RICHARD DAGOBERT BRAUER 1901–1977

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J. A. GREEN

Richard Dagobert Brauer, Emeritus Professor at Harvard University and one of the foremost algebraists of this century, died on April 17, 1977, in Boston, Massachusetts. He had been an Honorary Member of the Society since 1963.

Richard Brauer was born on 10 February, 1901, in Berlin–Charlottenburg, Germany; he was the youngest of three children of Max Brauer and his wife Lilly Caroline. Max Brauer was an influential and wealthy businessman in the wholesale leather trade, and Richard was brought up in an affluent and cultured home with his brother Alfred and his sister Alice.

Richard Brauer's early years were happy and untroubled. He attended the Kaiser–Friedrich–Schule in Charlottenburg from 1907 until he graduated from there in 1918. He was already interested in science and mathematics as a young boy, an interest which owed much to the influence of his gifted brother Alfred, who was seven years older than Richard.

His youth saved him from service with the German army during the first World War. He graduated from high school in September 1918, and he and his classmates were drafted for civilian service in Berlin. Two months later the War ended, and in February 1919 he was able to enrol at the Technische Hochschule in Berlin–Charlottenburg (now the Technische Universität Berlin). The choice of a technical curriculum had been the result of Richard's boyhood ambition to become an inventor, but he soon realised that, in this own words, his interests were " more theoretical than practical ", and he transferred to the University of Berlin after one term. He studied there for a year, then spent the summer semester of 1920 at the University of Freiburg —it was a tradition among German students to spend at least one term in a different university—and returned that autumn to the University of Berlin, where he remained until he took his Ph.D. degree in 1925.

The University of Berlin contained many brilliant mathematicians and physicists in the nineteen-twenties. During his years as a student Richard Brauer attended lectures and seminars by Bieberbach, Carathéodory, Einstein, Knopp, von Laue, von Mises, Planck, E. Schmidt, I. Schur and G. Szegö, among many others. In the customary postscript to his doctoral dissertation **[1]**, Brauer mentions particularly Bieberbach, von Mises, E. Schmidt and I. Schur. There is no doubt that the profoundest influence among these was that of Issai Schur. Schur had been a pupil of G. Frobenius, and had gradulated at Berlin in 1901; he had been " ordentlicher Professor " (full professor) there since 1919. His lectures on algebra and number theory were famous for their masterly structure and polished delivery. Richard Brauer's first published paper arose from a problem posed by Schur in a seminar on number theory in the winter semester of 1921. Alfred Brauer also participated in this seminar. He was less fortunate than Richard, in that his studies were seriously interrupted by the War; he had served for four years with the army and been very badly wounded. The Brauer brothers succeeded in solving Schur's problem in one week, and in the same week a completely different solution was found by Heinz Hopf. The Brauer proof was published in the book by Polya and Szegö (1925; p. 137,

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pp. 347–350), and some time later the Brauers and Hopf combined and generalized their proofs in their joint paper [**2**].

Richard Brauer also participated in seminars conducted by E. Schmidt and L. Bieberbach on differential equations and integral equations—a proof which he gave in a talk at this seminar in 1922 appears, with suitable acknowledgment, in Bieberbach's book on differential equations (1923; p. 129). But Brauer became more and more involved in Schur's seminar. As a participant in this, he reported on the first part of Schur's paper " Neue Anwendungen der Integralrechnung auf Probleme der Invariantentheorie " (1924), which shows how Hurwitz's method of group integration can be used for the study of the linear representations of continuous linear groups. In the second part of this work, Schur applied his method to determine all the irreducible (continuous, finite-dimensional) representations of the real orthogonal and rotation groups. He suggested to Brauer that it might also be possible to do this in a more algebraic way. This became Brauer's doctoral thesis [**1**], for which he was awarded his Ph.D. *summa cum laude* on March 16, 1926.

On September 17, 1925 Richard Brauer married Ilse Karger, a fellow-student whom he had first met in November 1920 at Schur's lecture course on number theory. Ilse Karger was the daughter of a Berlin physician. She studied experimental physics and took her Ph.D. in 1924, but she realized during the course of her studies that she was more interested in mathematics than in physics, and she took mathematics courses with the idea of becoming a school-teacher. In fact she subsequently held instructorships in mathematics at the Universities of Toronto and Michigan and at Brandeis University, and she eventually became assistant professor at Boston University. The marriage of Ilse and Richard Brauer was a long and very happy one. Their two sons George Ulrich (born 1927) and Fred Günther (born 1932) both became active research mathematicians, and presently hold chairs at, respectively, the University of Minnesota, Minneapolis, and the University of Wisconsin, Madison.

Brauer's first academic post was at the University of Königsberg (now Kaliningrad), where he was offered an assistantship by K. Knopp. He started there in the autumn of 1925, became Privatdozent (this is the grade which confers the right to give lectures) in 1927, and remained in Königsberg until 1933. The mathematics department at that time had two chairs, occupied by G. Szegö and K. Reidemeister (Knopp left soon after Brauer arrived), with W. Rogosinski, Brauer and T. Kaluza in more junior positions. The Brauers enjoyed the friendly social life of this small department, and Richard Brauer enjoyed the varied teaching which he was required to give. During this time he also met mathematicians from other universities with whom he had common interests, particularly Emmy Noether and H. Hasse.

This was the time when Brauer made his fundamental contribution to the algebraic theory of simple algebras. In [**4**], he and Emmy Noether characterized Schur's " splitting fields " of a given irreducible representation Γ of a given finite dimensional algebra, in terms of the division algebra associated to Γ. Brauer developed in [**3**], [**5**] and [**7**] a theory of central division algebras over a given perfect field, and showed in [**13**] that the isomorphism classes of these algebras can be used to form a commutative group whose properties give great insight into the structure of simple algebras. This group became known (to its author's embarrassment!) as the " Brauer group ", and played an essential part in the proof by Brauer, Noether and Hasse [**14**] of the long-standing conjecture that every rational division algebra is cyclic over its centre.

Early in 1933 Hitler became Chancellor of the German Reich, and by the end of March had established himself as dictator. In April the new Nazi régime began to implement its notorious antisemitic policies with a series of laws designed to remove Jews from the " intellectual professions " such as the civil service, the law and teaching. All Jewish university teachers were dismissed from their posts. Later some exemptions were made—it is said at the request of Hindenburg, the aged and by now virtually powerless President of the Reich—to allow those who had held posts before the first World War, and those who had fought in that War, to retain their jobs. Richard Brauer came into neither of these categories, and was not reinstated. It is tragically well known that the " clemency " extended to those who were allowed to remain at their posts was short-lived. Alfred Brauer, whose war service exempted him from dismissal in 1933, eventually came to the United States in 1939. Their sister Alice stayed in Germany and died in an extermination camp during the second World War.

The abrupt dismissal of Jewish intellectuals in Germany in 1933 evoked shock and bewilderment abroad. Committees were set up and funds raised, particularly in Great Britain and the United States, to find places for these first refugees from Nazism. Through the agency of the Emergency Committee for the Aid of Displaced German Scholars, which had its headquarters in New York, and with the help of the Jewish community in Lexington, Kentucky, enough money was raised to offer Richard Brauer a visiting professorship for one year at the University of Kentucky. He arrived in Lexington in November 1933, speaking very little English, but already with a reputation as one of the most promising young mathematicians of his day. His arrival was greeted with sympathetic curiosity; the local paper reported an interview with the newcomer, conducted through an interpreter, and recorded Brauer's first impressions of American football. Ilse Brauer and the two children, who had stayed behind in Berlin, followed three months later. The friendly welcome which the Brauers found in Lexington, and their own adaptability, made the transition to life in the United States an easy one.

In that same academic year 1933–34 the Institute for Advanced Study at Princeton came into full operation. Among its first permanent professors was Hermann Weyl. Brauer did not know Weyl personally, but had always hoped to do so from the time when he had been writing his thesis on the rotation group; Weyl's classic papers, in which he combined the infinitesimal methods of Lie and E. Cartan with Schur's group integration method to determine the characters of all compact semisimple Lie groups, appeared in 1925–26. It was therefore the fulfilment of a dream for Brauer to be invited to spend the year 1934–35 at the Institute as Weyl's assistant. Brauer's great admiration and respect for Weyl were returned. Many years later Weyl wrote that working with Brauer had been the happiest experience of scientific collaboration which he had ever had in his life. The famous joint paper on spinors [**19**] was written during this year, and also Brauer's paper [**21**] on the Betti numbers of the classical Lie groups. Pontrjagin had recently determined these numbers by topological means (1935), and Brauer, in response to a question by Weyl, was able to give in a few weeks an alternative purely algebraic treatment based on invariant theory. The references to Brauer in Weyl's book *The Classical Groups* (1939) make evident the esteem in which he held his younger colleague. Brauer collaborated with N. Jacobson, who had been Weyl's assistant during the second half of 1933–34, in writing up notes of Weyl's lectures on Lie groups, and of some of the seminar talks which followed. These appeared under the title *The Structure and Representation of Continuous Groups* (Princeton, 1934–1935).

