$$

In [1959a, Theorem 4] Bosanquet gives the elegant convexity result:

If $-\infty < \sigma_1 < \sigma < \sigma_2 < \infty$, *if* $W(n)$, $V(n)$ *are positive and non-increasing, and if*

$$\Delta^{\sigma_1} u_n = O(W), \quad \Delta^{\sigma_2} u_n = O(V) \quad (-\infty < n < \infty),$$

then

$$\Delta^\sigma u_n = O\{W^{(\sigma_2-\sigma)/(\sigma_2-\sigma_1)} V^{(\sigma-\sigma_1)/(\sigma_2-\sigma_1)}\} \quad (-\infty < n < \infty).$$

In a joint paper [1976], Bosanquet and Russell consider the *Riesz* (R, λ_n, σ)*-mean* of a series Σa_k, namely $t^{-\sigma} A_\sigma(t)$ where $A_\sigma(t)$ is defined by (14); in this transform, t is a continuous real variable. But for many purposes (especially where an inversion is required) a discrete (that is, matrix) transform of a sequence or series is more tractable, and they construct such a matrix, defining the 'generalized Cesàro (C, λ_n, σ)-mean' of Σa_k. The principal result [1976, Theorem 2] is that (R, λ_n, σ) and (C, λ_n, σ) summabilities are equivalent for any series Σa_k, for any choice of $\sigma \geqslant 0$ or of the fixed sequence (λ_n): $0 \leqslant \lambda_0 < \lambda_1 < \ldots < \lambda_n \to \infty$. Incidentally, the Riesz sums (14) have direct connection with approximation theory: for σ a positive integer, $A_\sigma(t)$ represents a spline function (piecewise polynomial) of degree σ with knots (or nodes) at the points λ_k, and the connection with (C, λ_n, σ) has important implications for interpolation problems. Other implications of Riesz summability for approximation theory have yet to be explored. For a recent application, see [**17**].

Finally, Bosanquet's talent for *surveys and historical summaries* should be mentioned. We have already pointed out (see §4 above) the annotations in relation to [**5**], [**6**, Volume 6] and [**11**, §2*d*]. Attention should also be drawn to the short surveys in [1950a], [1966a] and [1982], as well as the extensive lecture notes [1965] and [1970].

5.7. *Conclusion*

Stephen Bosanquet would occasionally remark ruefully that he was not in fashion any longer. The fact is that his work, besides containing a large number of definitive results, continues to be a source of information on many topics, and there is still a great deal of it which is going to provide motivation for more general theories. He had an uncanny ability, refined and developed all through his career, to pare down hypotheses to their essential minimum, and he never shirked the inconvenient cases. All of his research papers are masterpieces of exposition and integrity which never fail to place the results in their proper context in relation both to previous and contemporary work.

6. *Acknowledgments*

First and foremost, I am grateful to Isobel and Caroline Bosanquet for many helpful letters and conversations and for the supply of a great deal more biographical material than I have been able to include. The photograph of Hardy's cricket team (dubbed by him 'Mathematicians *v.* The Rest of the World') was taken at New College, Oxford on the occasion of a British Association meeting (and appeared in the Daily Mirror of 11 August 1926); see also [16]. I also had access to the address given by Dr. J. C. Burkill at a memorial service in Cambridge, and conversation and correspondence with Dame Mary Cartwright has been stimulating and informative. Otherwise, Stephen Bosanquet's life and work reflects its own quality: he was a modest, gentle person—a fine analyst, a dedicated teacher and a generous friend.

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