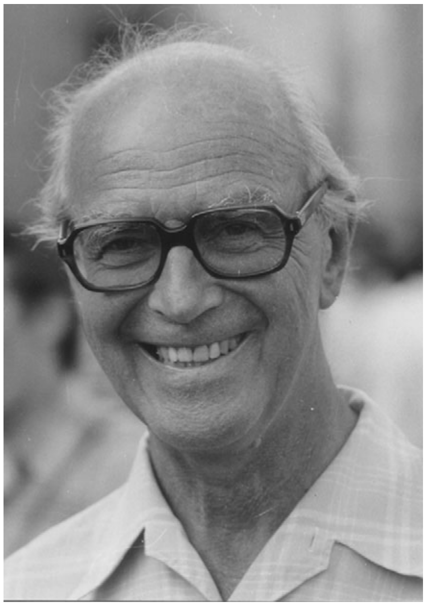
Bull. London Math. Soc. 42 (2010) 341–370
doi:10.1112/blms/bdq001

OBITUARY

Gustave Choquet, 1915–2006



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1. *Life* [131, 137, 155], ⟨12, 29, 38, 49, 50, 53, 54, 60, 61⟩

Gustave Choquet, who was the Hardy Lecturer of the London Mathematical Society in 1969 and who was elected an Honorary Member of the Society in 1988, died on 14 November 2006, aged 91. He will be remembered not only for his fundamental contributions to mathematics, but also as a truly inspiring teacher. His outstanding talents were allied to great kindness and deep humanity, and he won the respect and the warm affection of his many pupils and of scientific colleagues worldwide.

His work in functional analysis and potential theory profoundly marked the development of mathematical analysis in the second half of the twentieth century. In particular he created the theory of capacities, as well as that of integral representations in convex sets. He was a person of great mathematical learning, and his research and books brought him international fame. In addition, he was passionately concerned to promote good mathematical teaching, and was for many years president of an international commission for its improvement.

Gustave Alfred Arthur Choquet was born on 1 March 1915 in Solesmes, near Valenciennes in the Nord département of France. His parents and grandparents were all from the Valenciennes region. He was the second of three children, and the only son, of Gustave Choquet and Marie Choquet (née Fosse).

His father had distinguished himself at primary school when taking his Certificat d'Études by winning the top prize of the département, but had then chosen to leave school in order to work. Since July 1914, however, he had been at the front, and so his wife was obliged to look after the young Gustave and his sister during the hardships of the war. They were in due course evacuated with great difficulty, via Switzerland and Paris, to the Vendée. Details of the family's peaceful life in a tiny hamlet there remained among the son's earliest memories. After the war the family were briefly at Valenciennes, and then moved a short distance to the village of Saultain, where his father became Chief Accountant of a firm making coarse textile products.

His father had a sense of duty and seriousness about work which made a lasting impression on his son, but he also took great joy in playing the clarinet in the local band. Encouraged by his father, the young Gustave learned to play the flute, and was able to join in concerts and festivals in the local villages. Another occupation shared with his father was the cultivation of the family garden, from which he derived a lifelong love of gardening. His mother contributed great human warmth to the life of the family. She had a taste for music, flowers, and poetry, and herself wrote short poems.

At Saultain, there were all the normal pastimes of a country boy, but at his primary school he encountered a remarkable master, M. Flamant, who knew how to communicate by example his own curiosity and enthusiasm for all kinds of scientific experiments. M. Flamant also taught his pupils to find simple geometrical interpretations of problems in arithmetic, and the search for geometrical approaches to mathematical problems was to become an enduring predilection for Choquet.

Secondary education followed, at the lycée of Valenciennes. He had a number of excellent teachers, among them his mathematics master, M. Mas, whose mathematical style he found very congenial and who again encouraged the habit of looking for geometrical approaches to mathematical problems. Choquet derived particular satisfaction at school from solving difficult problems in geometry. Another source of great fascination was an elementary calculus book lent by a classmate, which he studied avidly. Looking back, he thought it could well have been at this time that he decided to make mathematics, if possible, a lifelong pursuit.