The year at Princeton was very productive of new mathematical contacts for Brauer. The Institute was already a brilliant centre for mathematics. Besides its permanent professors (J. W. Alexander, A. Einstein, J. von Neumann, O. Veblen and Weyl) there were in the School of Mathematics that year four assistants and thirty-four " workers " (i.e. visiting members). Among the latter were W. Magnus, C. L. Siegel and O. Zariski, all of whom were to become lifelong friends of the Brauers. Brauer's mathematical contact with Siegel was particularly close, and bore fruit later in **[52]**. In addition to the mathematicians at the Institute, the mathematics faculty at Princeton University (who were then housed in the same building) included Bochner, Lefschetz and Wedderburn. The Brauers were also able to see Emmy Noether regularly, because she was giving a weekly seminar at Princeton that year. Emmy Noether was another refugee from Nazism, and held a post as visiting professor at Bryn Mawr College, Pennsylvania, from 1933 until her death in the spring of 1935.

It was as a result of the account of him given by Emmy Noether when she visited the University of Toronto that Brauer was offered an assistant professorship there. He took up this post in the autumn of 1935, and was to remain in Toronto, holding in due course positions as associate and then full professor, until 1948. At Toronto, Brauer developed his famous modular representation theory of finite groups, which will probably always be regarded as his most original and characteristic contribution to mathematics. Some of the preliminaries to this theory appeared in 1935 in **[18]**, but the first full treatment of modular characters, decomposition numbers, Cartan invariants and blocks was published jointly with C. J. Nesbitt in 1937 (**[27]**). Nesbitt was Brauer's doctoral student at Toronto from 1935–37, and he has given this interesting account of their collaboration. " Curiously, as thesis advisor, he did not suggest much preparatory reading or literature search. Instead, we spent many hours exploring examples of the representation theory ideas that were evolving in his mind. Eventually, I pursued a few of these ideas for thesis purposes, they received some elegant polishing by him, and later were abstracted and expanded by another great friend, Tadasi Nakayama. Professor Brauer generously ascribed joint authorship to several papers that came out of these discussions but my part was more that of interested auditor."

One of these joint papers with Nesbitt " On the modular characters of groups " **[34]** appeared in 1941 and remained for many years the only readily available reference for modular theory. An essential part of this theory was a new general representation theory of algebras, initiated by Brauer and developed by him, Nesbitt and Nakayama during this period.

Brauer's teaching contribution to mathematics at Toronto was considerable; his lectures and seminars were well-attended, and he had several Ph.D. students apart from Nesbitt, including R. H. Bruck, S. A. Jennings, N. S. Mendelsohn, R. G. Stanton and R. Steinberg. Brauer was elected to the Royal Society of Canada in 1945. With his Toronto colleagues H. S. M. Coxeter and G. de B. Robinson he was involved in the Canadian Mathematical Congress and the founding of the Canadian Journal of Mathematics. During his years in Canada he kept up many contacts with the United States; he was visiting professor at the University of Wisconsin in 1941, and a visiting member of the Institute of Advanced Study in 1942. In 1942 he also spent some time with Emil Artin at Bloomington, Indiana. Brauer had met Artin briefly in Hamburg, but this was their first real mathematical and personal contact. Their discussions and correspondence over the ensuing years resulted in Brauer's famous proof **[51]** of Artin's L-function conjecture, and a series of subsequent papers relating to class-field theory, for which he received the American Mathematical Society's Cole Prize in 1949. Artin and Brauer were to remain close friends until Artin's death in 1962.

By 1948 Brauer was becoming one of the leading figures on the international mathematical scene, and it can have surprised no one when he moved back to the United States in that year, to a chair at the University of Michigan, Ann Arbor. Nesbitt was on the faculty there, but by then had moved into another area of mathematics, and the few graduate courses in algebra were being taught by R. M. Thrall, who already had considerable contact with the work of Artin, Brauer and Nakayama. Brauer at once set about enlarging the graduate programme in algebra and number theory, and he took on a big personal load of advanced lectures, seminars and Ph.D. supervision. There was no National Science Foundation to support research in those days, but many of the best international researchers were prepared to lecture at summer schools in the United States. Michigan had always had a particularly good and well-attended summer programme in mathematics, which was now enhanced by the attraction of Brauer. When Brauer was not involved in such an Ann Arbor summer, he and Ilse would take vacations at Estes Park, Colorado, where there were usually other algebraists present—for example Reinhold Baer, a former school-fellow of Brauer's in Berlin, and now at the University of Illinois at Urbana. Michigan became one of the liveliest centres of algebra, with a remarkable young generation—Ph.D. students of Brauer's included K. A. Fowler, W. Jenner and D. J. Lewis; and W. Feit, J. P. Jans and J. Walter were students while Brauer was at Michigan, although they did not take their doctorates with him. A. Rosenberg was a post-doctoral fellow at Michigan during this time, and the junior faculty included M. Auslander and J. McLaughlin.

About 1951 Brauer, together with his pupil K. A. Fowler, found the first group-theoretical characterization of the simple groups $LF(2, q)(q \geqslant 4)$. At nearly the same time, M. Suzuki in Japan had proved a similar theorem for the case $q = p$ (prime), and later introduced important simplifications in the proof of the general case with his method of " exceptional characters ". G. E. Wall, who was then at Manchester, had also arrived at Brauer's theorem independently by about 1953. The final version, a joint paper by Brauer, Suzuki and Wall [**70**], did not appear until 1959. This work, together with Brauer and Fowler's paper " On groups of even order " [**64**], marked the beginning of a new advance in the theory of finite groups. A few years later W. Feit and J. Thompson made another breakthrough with their long proof (Feit, Thompson 1963) of the old conjecture of Burnside that every non-Abelian finite simple group has even order. Most of the great progress in the understanding and classification of finite simple groups, which has dominated algebra in the past 25 years, can be traced to these pioneering achievements. Brauer was to remain a leading contributor to this progress.

The Brauers were very happy at Ann Arbor, and expected to stay there for the rest of their lives. However in 1951 Brauer was offered a chair at Harvard University, which he accepted. He took up this post in 1952, and stayed at Harvard until he retired in 1971; he and Ilse lived at Belmont, Massachusetts until his death in 1977.

Brauer was fifty-one years old when he went to Harvard. It is a striking fact of his career that he continued to produce original and deep research at a practically constant rate until the end of his life. About half of the 127 publications which he has left were written after he was fifty; the years 1964–77 produced 44 papers. The mathematical atmosphere at Harvard and at the neighbouring Massachusetts Institute of Technology was very congenial to Brauer, who had many contacts at both places. He had an impressive catalogue of successful students at Harvard, including D. M. Bloom, P. Fong, M. E. Harris, I. M. Isaacs, H. S. Leonard, J. H. Lindsey, D. S.

Passman, W. F. Reynolds, L. Solomon, D. B. Wales, H. N. Ward and W. Wong—and this list, like those which we have given of Brauer's students at Toronto and Michigan, is farf rom complete. Beside students, there were many visitors who came to Harvard because Brauer was there. The Brauers were a hospitable couple, and had always liked to entertain colleagues and students in their home. Everyone who had contact with Brauer in his years at Harvard, whether as student, colleague or visitor, has spoken of the great warmth and personal interest which he and Ilse brought to the mathematical community in the Boston area.

The Brauers travelled abroad regularly, usually to Europe where there were old friends. They visited the Baers in Frankfurt, after they had returned to Germany in 1956. They regularly spent summer vacations at Pontresina in Switzerland with C. L. Siegel, and also visited him in Göttingen—Brauer held the Gauss professorship at the Akademie der Wissenschaften there for a semester in 1964. In 1959–60 he was visiting professor at Nagoya University in Japan at the invitation of T. Nakayama, whom the Brauers had known for many years. They visited England frequently to stay with the Rogosinskis in Newcastle. Brauer was made honorary member of the London Mathematical Society in 1963, and was Hardy Lecturer in 1971. He and Ilse spent a term at Warwick in 1973, which is remembered there with great pleasure; Brauer's paper **[126]** had its origin in the seminar on modular representations which he held on this occasion. In 1972 Brauer was visiting professor at Aarhus University in Denmark.

Early in 1969 Brauer began to suffer from myasthenia gravis, a neurological disease which causes a selective weakening of the muscles, in his case the muscles of the eye. Although he could still read, this partial paralysis impaired his side vision and made him see double from beyond a certain distance. He adjusted himself with great fortitude to this distressing condition, and managed to lead an almost normal life in spite of it.

Brauer received many honours in the course of his life, and a list of these is given at another place in this notice. We mention here his election to the National Academy of Sciences in 1955, the Cole Prize of the A.M.S. in 1949 for his work on class-field theory, and the National Medal for Scientific Merit awarded to him by the President of the United States in 1971.

In 1976 Brauer became sufficiently ill to require hospital treatment on two occasions—in his own words, " For the first time in my life I have seen hospital rooms at night." He made a good recovery and continued his busy working life. But in the middle of March 1977 he had to be rushed to hospital again. He was suffering from aplastic anaemia, a condition in which the body no longer produces enough blood cells, and consequently loses its natural defences against infection. He knew that he was very ill, but did not doubt that he would recover eventually. He continued to deal with his correspondence from his hospital bed, dictating letters to Ilse, who stayed with him throughout his illness. A general sepsis led to his death on 17 April.

Richard Brauer has been one of the most consistent and effective influences in algebra this century. His work provides an example of mathematical research and scholarship at its best. He solved important problems which had been long outstanding in group representation and number theory, and in the process he made major theoretical advances which have since become incorporated into the groundwork of algebra. We shall discuss Brauer's work in more detail later, and so mention here only one example, the theory of linear associative algebras. This was enriched by Brauer in two ways: first by his theory of simple algebras, which led to the paper by him, Noether and Hasse on rational division algebras, and which was the result of Brauer's studies on the Schur index of a representation. His second contribution to the theory of algebras was his analysis of the regular representations of a non-semisimple algebra, which led to the idea of projective and injective modules, the local (p-adic) theory of orders in a semisimple algebra, and to Nakayama's researches on Frobenius algebras. This work was one of the by-products of Brauer's theory of group representations over a field of finite characteristic.

The progress of this " modular " theory of group representations shows all of Brauer's remarkable mathematical qualities at work. Frobenius and Burnside had revolutionized the theory of finite groups in the first decade of this century, and some of their deepest results were those obtained by the application of the new theory of group characters. The idea of a modular theory of group representations was not new; Dickson had already done some pioneering work in the early 1900's (Dickson 1902, 1907). Schur suggested, in lectures at Berlin, an " arithmetic " approach: a given rational prime p generates, in the integral group ring ZG of a given finite group G, an ideal whose prime divisors, in a suitable order containing ZG, correspond to the types of irreducible representations of G over a field of characteristic p. But it was Brauer who solved, one by one, the enormous technical and conceptual problems which stood between Schur's idea and a theory which could contribute to the understanding of the structure of the group G. Brauer always considered that the aim of his theory was to give information about the structure of groups; more particularly, he used modular theory as a way of obtaining refined " local " information on the *ordinary* character table of G—his beautiful theory of blocks being the principal means to this end. His judgment was brilliantly vindicated in the event, and it is hard to imagine any other contemporary algebraist with the superb creative and technical resource necessary to carry through Brauer's programme.

Brauer's many students, and others who were influenced by his teaching at an early stage in their careers, are now to be found at universities throughout the United States and Canada. To them he transmitted a fine tradition of German algebra and number theory which can be traced back through Schur to Frobenius and Kronecker. Brauer's lectures were carefully prepared and undramatic; he was very concerned to give proofs in complete detail (in contrast to the prevailing fashion), and would sometimes go back and rephrase an argument two or three times in order to make things clearer. Some students found this tedious, but there were others who came to realize that Brauer had few equals as an expositor, both of mathematical ideas and of techniques. A former student at Michigan has said of his lectures, " You had the feeling you were seeing a magnificent structure being built before your eyes by a master craftsman, brick by brick, stone by stone." Many people have expressed the hope that some of Brauer's lecture courses might eventually be published.

It is not possible to separate Richard Brauer's mathematical qualities from his personal qualities. All who knew him best were impressed by his capacity for wise and independent judgment, his stable temperament and his patience and determination in overcoming obstacles. He was the most unpretentious and modest of men, and remarkably free of self-importance. He was embarrassed to find his name attached adjectivally to some of his discoveries, and rebuked a student, gently but seriously, for referring to " Brauer algebra classes " in the theory of simple algebras—this was at Harvard, and the offending terminology had been standard for at least twenty years!

Brauer's interest in people was natural and unforced, and he treated students and colleagues alike with the same warm friendliness. In mathematical conversations, which he enjoyed, he was usually the listener. If his advice was sought, he took this as a serious responsibility, and would work hard to reach a wise and objective decision.

Richard Brauer occupied an honoured position in the mathematical community, in which the respect due to a great mathematician was only one part. He was honoured as much by those who knew him for his deep humanity, understanding and humility; these were the attributes of a great man.

Acknowledgments

In preparing these biographical and personal notes I have relied on the generous help of many people. I should like to thank particularly Professor Alfred Brauer for his recollections of his brother's early life. I have had letters and information also from R. and M. Baer, J. C. Beidleman, H. S. M. Coxeter, W. Feit, P. Fong, N. Jacobson, W. Ledermann, D. J. Lewis, G. Mackey, W. Magnus, D. Montgomery, C. J. Nesbitt, W. F. Reynolds, G. de B. Robinson, C. L. Siegel, M. Suzuki, J. Tate, G. E. Wall, W. J. Wong and O. Zariski; to all these, and to others who have given me their kind assistance, I am much indebted.

P. Fong and W. Wong have been good enough to allow me to use their bibliography of Richard Brauer's collected papers, which they have edited for publication by the Massachusetts Institute of Technology Press, in the series " Mathematicians of Our Time ". This project was begun in 1972, and was already substantially completed before Brauer's death.

Above all my thanks are due to Mrs. Ilse Brauer. She has been willing, even in the early days of her sad bereavement, to give the generous, detailed and patient help which has made it possible for me to write this notice.

Honours and honorary posts held by Richard Brauer

Elected memberships of learned societies

Royal Society of Canada 1945
American Academy of Arts and Sciences 1954
National Academy of Sciences 1955
London Mathematical Society (Honorary Member) 1963
Akademie der Wissenschaften Göttingen (Corresponding Member) 1964
American Philosophical Society 1974

Prizes, etc.

Guggenheim Memorial Fellowship 1941–42
Cole Prize (American Mathematical Society) 1949
National Medal for Scientific Merit 1971

Honorary doctorates

University of Waterloo, Ontario 1968
University of Chicago 1969
University of Notre Dame, Indiana 1974
Brandeis University, Massachusetts 1975

Presidencies of Mathematical Societies

Canadian Mathematical Congress 1957–58
American Mathematical Society 1959–60

Editorships of learned journals

Transactions of the Canadian Mathematical Congress 1943–49
American Journal of Mathematics 1944–50
Canadian Journal of Mathematics 1949–59
Duke Mathematical Journal 1951–56, 1963–69
Annals of Mathematics 1953–60
Proceedings of the Canadian Mathematical Congress 1954–57
Journal of Algebra 1964–70

Mathematical Work of Richard Brauer

This survey of Brauer's mathematical work can be no more than an outline. I have attempted to describe those of Brauer's ideas which have had the greatest influence on contemporary mathematics; I have made no attempt to review separately each of his major papers.

Some of the gaps in my account can be filled by reading W. Feit's obituary article on Brauer in the Bulletin of the American Mathematical Society.

The section on number theory is adapted from a manuscript which D. J. Lewis has kindly given me. I should also like to thank W. Feit, M. Suzuki, J. Tate and G. E. Wall for their valuable help.

Contents

1. *Representations of continuous groups*

In his thesis **[1]**, Brauer calculated the characters of the irreducible representations of the groups $D = SO(n)$ and $D' = O(n)$ (D' is the group of all real orthogonal transformations of n variables; D the subgroup of those whose determinant is 1). By a " representation " of a linear group Γ, such as D or D', is meant a continuous homomorphism $H : \Gamma \to GL(N, \mathbb{C})$, whereby each element s of Γ is represented by a non-singular complex matrix or linear transformation $H(s)$ of some finite degree N.