In 1933 Choquet took part in the national school mathematics contest (the concours général) and won the first prize. He then left Valenciennes for Paris to prepare at the lycée Saint-Louis for entry to the grandes écoles. There he immediately joined the preparatory class (the 'taupe'), having been excused the initial stage (the 'hypotaupe') because of his performance in the concours général. At the entrance examination for the École Normale Supérieure (ENS) there were approximately 400 candidates for twenty science places. Choquet was admitted. He studied there during 1934 to 1938, and was ranked first in the mathematical sciences agrégation in 1937. His year was particularly brilliant, and included three future members of the Académie des Sciences (in addition to Choquet, Laurent Schwartz and Blanclapierre) and one future corresponding member (Félici).

His time at the ENS afforded all that he could have wished for in opportunities for learning. His interest in differential geometry was deepened by the elegant lectures of Garnier and, especially, those of Darmois, and by reading E. Cartan's book on Riemannian geometry. He had earlier acquired a taste for differential geometry when reading Darboux on the theory of surfaces. However, what finally diverted him from any thought of a career in differential geometry was the discovery in the ENS library of Baire's *Leçons sur les fonctions discontinues* and the French translation of Cantor's *Beiträge zur Begründung der transfiniten Mengenlehre*. In 1933 he had already studied Borel's *Leçons sur la théorie des fonctions*, but these two works were a revelation to him and helped set him in the path he was to follow. He gave an account of some of this reading in a seminar for fellow-students organized jointly by himself and Laurent Schwartz.

After the agrégation, he took the advice of Darmois to study Hobson's *Theory of functions of a real variable* and Carathéodory's *Vorlesungen über reelle Funktionen* and then to approach Denjoy. He found in Denjoy the ideal mentor, whose mathematical interests and outlook accorded well with his own, and with whom he could fruitfully discuss both his own current research as well as that of Denjoy. He never dreamt of asking for a research topic, nor did Denjoy ever seek to impose one, but by 1938 he had enough original material for three notes, which were published as [**1–3**].

He spent the year 1938–39 at Princeton, where new mathematical perspectives opened to him. He profited particularly from the courses of Church, and from the work of Gödel , Turing, and Kleene. A Jane Eliza Proctor bursary entitled him to stay for two years; but in 1939 the outbreak of war obliged him to give up the intended second year. Instead, he was called up and sent, in the company of a hundred or so former pupils of the grandes écoles, to a camp at Biscarosse for anti-aircraft instruction. His critical attitude to the proposed means of defence earned him an early posting to a horse artillery unit, the 7^e R.A.D., and in 1940 he took part in the Battle of France, first in Lorraine, then from the banks of the Aisne to the banks of the Creuse. In August 1940 he was demobilized at Limoges and shortly afterwards returned to Paris with his fiancée. They were married in January 1941.

During the war the couple lived frugally in a little two-room apartment in Paris, where their two sons were born. In the years 1941–46 his research was supported by a C.N.R.S. stipend. The amount was modest, but he was allowed complete freedom to pursue his research as he saw fit, and this proved to be an extremely productive time for him. By 1945 he had published 30 or so articles, but had never presented himself for a doctorate. Nevertheless, in that year he learnt of a one- to two-year teaching position at the French Institute in Cracow, for which a doctorate was a necessary condition. He had for some time read extensively in *Fundamenta Mathematicae*, and had come to look upon Poland as a veritable mathematical paradise. He therefore set to work on a doctoral thesis, which he completed in three months. He duly obtained the Docteur ès Sciences Mathématiques degree as well as the appointment to the teaching post in Cracow. He and his family accordingly moved to Poland in 1946. Although some famous mathematicians had died, he found Polish mathematical life flourishing, despite the political situation, and much to his taste. He visited the principal universities and made many valuable contacts, notably with Sierpiński, Kuratowski, Steinhaus, and Nikodým. In 1947 he left Poland, with some regret, to become maître de conférences at Grenoble, where he spent the next two years. Here commenced his collaboration with Marcel Brelot, and here, in 1948, a daughter was added to the family. After Grenoble he returned to Paris as maître de conférences in the university, becoming professor in 1952.