Schur had shown (1924 I, II, III) that his own classical treatment (1905) of the character theory of a finite group Γ can be extended to a continuous linear group Γ on which a finite invariant integral can be defined. Hurwitz (1897) had introduced invariant integrals as a method of calculating polynomial invariants, and had determined such an integral for D. Schur (1924 II) used Hurwitz's integral to give explicit formulae for the irreducible characters of D'. He suggested to Brauer that it might be possible to recover these formulae by purely algebraic methods.

Any character χ of D is given by the function which expresses $\chi(s)$ (s an arbitrary element of D) as polynomial in the eigenvalues of s. The terms in this polynomial can be ordered in a " lexical " ordering. Brauer showed that the *irreducible* characters χ are uniquely specified by their leading terms; he proved then that these irreducible characters $\gamma(k_1, \ldots, k_\nu)$ can be parameterized by integers $k_1, \ldots, k_\nu$ satisfying $k_1 \geqslant \ldots \geqslant k_\nu \geqslant 0$ (in case $n = 2\nu+1$ is odd), or $k_1 \geqslant \ldots \geqslant k_{\nu-1} \geqslant |k_\nu|$ (in case $n = 2\nu$ is even).

The problem now was to find explicit expressions for the $\gamma(k_1, \ldots, k_\nu)$. Brauer first recovered Schur's formulae for the characters of D', which are easily expressed in terms of the $\gamma(k_1, \ldots, k_\nu)$, using an ingenious induction on n, and also a theorem of E. Study (1897) which gives the polynomial invariants of D. Then Brauer found analogous formulae for the $\gamma(k_1, \ldots, k_\nu)$ (which had eluded Schur) by another inductive argument which turns on a beautiful formula for the product of characters of D.

Brauer's inductive arguments required extensive manipulation of delicate determinantal identities, which meant that the price paid, in order to avoid the analytic element in Schur's integral method, was quite heavy. While Brauer was writing his thesis, Hermann Weyl was working on his famous papers on the representations of semisimple groups (Weyl 1925, 1926), and he too found the formulae for the characters for $D = SO(n)$ (Weyl, Selecta, pp. 322, 323)—Brauer and Weyl arrived at these formulae independently, although of course both had Schur's papers as common starting point.

Weyl's work was a triumph for the analytic method. For any connected complex-analytic group Γ, whose Lie algebra $\mathfrak{g}$ is semisimple, he constructed *via* $\mathfrak{g}$ a compact real-analytic (i.e. Lie) subgroup Γ_u of Γ; the representation theory of Γ_u coincides with the analytic representation theory of Γ. Weyl constructed an invariant integral for the simply-connected covering group Γ_u^0 of Γ_u, and was able to use Schur's methods, combined with E. Cartan's classification (1913) of the representations of $\mathfrak{g}$, to give his famous formula (Selecta, p. 358) for the irreducible characters of Γ_u^0. The irreducible characters of Γ_u can be identified with a subset of those of Γ_u^0, since Γ_u can be regarded as the factor group of Γ_u^0 by a suitable central subgroup Z.

Brauer's work on the representations of semisimple groups, like all other research in this field since 1926, has to be seen against the background of Weyl's massive achievement. Brauer's contribution was that he continued to press the case for purely algebraic methods, a case to which Weyl himself was very sympathetic.

The classic joint paper [**19**] by Brauer and Weyl on spinors gives a beautifully explicit algebraic realization of the " two-valued " representation Δ of $D = SO(n)$ of dimension 2^ν ($n = 2\nu$ or $2\nu+1$, as before), whose existence had been proved by É. Cartan (1913). In accordance with Weyl's theory, Δ can also be regarded as a genuine representation of the simply-connected covering group D^0 $(= \mathrm{Spin}(n))$ of $D = D^0/Z$, whose kernel does not contain Z (Z has order 2 for $n \geqslant 3$). The construction starts by realizing $O(n)$ as a group of automorphisms of a certain 2^n-dimensional complex linear algebra which had first been studied by W. K. Clifford (1878), and then uses a matrix representation of this algebra invented by Dirac in his paper (1927) on the spin of an electron.

É. Cartan (1929) showed that the Betti number $B_p (p \geqslant 0)$ of a compact semisimple Lie group G (considered as a manifold) is equal to the dimension ν_p of the space of invariant differential forms ω of order p on G. These ω are determined by their behaviour at the identity element of G, and correspond to those elements of the pth exterior (alternating) power $E_p = \mathfrak{g}^* \wedge \ldots \wedge \mathfrak{g}^*$ ($\mathfrak{g}$ is the Lie algebra of G) which are invariant under the action of G on E_p which derives from the adjoint action of G on $\mathfrak{g}$. Thus the problem of finding the v_p—that is of calculating the Poincaré polynomial $1+v_1 t+v_2 t^2+\ldots$ of G—reduces to a problem of algebraic invariant theory. Brauer solved this problem for the classical groups (unitary, symplectic, orthogonal); an outline of the proof appears in **[21]**, and the complete proof for the unitary group is given by Weyl in his book (1946, pp. 232–238). These Poincaré polynomials had also been calculated by direct topological methods by Pontrjagin (1935). The other compact groups G, corresponding to the "exceptional" simple Lie algebras, were treated by Chevalley (1950).

[25] was Brauer's last substantial paper on continuous groups, and gives a glimpse of a general representation theory of continuous groups, based on invariant theory, and of a strictly "algebraic" nature. Unfortunately a promised sequel (**[25**, p. 858]) never appeared. Many of the ideas in **[25]** appeared, with generous acknowledgment, in Weyl's book (1939).

2. *Simple algebras and splitting fields*

Brauer's researches on simple algebras had their origin in Schur's "arithmetic" theory of irreducible groups of matrices. Let K be a fixed ground field, $\bar{K}$ an algebraically closed extension of K, and f a positive integer. We write R_f for the ring of all $f \times f$ matrices over a given ring or algebra R.

Let $\mathfrak{H}$ be an irreducible subset of $\bar{K}_f$ which is also a semigroup, i.e. $\mathfrak{H}$ is multiplicatively closed and contains the identity matrix. $\mathfrak{H}$ is said to be *rationally representable* over a field L ($K \subseteq L \subseteq \bar{K}$) if there exists some matrix $R \in GL(f, \bar{K})$ such that $R^{-1}\mathfrak{H}R \subseteq L_f$. Then L certainly contains the character χ of $\mathfrak{H}$, that is, $\chi(H) = \text{trace}(H)$ lies in L, for all H in $\mathfrak{H}$. From now on we shall assume that the ground field K contains χ, and also that $\mathfrak{H}$ is rationally representable over some L of finite degree $(L : K)$. Such a field L is called a *splitting field for* $\mathfrak{H}$ *(or for* χ*) over* K, and the minimal degree $(L\colon K)$ of all these splitting fields is Schur's *index* $m = m_K(\mathfrak{H}) = m_K(\chi)$. In two papers (1906, 1909) Schur proved the following theorems in the case $\bar{K} = \mathbb{C}$.

I. m divides f.

II. m divides the degree $(L\colon K)$ of any splitting field L.

III. If $\mathfrak{H}^{(m)}$ is the semigroup of all mf-rowed matrices

$$\begin{pmatrix} H & & & 0 \\ & H & & \\ & & \ddots & \\ 0 & & & H \end{pmatrix} \quad (H \in \mathfrak{H}),$$

then $\mathfrak{H}^{(m)}$ is rationally representable over K.

Schur's ideas are often expressed in terms of linear algebras. Our assumptions imply that the K-linear closure $A = K\mathfrak{H}$ of $\mathfrak{H}$ is a finite-dimensional central simple algebra over K ("central" or "normal" means that the centre of A contains only the scalar multiples of the identity). A given field L (we assume always $K \subseteq L \subseteq \bar{K}$ and $(L : K) < \infty$) is a splitting field for $\mathfrak{H}$, if and only if it is one for A. Moreover

$L\mathfrak{H}$ is isomorphic to $L \otimes_K A$, which is a simple algebra over L, and it follows easily that L is a splitting field for A if and only if

$$L \otimes_K A \cong L_f.$$

This condition depends only on the abstract structure of A as algebra over K; accordingly L can be described as a splitting field for this abstract algebra. Wedderburn's structure theorem (1907) says that $A \cong D_t$, where $t \geqslant 1$ is an integer, and D is a central division algebra (algebras are now assumed to be over K), which is determined up to isomorphism by A. The splitting fields for A are the same as those for D, therefore these fields are characteristic of the *algebra class* $[A]$ of A; two central simple algebras A, B are put into the same class if they determine isomorphic division algebras.