Although Bourbaki volumes had been coming out since 1939 and van der Waerden's *Moderne Algebra* had appeared in 1930, such developments had at that epoch made little or no impression on the great majority of French university mathematics courses, which were overdue for modernization. A case in point was the Paris deuxième cycle course in differential and integral calculus, which for decades had been given in the spirit of the elementary parts of Goursat's *Traité d'analyse*. At that time the course was given by Valiron, but it became necessary to find a replacement because of Valiron's failing health. Henri Cartan, who had given pioneering modern courses of analysis as early as 1939 in Strasbourg, and 1940 in Clermont-Ferrand, on this occasion could not help directly because he was at the ENS; but on his suggestion Choquet was chosen. He duly took charge in 1954, and at once carried out a root-and-branch reform of the entire course. It now started with set theory, followed by the algebra of groups, rings, and fields, the construction of the real and complex numbers, linear algebra, then topological spaces, normed vector spaces, and Hilbert space. Finally, there was a resolutely new account of differential equations that made free use of the foregoing fundamental notions of functional analysis. (See § 2.11 and [**97**, **173**] for more particulars.) When the extent of the changes became apparent, there was widespread consternation. The dismay of the students who were being required to take the course a second time was especially acute. Nevertheless, Choquet soon had new colleagues to support him: in 1955 the university appointed Chevalley, Dixmier, Ehresmann, Godement, Pisot, and Zamansky. The provincial universities were quick to follow the lead given by Paris, and within three years all had renovated their courses in the spirit of the Paris reforms. Choquet's revolution was thus the origin of a renewal of undergraduate mathematical teaching throughout France (see ⟨**50**⟩ for a fuller account).

The year 1955–56 saw another chapter in the collaboration, begun in 1944, between Choquet and Deny. They had been independently invited to the Institute for Advanced Study in Princeton, where they found themselves sharing the study vacated at his recent death by Einstein. They discovered that they had arrived at similar ideas about future research in potential theory, which they now proceeded to develop in a series of joint articles.

Choquet was a co-founder in 1957 of the *Séminaire Brelot–Choquet: Théorie du potentiel*, which met once a week and published its proceedings; the following year it became the *Séminaire Brelot–Choquet–Deny*. In the early 1960s Choquet founded the *Séminaire Choquet: Initiation à l'analyse* which also met weekly and published its proceedings. These two seminars became an important and internationally recognized focus for research and advanced exposition. The circle of researchers around Choquet acquired official status as the *Équipe d'analyse* in 1972. The Équipe (recently renamed *Équipe d'analyse fonctionnelle*) and the Séminaire Choquet are still going concerns today.

From 1960 Choquet held, in addition to his university post, first a maîtrise de conférences at the École Polytechnique, and then, during 1965–69, a professorship there. After the events of 1968, the University of Paris was divided into thirteen units and it was also decided that posts at the École and the University could no longer be held jointly. Choquet chose to stay at the University, where he remained, now in Paris VI, until he moved to a post in Orsay a few years before his retirement in 1984. Over the years he made many extended visits to foreign universities, in particular to Kansas, Cornell, Seattle, Berkeley, Maryland, Utah, and Princeton, and to the Institute for Advanced Study. He also paid many shorter visits to other countries, including England, Australia, Japan, Korea, and China. In retirement he became emeritus professor in the Universities of Paris VI and XI.