In the late 1920's Brauer and Emmy Noether, working independently and using quite different methods, showed that Schur's theorems hold in arbitrary characteristic; moreover if A has Schur index m, then $\dim_K D = m^2$, and the splitting fields L of degree $(L : K) = m$ coincide, up to isomorphism, with the maximal subfields of D. After Brauer and Noether had become aware of each other's work, Brauer was able to improve this last theorem to

IV. Every splitting field of degree mr (see II) is isomorphic to a maximal subfield of D_r. Conversely, every maximal subfield L of D_r is a splitting field, and $(L: K) = ms$ for some divisor s of r.

Brauer proved IV under the assumption that K was perfect; Noether was later able to remove this restriction. They announced this and other common discoveries in [**4**, (1927)]. Noether's proofs used her new structure theory of algebras (1929, 1933), and were based on the systematic use of representation modules. Brauer's proofs appeared in three papers ([**3**, (1926)], [**5**, (1928)], [**7**, (1929)]). They were based on his theory of *factor-sets* of separable field extensions.

Suppose $L = K(\theta)$ is separable over K, and that $\{\theta_\alpha\}_{\alpha=1,\ldots,r}$ are the conjugates of θ over K. To each central simple algebra A which has L as splitting field, Brauer associated a factor-set $(c_{\alpha\beta\gamma})_{\alpha,\beta,\gamma,=1,\ldots,r}$, whose values $c_{\alpha\beta\gamma}$ are non-zero elements of the normal closure of L over K. The $c_{\alpha\beta\gamma}$ satisfy certain "cocycle" conditions (of course, the cohomological language was not used until much later), and the set of all such "cocycles", taken modulo suitable "coboundaries", forms a multiplicative group which we will denote $H_L(K)$. The main theme in [**3, 5, 7**] is that the correspondence $A \to (c_{\alpha\beta\gamma})$ induces an isomorphism $B_L(K) \cong H_L(K)$; here $B(K)$ is the "Brauer group", whose elements are the classes $[A]$ of all central simple algebras A over K, multiplied by the rule $[A][B] = [A \otimes_K B]$, and $B_L(K)$ is the subgroup consisting of those $[A]$ for which L is a splitting field. The unit element of $B(K)$ is $[K]$, the class of all A which are isomorphic to some $K_f (f \geqslant 1)$. The group $B(K)$ did not appear explicitly until [**13**], which was concerned with Noether's non-commutative Galois theory (1933). But the results in the early papers [**3, 5, 7**] are proved by using the interplay between an algebra A and its factor sets. We mention here only one such theorem. The *exponent* l of A can be regarded as the order of $[A]$ as element of $B(K)$. Schur's theorem III can be read as $[A]^m = 1$, hence l divides m. In [**3**], Brauer showed that every prime divisor of m also divides l, by an argument which appeared later in the famous joint paper with Hasse and Noether [**14**] on central division algebras over an algebraic number field.

At the heart of Brauer's theory is a construction [5, 7] which shows how to make a central simple algebra A, with L as splitting field and having a prescribed factor-set $(c_{\alpha\beta\gamma})$. When L is a Galois (= normal and separable) extension of K, the algebra A reduces to a *crossed-product* algebra (= verschränktes Produkt; this term is due to Noether), and the factor-set $(c_{\alpha\beta\gamma})$ reduces to a *Noether factor-set* $(r_{S,T})$ indexed by the elements S, T of the Galois group $G = \mathrm{Gal}\,(L/K)$ ([15]; see also the excellent account of the Noether and Brauer theories in van der Waerden (1937)). Each factor-set $(r_{S,T})$ determines a group-extension of G by the multiplicative group L^* of L, and $H_L(K)$ can be identified with the usual cohomology group $H^2(G, L^*)$. But if it happens that all the $r_{S,T}$ are roots of unity, then one can make a *finite* group extension G_1 of G by the cyclic group generated by the $r_{S,T}$. The study of these finite extensions led Brauer to some of his deepest work on the structure of division algebras ([15], [50]). Brauer's isomorphism $H^2(G, L^*) \cong B_L(K)$, together with Hilbert's " theorem 90 " (whose cohomological formulation is $H^1(G, L^*) = 0$), has formed the basis of *Galois cohomology*, which has had a great influence in number theory—particularly through Tate's work on class-field theory (Tate, see Cassels and Fröhlich 1967)—and, more recently, in the theory of commutative rings. Azuyama (1951) and Auslander and Goldman (1960) defined a Brauer group $B(R)$ for an arbitrary commutative ring R; Auslander and Goldman gave a generalized version of the isomorphism $H^2(G, L^*) \cong B_L(K)$. A great deal of further generalization has followed—see particularly Chase, Harrison and Rosenberg (1965), and for recent literature, see the proceedings of a conference on Brauer groups held in 1975 at Evanston (Lecture Notes in Mathematics no. 549, Springer, Berlin 1976).

Schur's original problem had been to calculate the Schur index $m_K(\chi)$, over a field K of characteristic zero, of a given irreducible character χ of a given finite group G. A related problem was to find *splitting fields* for G, that is, fields K such that $m_K(\chi) = 1$ for all irreducible characters χ of G. In [47] Brauer verified a long-standing conjecture by proving

V. Let ε be a primitive $|G|$th root of unity, where $|G|$ is the order of G. Then $\mathbb{Q}(\varepsilon)$ is a splitting field for G.

The proof in [47] used modular characters. A quite different proof, and some sharper versions of V, resulted in [53] from the application of Brauer's " induction theorem "—we shall describe this below. Using the same ideas, Brauer gave in [60] a profound reduction of Schur's index problem: he showed that all the Schur indices for a given finite group G can be found, if the same can be done for all the "hyper-elementary " subgroups H of G. A group H is hyper-elementary if, for some prime p, there is a cyclic normal subgroup H_0 of H such that H/H_0 is a p-group.

Brauer first proved his induction theorem in his famous paper [51] on Artin's L-series (see p. 331). In [62] he proved the " characterization of characters ", and showed that this was equivalent to the induction theorem. Roquette (1952) gave a proof much simpler than those in [51] and [62], and this was further simplified by Brauer and Tate [63] to give the elegant proof which is now standard. None of these proofs uses modular methods, but they are all based on the idea of induction from elementary subgroups of G, and this idea appeared in Brauer's earliest paper [18] on modular representations. A finite group E is called elementary if $E = A \times B$, where A is cyclic, and B is a p-group for some prime p. We write $R(G)$ for the set of all "generalized characters" of G, i.e. integral combinations $z_1\chi_1 + \ldots z_s\chi_s$ of the irreducible characters $\chi_1, \ldots, \chi_s$ of G.

Brauer's Induction Theorem. Every character χ of G can be written as a linear combination $\chi = \sum c_i \psi_i^*$, where each c_i is an integer and ψ_i^* is the character of G induced from a linear character ψ_i of some elementary subgroup E_i of G.

The Characterization of Characters. Let θ be a complex-valued class-function on G. Then θ lies in $R(G)$ if and only if the restriction $\theta_{|E}$ lies in $R(E)$ for every elementary subgroup E of G.

These must be the most widely-quoted of all Brauer's theorems. He applied them himself to class-field theory, to the theory of Schur indices (as we have seen) and to modular and ordinary character theory. Of the many generalizations and applications made by others, we might mention particularly Swan's induction theorems for integral representations (1960), and Atiyah's paper (1961) on the connection between $R(G)$ and the integral cohomology of G. Serre (1971) gives a very good discussion of the induction theorem and of its application to character theory.

3. *Modular representations*

As early as 1902, L. E. Dickson showed that Frobenius's theory (1896) of characters of a finite group G holds in an algebraically closed field k of prime characteristic p, provided p does not divide the order $|G|$ of G. In later papers Dickson (1907a, b) considered the case where p divides $|G|$. In this case the group-algebra $A = kG$ is not semisimple. A representation $F : G \to GL(n, k)$ is in general not completely reducible, and is very imperfectly described by its natural character $\chi_F (= \text{trace } F)$. Dickson found some interesting facts about such "modular" representations, but they did not amount to a general theory.