Choquet's research spanned more than half a century and continued well into his retirement. His work manifested his desire for 'une vision directe et géométrique des problèmes', and often achieved great elegance. His preference was for problems that could be reformulated in a broadened setting and whose solution would give rise to new concepts of very wide application. His solution of the capacitability problem, to mention just one example, is a fine illustration of this. He found in potential theory an inexhaustible source of ideas, and in this connection he counted as a piece of exceptional good fortune the opportunity of collaborating with Brelot and Deny. He also made original contributions to many other areas of mathematics: general topology, real analysis, measure theory, functional analysis (especially integral representation theory), topological dynamics, and the axiomatization of geometry. Mathematical entities named after him include Choquet capacities, Choquet theory, the Choquet integral, the Choquet expected utility, the Choquet boundary, and Choquet simplexes. He was an outstanding supervisor of research, and many of his students have gone on to distinguished academic careers. Two of his most brilliant students (Brézis and Talagrand) are now members of the Académie des Sciences.

Choquet was not only the creator of a very large and important body of mathematical work, but also a truly exceptional teacher at all levels, whose courses inspired the enthusiasm of many. His engaging personality and extraordinary talent won him the respect and affection of generations of students. (Several former students have written lengthy tributes to his teaching — see, for instance, ⟨**29**, **30**, **49**, **50**, **60**, **61**⟩.) The fundamental innovations of his 1954–55 undergraduate course and their influence have been noted above. His presidency 1950–58 of the International Commission for the Study and Improvement of the Teaching of Mathematics (the Gattegno Commission) and the illuminating expository style of his books [**97**, **98**, **111–113**, **117**, **173**] both testify to his enduring concern for excellence in teaching.

His concerns about teaching and the mentoring of research students coloured his view of the Bourbaki project. Although he was never a member of the Bourbaki group, he was, like most mathematicians of his generation, profoundly influenced by their work. Nevertheless, he had serious reservations about Bourbaki as a resource for young mathematicians. He noted that many of the most valuable things in Bourbaki were to be found in the exercises, where they could easily be overlooked. He observed, moreover, that the very polished but unmotivated exposition in the main text of the Bourbaki volumes conveyed little idea of the creative processes of mathematical research.

During his career Choquet received many honours. The Académie des Sciences awarded him four prizes: the Houllevigne (1945), the Dickson (1951), the Carrière (1956), and the Grand Prix des sciences mathématiques (1968); he was elected a member of the Académie in 1976. In 1966 he was made Chevalier of the Légion d'Honneur, later becoming Officier. He was a corresponding member of the Bavarian Academy, and in 2002 was made Honorary Doctor of Science by the Charles University of Prague. His honorary membership of the LMS has already been noted above.

His leisure pursuits included gardening, mountain walking, and swimming, and he was devoted to his children. His first marriage was dissolved, and in 1961 he married a fellow mathematician, Madame Yvonne Bruhat, with whom he was to enjoy a happy partnership for the rest of his life. (Madame Choquet-Bruhat has herself had a very distinguished career, and is the first woman to have been elected a member of the Académie des Sciences). He is also survived by the five children from his two marriages.

2. *Mathematical work*

2.1. *Early work and Thesis* [**1**–**5**, **10**–**14**, **17**, **22**, **23**, **26**, **30**, **31**, **33**, **167**]

Most of Choquet's earliest work was published in the form of short Notes, usually without proofs, in the *Comptes Rendus*. The number and variety of his early publications do not allow them all to be summarized here, but a few indications can be given.

In [**2**] Choquet rediscovered Borůvka's (1926) algorithm for constructing a minimum spanning tree in a connected graph having edge-weights that are all different. Later, in 1951, it was rediscovered again by five other mathematicians. In [**26**] it is shown that if a graph is isomorphic to its complement, then the cardinality of its vertex set is either infinite or of the form $4n$ or $4n + 1$.

In [**1**, **3**, **4**] various questions about homeomorphisms between plane sets of points are studied. For example, there is a characterization of the compact subsets H of $\mathbb{R}^2$ for which every homeomorphism of H onto another compact subset of $\mathbb{R}^2$ can be extended to the whole of $\mathbb{R}^2$. In [**1**] a point P of a closed subset F of $\mathbb{R}^2$ is said to be *accessible* if there exists a simple Jordan arc γ such that $\gamma \cap F = \{P\}$. This notion is applied in [**1**] to the study of homeomorphisms between curves that divide the plane into two regions, and in [**10**] to the study of homeomorphisms between plane domains. The topology of plane sets of various kinds is further studied in [**11**, **12**], together with that of certain conformal transformations. In [**23**] it is proved that, if ϕ is a homeomorphism of a Jordan curve γ onto a closed convex curve Γ, then ϕ has a harmonic extension that is a homeomorphism of the bounded domains determined by γ and Γ. The note [**5**] is a study of the fixed points of certain classes of transformations of a plane continuum into itself.

The long article [**17**] contains many results about the finite subsets of metric spaces, of which only a few simple specimens can be given here. For example, if a compact metric space E is not homeomorphic to a closed subset of [0,1] and $0 < \theta \leqslant \pi/3$, then E contains infinitely many isosceles triplets of the vertex angle θ. The *curvature* of a triplet in a metric space E is defined as the sum of the two smaller angles in the associated triangle. At a cluster point P, the space E is said to be *flat* if the curvatures of the triplets approach zero as their points approach

P, and we say simply that E is flat if this is so at every cluster point. Flat metric spaces are investigated, and it is shown, for instance, that every compact connected flat metric space is a simple closed curve or is homeomorphic to a compact interval of $\mathbb{R}$. In the same spirit a notion of *semiflat* metric space is also defined and investigated.

A well-known theorem states that if S is a C^2 surface all of whose points are umbilics, then S is part of a sphere or of a plane. Analysing this result from the standpoint of Bouligand's direct infinitesimal geometry ⟨**13**⟩, Choquet [**13**] is able to reach the same conclusion under considerably weakened hypotheses.

Choquet's thesis [**30**] (summarized in [**33**]) is a study of the differentiability properties of subsets of Euclidean spaces, and is a pioneering contribution to non-smooth analysis which reveals profound relations between certain differentiable and topological structures. The best known result of this work is his solution to a famous problem of Lebesgue, namely that of finding a characterization of functions that are derivatives. Recall that if a real function f on $[0,1]$ is a derivative, then (i) f is of the first Baire class and (ii) f has the Darboux property, that is to say, it maps each subinterval of $[0,1]$ onto an interval. The converse is trivially false, but Choquet shows that if f is a real function on $[0,1]$ that has the properties (i) and (ii), then, for some increasing homeomorphism α of $[0,1]$, the function $f\circ\alpha$ is a derivative. Another problem of Lebesgue concerned the derivative $g = df/d\alpha$, where f and α are given real continuous functions on $[0,1]$. The problem is to determine f, given α and g, and is solved in [**14**, **30**] by an extension of Denjoy's totalization. A further topic studied in [**30**] (see also [**22**]) is that of the differentiable parametrization of curves and varieties. For example, Choquet solved the problem, proposed by Fréchet, of characterizing in Euclidean space those parametrized curves $[0,1]\ni t\mapsto f(t)$ that admit an equivalent parametrization $t\mapsto f(\alpha(t))$ (where α is a homeomorphism of $[0,1]$) that has a non-null derivative at every point.