The subject lay dormant until the middle 1930's, when Brauer laid the foundations of his modular representation theory in three fundamental papers **[18]**, **[27]**, **[28]**; the two last were written jointly with C. Nesbitt. **[27]**, a short memoir published by the University of Toronto Press, contains in 21 pages all the main ingredients of the mature theory; the proofs are complete, except for some important theorems on the regular representations of algebras which were announced in **[28]** and proved by Nesbitt in his thesis (Nesbitt 1938). Nakayama (1938) gave alternative proofs for some of the theorems in **[27]** and **[28]**. Subsequent accounts of modular theory appeared in **[34]**, **[65]** and **[73]**.

Let G_0 denote the set of all p'-elements of G (i.e. elements whose order is prime to p). A conjugacy class of G is called a p'-class (or p-regular class) if it lies in G_0. The "modular character" ϕ_F of a representation $F : G \to GL(n, k)$ (since known as the "Brauer character") is a complex-valued class-function on G_0—it is a kind of "complexified" version of the natural trace function χ_F. It was defined in **[27]**. If $F_1, \ldots, F_l$ is a full set of irreducible modular representations, their Brauer characters $\phi_1, \ldots, \phi_l$ are linearly independent. For any modular representation F, one has $\phi_F = \sum n_i(F) \phi_i$, where $n_i(F)$ is the multiplicity with which F_i appears as a composition factor in F. This was used in **[27]** to prove

I. The number l of irreducible modular representations F_i of G, is equal to the number of p'-classes of G.

Brauer had already proved this beautiful theorem in **[18]** in a different way. For a third proof, see **[65]**.

The most important and useful feature of modular theory is that it relates "ordinary" (characteristic zero) representations to modular ones. Let K be a field of characteristic zero, which is a splitting field for G. Let R be a subring of K having K as quotient; we assume that R is a principal ideal domain, and that it has a prime ideal $\mathfrak{p}$ containing p. Identify $\bar{R} = R/\mathfrak{p}$ with a subfield of k. Any ordinary character χ of G can be realized by a representation $X: g \to X(g)$ by matrices $X(g)$ all of whose coefficients lie in R. Taking these mod $\mathfrak{p}$, we get a modular representation $\bar{X}: g \to \overline{X(g)}$ of G. The equivalence class of $\bar{X}$ is not uniquely determined by χ, but its Brauer character is, and in a very simple way: $\phi_{\bar{X}}$ is just the restriction to G_0 of χ. Therefore there hold equations

$$\chi_\sigma(g) = \sum_{i=1}^{l} d_{\sigma i}\phi_i(g) \quad (\sigma = 1, \ldots, s;\ g \in G_0) \tag{1}$$

with non-negative integer coefficients $d_{\sigma i}$. The $d_{\sigma i}$ are the *decomposition numbers* for G with respect to p.

$\{F_1, \ldots, F_l\}$ can be put into natural bijective correspondence with a full set $\{U_1, \ldots, U_l\}$ of inequivalent indecomposable summands of the regular representation of $A = kG$—this follows from one of the "new wave" of theorems on algebras announced in [**28**]. If ξ_i is the Brauer character of U_i we have

$$\xi_i = \sum_{j=1}^{l} c_{ij}\phi_j \quad (i = 1, \ldots, l), \tag{2}$$

where the $c_{ij} = n_j(U_i)$ are the Cartan invariants for kG. Cartan invariants are defined for any algebra A (Cartan 1898), but in case $A = kG$ they are related to the decomposition numbers by

$$c_{ij} = \sum_{\sigma=1}^{s} d_{\sigma i}d_{\sigma j} \quad (i, j = 1, \ldots, l). \tag{3}$$

Formula (3) was proved in [**27**] using a determinant of Frobenius. Nakayama (1938) and Brauer [**31**] gave another proof, based on the fact that if K is a complete discrete valuation ring and R its ring of valuation integers, then each U_i can be "lifted" to a representation $\hat{U}_i$ over R. $\hat{U}_i$ has an ordinary character, η_i say, whose restriction to G_0 is ξ_i. It can be proved that

$$\eta_i = \sum_{\sigma=1}^{s} d_{\sigma i}\chi_\sigma \quad (i = 1, \ldots, l), \tag{4}$$

and then (3) follows by applying (1) and (2).

Modular character relations for the ϕ_i and η_i can be found by applying Frobenius's ordinary character relations to these formulae (1)–(4). They have striking consequences, for example that $\eta_i(1) = \dim U_i$ is divisible by the order p^a of a Sylow p-subgroup of G, and $\eta_i(g) = 0$ for any g in $G - G_0$. Another consequence is that the Cartan matrix $C = (c_{ij})$ is non-singular ([**27**], [**34**]). Brauer proved in [**33**] a much deeper theorem, namely

II. det C is a power of p.

This theorem was important in later applications of modular theory. A relatively elementary proof, based on the characterization of characters (see p. 327) appeared in [**62**].

In [**32**] Brauer announced the first applications of modular theory to the structure of finite groups. The main theorem was

III. Let G be a finite subgroup of $GL(n, \mathbb{C})$ whose order is divisible by a prime p, but not by p^2. If $n < \frac{1}{2}(p-1)$, then G has a normal Sylow p-subgroup.

(Feit and Thompson (1961) removed the restriction $p^2 \nmid |G|$; however their proof still required III.) The proof of III was extremely original. It appeared in **[40]**, at the end of a series of papers **[37]**, **[38]**, **[39]** which set out some fundamental new theory and techniques for group characters. In this work modular theory is used mainly to give information about ordinary characters: the objective is usually to apply some version of the elementary criterion of Frobenius, that an element $g(\neq 1)$ of a finite group G lies in a proper normal subgroup of G, if $\chi_\sigma(1) = \chi_\sigma(g)$ for some $\chi_\sigma(\neq 1_G)$. If g is a p'-element, formulae (1) hold out some hope of calculating $\chi_\sigma(g)$ if the irreducible modular characters ϕ_i of G are known. Unfortunately it is usually much harder to find the ϕ_i than to find the χ_σ, but Brauer changed the situation by extending formulae (1), so that they applied to *all* elements g of G. Every g in G has a unique expression $g = \pi v = v\pi$, where v is a p'-element, and π is a p-element of G (i.e. the order of π is a power of p). Fix π, and let $\{\phi_i^{(\pi)}\}$ be the irreducible modular characters of the centralizer $C_G(\pi)$ of π. It was shown in **[37]** that there hold equations

$$\chi_\sigma(\pi v) = \sum_i d_{\sigma i}^{(\pi)} \phi_i^{(\pi)}(v) \qquad (\sigma = 1, \ldots, s;\ v \in C_G(\pi)_0). \tag{5}$$

The *generalized decomposition numbers* $d_{\sigma i}^{(\pi)}$ are certain algebraic integers, independent of v. In case $\pi = 1$, $d_{\sigma i}^{(\pi)} = d_{\sigma i}$ and (5) reduce to (1). Formulae (5) give a chance of calculating $\chi_\sigma(g)$ when $g = \pi v$ is "p-singular", for then $\pi \neq 1$ and $C_G(\pi)$ may be a subgroup of G whose modular theory is accessible. Information about the $d_{\sigma i}^{(\pi)}$ comes from formulae which generalize (3). But to extract precise results from this method Brauer had to use the theory of *blocks*.

Blocks were defined in **[27]**, and their study occupies a large part of Brauer's works. After he had used block theory to prove theorems such as III, Brauer continued for the rest of his life to develop both theory and applications in numerous papers—we might mention for example a series which appeared in the Journal of Algebra **[85]**, **[86]**, **[92]**, **[112]**, **[121]**. Blocks are most easily defined by taking a decomposition $1 = e_1 + \ldots + e_t$ of 1 into primitive idempotents e_τ of the centre $Z(kG)$ of kG. This can be "lifted", uniquely, to a similar decomposition $1 = \hat{e}_1 + \ldots + \hat{e}_t$ in $Z(RG)$. We say that an ordinary (or modular) irreducible character ψ of G *belongs to the p-block B_τ of G*, if $\hat{e}_\tau$ (or e_τ) is not represented by zero in a representation corresponding to ψ. By this rule both the sets $\{\chi_1, \ldots, \chi_s\}$ and $\{\phi_1, \ldots, \phi_l\}$ are partitioned among the t $(p-)$ blocks $B_1, \ldots, B_t$ of G.

With each block B_τ is associated a conjugacy class of p-subgroups of G called the *defect groups* of B_τ (**[48]**). If p^d is the order of a defect group, d is the *defect* of B_τ. The advantage of working within a given block B_τ, is that the number s_τ of ordinary irreducible characters in B_τ is bounded, by a bound depending only on p^d. Brauer gave one such bound in **[49]**, and later he and Feit **[72]** proved

IV. $s_\tau \leqslant \frac{1}{4}p^{2d} + 1$.

A conjecture $s_\tau \leqslant p^d$(**[49]**) is still unresolved, except for small d.

Let D be a fixed p-subgroup of G, and H a subgroup of G such that $D.C_G(D) \leqslant H \leqslant N_G(D)$. For each block b of H can be defined a block $B = b^G$ of G (this is rather like the construction of an induced character). The "first main theorem" of block theory is as follows.

V. $b \to b^G$ defines a bijective map between the set of all blocks b of $H = N_G(D)$ which have defect group D, and the set of all blocks B of G which have defect group D.

This was announced in **[43]**, **[48]**; the proof appeared (10 years later) in **[65]** (see also Osima (1955)). The "second main theorem" (below) was announced in **[49]**; the proof appeared (this time after 13 years!) in **[73]**. Later Nagao (1963) gave a simpler proof.

VI. Suppose that χ_σ belongs to a block B of G. Then the decomposition numbers $d_{\sigma i}{}^{(\pi)}$ in (5) are zero, except for those $\phi_i{}^{(\pi)}$ which belong to blocks b of $H = C_G(\pi)$ for which $b^G = B$.

Brauer was sometimes able to amass an astonishing amount of detailed facts about the ordinary characters in a block B of G, given only very scanty information about B, such as the structure of a defect group D, and perhaps also the structure of $N_G(D)$, $C_G(D)$. The best case, not surprisingly, was where D had order p. This was treated in the key paper **[38]** (in which the famous "Brauer tree" made its appearance). Dade (1966) generalized this to the case of an arbitrary cyclic defect group. **[39]** is a rich mine of techniques, based on the results of **[38]**, for calculating characters of a group G which has a Sylow subgroup of order p— these techniques are much used in constructing character tables. Perhaps the most beautiful application involving a non-cyclic defect group, is the proof in **[86]** of the theorem below. This theorem was first announced by Brauer and Suzuki in **[74]**; Glauberman subsequently (1974) gave a proof not using modular theory.

VII. Let G be a finite group with $O_{2'}(G) = 1$, whose Sylow 2-subgroups are quaternion groups. Then the centre of G has order 2.

4. *Number Theory*†

The most significant contribution of Brauer to number theory was his work on the Artin L-functions and the consequences which followed. Heilbronn always held this to be a magnificent monumental piece of work, which clearly demonstrated the need of number theorists to be aware of the developments in modern algebra and to be prepared to use them.

Papers **[51]**, **[52]**, **[58]**, **[59]**, **[66]**, **[119]** are concerned with Artin L-functions, Dirichlet L-functions, zeta-functions and related matters. Let K be a Galois extension of an algebraic number field F. Let M be a complex matrix representation of the Galois group G of K over F, and let χ be the character of M. Let $[(K/F)/\mathfrak{P}]$ denote the Frobenius automorphism associated with an unramified prime $\mathfrak{P}$ of K, and let $\mathfrak{p}$ be the prime of F under $\mathfrak{P}$. The Artin L-series is defined as follows

$$L(s, \chi, K/F) = \prod_{\mathfrak{p}} \frac{1}{\det\left(I - M\left(\left[\frac{K/F}{\mathfrak{P}}\right]\right) N\mathfrak{p}^{-s}\right)}.$$

(The product is over *all* primes $\mathfrak{p}$ of F; the factors on the right require suitable interpretation for ramified $\mathfrak{p}$.) Artin (1924, 1931) had proved the following facts.

I. If χ is a linear combination $\sum c_\nu \phi_\nu$ of characters ϕ_ν with rational coefficients c_ν, then $L(s, \chi, K/F) = \prod L(s, \phi_\nu, K/F)^{c_\nu}$. Moreover every character χ of G is expressible as a rational combination of characters ϕ_ν which are induced by cyclic subgroups of G (this latter fact is "Artin's induction theorem").

II. Let Ω be a subfield of K containing F, and let H be the subgroup of G fixing Ω.

† This section is based on a manuscript by D. J. Lewis; I have also incorporated some remarks by J. Tate.

If ψ is a character on H and ψ^* the corresponding induced character of G, then $L(s, \psi^*, K/F) = L(s, \psi, K/\Omega)$.

III. If N is the kernel of the representation M which affords the character χ, and if Φ is the subfield of K fixed by N, then $L(s, \chi, K/F) = L(s, \chi, \Phi/F)$, where on the right we view χ as a character of G/N.

IV. If K is abelian over F and if χ is an irreducible character, then $L(s, \chi, K/F)$ coincides with some Dirichlet L-series of K/F. It follows from results of Hecke that in this case $L(s, \chi, K/F)$ is a meromorphic function satisfying a certain functional equation.

It followed immediately from I, II and IV that for any χ, and whether the extension K/F is abelian or not, $L(s, \chi, K/F)$ can be continued analytically over the whole complex plane and that a suitable integral power $L(s, \chi, K/F)^m$ is meromorphic. Moreover $L(s, \chi, K/F)$ again satisfies a functional equation, as in IV. But since m could be greater than 1, this does not show that $L(s, \chi, K/F)$ is single-valued. Artin's conjecture that $L(s, \chi, K/F)$ is in fact single-valued was proved by Brauer in **[51]**. Brauer's proof was an immediate consequence of his induction theorem (see p. 327).

Brauer's proof of this conjecture of Artin represented a decisive step forward in the generalization of class-field theory to non-abelian fields—one of the most difficult problems and certainly one of great importance in modern number theory. At the Princeton Bicentennial Conference in 1946, after Brauer had given an exposition of his result, Artin stated " My own belief is that we know already—though no one will believe me—that whatever can be said about non-abelian class field theory follows from what we know now, since it depends on the behaviour of the broad field over the intermediate fields, and there are sufficiently many abelian cases. Our difficulty is not in the proofs, but in learning what to prove." Despite this guarded optimism, progress along these lines has not been great. Today most efforts relative to non-abelian class field theory are via automorphic functions, as represented by the work of Langlands, Shimura and Weil. Put simply, what they try to do is to show that Artin (and other) L-functions are Mellin transforms of automorphic forms.

In **[52]** Brauer used his induction theorem, along with earlier techniques of Artin, to give a new proof of the following theorem of Aramata (1933).

V. If K is a finite normal extension of an algebraic number field k, then $\zeta(s, K)/\zeta(s, k)$ is an entire function.

This enabled him to prove the following conjecture of Siegel:

VI. Consider all algebraic number fields of a fixed degree n. If k is such a field, let d be its discriminant, h its class number, and R its regulator. Then

$$\log(hR) \sim \log\sqrt{(|d|)} \quad \text{as} \quad |d| \to \infty. \tag{*}$$

In **[58]**, Brauer showed that (*) holds for every sequence of normal fields over $\mathbb{Q}$ for which $n/\log|d| \to \infty$. (Note, here $n = (k : \mathbb{Q})$ is no longer fixed.) In 1949 Brauer received the American Mathematical Society's Cole Prize for his work on Artin L-functions, specifically for papers **[51]** and **[52]**. In **[59]** Brauer used the results of **[51]** to get relations between the class number of an algebraic number field and the class numbers of its subfields. In **[66]**, in collaboration with N. C. Ankeny and S. Chowla, he used the results of **[51]** to show that there exist infinitely many number fields K such that $h(K) > |d(K)|^{\frac{1}{2}-\varepsilon}$. Landau (1918) had shown that $h(K) < c|d(K)|^{\frac{1}{2}+\varepsilon}$ for all K, and Brauer's result shows this bound to be quite sharp.

There are five other papers which are number theoretic: **[2]**, **[17]**, **[20]**, **[45]**, and **[87]**. **[2]** (with A. Brauer and H. Hopf) and **[20]** (with A. Brauer) deal with a problem on irreducibility of polynomials suggested by Schur. Paper **[17]** concerns the Klein form problem for a finite group of linear transformations or collineations. Brauer related this to his theory of factor-sets for simple algebras. Paper **[87]** is concerned with positive definite quadratic forms $Q = \sum c_{ij} x_i x_j$ with integral coefficients c_{ij}, particularly those where (c_{ij}) is the Cartan matrix of a p-block B of a finite group (see p. 329). Brauer showed that if B has defect d, then there is an equivalent form Q^* to Q, whose coefficients g_{ij} satisfy $|g_{ij}| \leqslant (\frac{9}{4})^{p^{2d-1}} p^{2d}$.

In [45] Brauer considers a system of homogeneous equations

$$f_i(x_1, \ldots, x_n) = 0 \qquad (i = 1, \ldots, h), \tag{1}$$

of degrees $r_1, \ldots, r_h$, over a field K. Generally, such a system will not have a non-trivial solution in K, but if $n > h$ it will have such a solution in some finite extension L of K. One question is whether L can be a soluble extension of K of not too large a degree. As Brauer indicated the answer is yes, if n exceeds some constant depending on the r_i. The actual theorem proved in full detail in **[45]** is

VII. Assume that K has the property

(D) For every integer $r > 0$, there exists an integer $\psi(r)$ such that for $n \geqslant \psi(r)$ every equation $a_1 x_1^r + \ldots + a_n x_n^r = 0$ with coefficients a_i in K has a non-trivial solution in K.

Then there is a function $\Omega(r_1, \ldots, r_h, m)$ such that if $n \geqslant \Omega(r_1, \ldots, r_h, m)$ every system (1) has an m-dimensional linear manifold of solutions defined over K.

Since p-adic fields $K_\mathfrak{p}$ have property (D), it follows that a projective algebraic variety defined over $K_\mathfrak{p}$, lying in an ambient space of dimension n, has a $K_\mathfrak{p}$-rational point provided n is sufficiently large compared to the degree of the variety.

Paper **[45]** motivated much work in the ensuing two decades by Birch, Davenport, Lewis and their students on rational points on algebraic varieties in large ambient spaces. Of particular importance in early work in this area was the diagonalization process, although that was later subsumed in statements on geometric obstructions. Perhaps the prettiest result along these lines was the paper by B. J. Birch (1957), where he used the diagonalization technique to show that a system of forms of odd degrees over $\mathbb{Q}$ in sufficiently many variables is also soluble in $\mathbb{Q}$.

5. *Simple Groups*

In 1954, at the International Congress of Mathematicians in Amsterdam, Brauer announced (**[68]**) some results which he had obtained with K. A. Fowler (**[64]**) on the structure of finite groups of even order, and proposed a programme which has had a great influence on the study of finite simple groups.

The underlying idea was surprisingly elementary. If G is a finite group of even order $|G|$, and if $K_1, \ldots, K_k$ are its conjugacy classes, then some of theses classes K_i consist of involutions, i.e. elements of order 2. Let M be the union of these classes of involutions. Let $[S]$ denote the sum, in the complex group algebra $\mathbb{C}G$, of the elements of a given subset S of G; thus $[K_1], \ldots, [K_k]$ form a basis of the centre of $\mathbb{C}G$. We have an equation

$$[M]^2 = \sum_{i=1}^{k} c_i[K_i], \tag{1}$$

where the c_i are non-negative integers—in fact, c_i is the number of pairs (x, y) of elements x, y of M, such that xy equals a given element g_i of K_i. Because the group generated by two involutions x, y is very easily described (it is a dihedral group), it is possible to give upper bounds for c_i in terms of the centralizer $C_G(g_i)$ of g_i in G. Applying these estimates to (1), Brauer and Fowler proved among other things the theorem

I. Let G be a simple group of even order, and let x be an involution in G. Then G has a proper subgroup of index $<\frac{1}{2}n(n+1)$, where $n = |C_G(x)|$. Hence $|G| < (\frac{1}{2}n(n+1))!$.

This implies that there is only a finite number of isomorphism types of simple groups G which contain an involution x such that $C_G(x)$ is isomorphic to a given abstract group H. This gives encouragement for Brauer's programme: given a group H with an involution x in its centre, to find all groups G (particularly simple ones) containing H as a subgroup, such that $H = C_G(x)$. The natural choice for H is to take the centralizer of an involution in some known simple group. This programme, with its variants, has been enormously successful. It has led to papers by dozens of authors giving characterizations of known simple groups, and it has led to the discovery of new simple groups. After Feit and Thompson (1963) had proved that every group of odd order is soluble, it was known that Brauer's ideas were available for all non-abelian simple groups. The tremendous progress in finite group theory in the past 25 years, which has brought within sight the classification of all finite simple groups, owes a great deal to the techniques which Brauer developed for the study of groups through their involutions.

Many of these techniques were first published in a joint paper with Suzuki and Wall **[70]**, which contains the proof of the following theorem (first announced in **[68]**).

II. Let G be a finite group of even order, with $G = G'$, and satisfying the condition (C) If A, B are two cyclic subgroups of G of even order, and if $A \cap B \neq \{1\}$, then there exists a cyclic subgroup Z of G which contains both A and B.

Then $G \cong PSL(2, q)$ for some prime-power $q \geqslant 4$.

The proof starts by showing that the Sylow 2-subgroups of G must be either (A) dihedral, or (B) elementary abelian; the same general methods apply in both cases, but the details are easier in case (B). The next step is to assemble information about the centralizer H of an involution in G, and about the conjugacy classes of G which meet H. Suzuki's powerful method of " exceptional characters " gives the values, at all elements of H except 1, of the irreducible characters $\chi_1, \ldots, \chi_s$ of G; it also gives congruences for the degrees $f_\sigma = \chi_\sigma(1)$. The " class relation " (1) is now used in several ways: first to calculate the f_σ exactly; then, in conjuction with a classical formula which expresses the coefficients c_i in terms of the χ_σ, it gives the value of $|G|$. A similar procedure also gives the orders of the centralizers of elements which are not conjugate to elements of H. A study of these centralizers reveals that G has a subgroup N of index $q+1$ (q a prime power which is odd in case (A), and even in case (B)). Finally a theorem of Zassenhaus (1936) is used to identify the action of G as permutation group on the cosets of N, with the action of $PSL(2, q)$ on the projective line of $q+1$ elements. The proof in [**70**] makes no use of modular methods, although in many other applications the involution techniques are combined with block theory—a beautiful example is the proof in [**86**] of the Brauer–Suzuki theorem which we have already mentioned in section 3 (VII), p 330.

It is worth saying something about the history of this important paper [**70**]†. In his thesis M. Suzuki (1951) had characterized the groups PSL$(2, p)$ (p prime) in terms of their subgroup structure, using a method which he later developed into the method of "exceptional characters". Suzuki conjectured II., and gave a proof for case (B); he sent his results to Brauer, asking for his comments. In his reply (dated April 1951) Brauer, while warmly encouraging Suzuki to publish his work, said that he already had a proof of II., and enclosed some notes. This proof was long, and used the block theory of groups with dihedral Sylow 2-subgroups. Brauer also explained that K. A. Fowler had a characterization of the groups PSL $(2, 2^a)$ (i.e. case (B)) which was intended for his Ph.D. thesis. As soon as Suzuki had read Brauer's manuscript, he saw how to make his own (non-modular) methods work in the general case, and he then had a proof of II. very close to that in [**70**].

At about the same time, and quite independently of both Brauer and Suzuki, G. E. Wall found another characterization of the PSL $(2, 2^a)$, closely related to that given by II. (case (B)). Wall started from a paper by Rédei (1950) which used the "involution counting" method to characterize the alternating group A_5. He combined this method with his own arguments using characters, to produce a proof very similar to that of case (B) in [**70**]. Wall submitted this for publication by the London Mathematical Society in May 1952. By an unfortunate error of judgment, the paper was rejected. Wall continued nevertheless to work on generalizations of this theorem. He became aware of Brauer's interest in these questions through a footnote in Suzuki's 1951 paper, and he sent Brauer (in 1953 or 1954) an account of a theorem, rather more general than II., which included a characterization of the groups PGL $(2, q)$ (q odd). Brauer acknowledged in his 1954 Congress lecture [**68**] the independent work of Suzuki and Wall. There followed a long delay—probably due to nothing more significant than that Brauer was very busy with other things—and [**70**] appeared finally in 1959.

We shall not attempt a survey of Brauer's long and productive "late period" from 1960–1977. He produced many important results on simple groups during this time, and introduced deep and subtle refinements to his modular methods. W. Feit (1978) has given an account of some of these papers, and we would refer the reader to his article.

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