The high point of the thesis is the *contingent–paratingent* theorem, which contains many earlier results. The general statement is quite abstract, and an initial idea of the theorem can perhaps best be obtained by looking at an important special case. Let X be a real separable Banach space, let E and F be subsets of X, and let $x\in X$. Then the *paratingent cone* $P_{E,F}(x)$ of E at x relative to F is defined by Choquet as follows:

$$\begin{aligned}P_{E,F}(x) &= \limsup_{\substack{(y,h)\to(x,0+)\\ y\in F}} h^{-1}(E-y)\\ &= \bigcap_{\substack{\delta>0\\ \eta>0}} \overline{\bigcup\{\,h^{-1}(E-y) : h\in(0,\delta),\ y\in B(x,\eta)\cap F,\ y\neq x\,\}}.\end{aligned}$$

If $F=\{x\}$, then this definition produces the Bouligand *contingent cone* $T_E(x)$ of E at X. That is, we have $T_E(x)=P_{E,\{x\}}(x)$. The Bouligand paratingent cone $P_E(x)$ of E at x is given by $P_E(x)=P_{E,E}(x)$.

A proposition $p(x)$ defined for each point x of a subspace S of X is said to be *generically* true in S if the set $\{x\in S:\neg p(x)\}$ is of the first Baire category in S. The fundamental 'contingent–paratingent' theorem of Choquet states that, for every pair E,F of subsets of X, the equation $T_E(x)=P_{E,F}(x)$, where $x\in F$, is generically true in F. (This formulation is to be found in Shi ⟨**55**⟩, though it is also a corollary of a more abstract version given earlier by Choquet in [**31**].)

The contingent–paratingent theorem has many applications. For example, one corollary states that, for a continuous real function f defined on an open subset of X, the following assertions are generically equivalent: (i) f is subdifferentially regular, (ii) f is strictly differentiable, (iii) f is regularly Gâteaux differentiable, and (iv) f is regularly Dini differentiable (see ⟨**55**⟩ and [**167**]). Another application noted in [**167**] is a direct proof of the following proposition. Suppose that F is a closed subset of a C^1 manifold M and that there exists a group of C^1 diffeomorphisms of M that leaves F invariant and acts transitively on F. Then F is a C^1 submanifold of M.

2.2. *Theory of capacities* [**35**, **37**, **39–44**, **63**, **65**, **67**, **71**, **131**, **160**]

Choquet's interest in the theory of capacities arose from the capacitability problem for Newtonian potential. In $\mathbb{R}^3$ the potential of a positive Radon measure μ at the point x is $U^\mu(x) = \int \|x-y\|^{-1}\,d\mu(y)$, and the Newtonian or electrostatic capacity $\mathrm{cap}(K)$ of a compact subset K of $\mathbb{R}^3$ is, by a formula of de la Vallée Poussin, the supremum of the total masses of the positive Radon measures μ on K that satisfy $U^\mu(x) \leqslant 1$ for all $x \in \mathbb{R}^3$. The inner capacity $\mathrm{cap}_*(X)$ of an arbitrary subset X of $\mathbb{R}^3$ is defined to be the supremum of the capacities of the compact sets contained in X; the outer capacity $\mathrm{cap}^*(X)$ of X is then defined as the infimum of the inner capacities of the open sets in $\mathbb{R}^3$ that contain X. A subset X of $\mathbb{R}^3$ is said to be *capacitable* if $\mathrm{cap}_*(X) = \mathrm{cap}^*(X)$. For example, compact sets are capacitable. In the late 1940s Brelot and H. Cartan remarked that it was still an important open question whether all Borel sets in $\mathbb{R}^3$ are capacitable. This *capacitability problem* seized the imagination of Choquet. To isolate the essential features of the problem he generalized it, replacing $\mathbb{R}^3$ by a Hausdorff topological space E, and the function $K \mapsto \mathrm{cap}(K)$ by an isotone map $\gamma : \mathcal{K}(E) \to \mathbb{R}_+$, where $\mathcal{K}(E)$ denotes the set of all compact subsets of E. He also assumes that γ is continuous on the right in the following sense: for each $K \in \mathcal{K}(E)$ and $a \in \mathbb{R}$ such that $\gamma(K) < a$ there exists an open set $U \supseteq K$ such that $\gamma(L) < a$ for all $L \in \mathcal{K}(E)$ such that $K \subseteq L \subseteq U$. A function $\gamma : \mathcal{K}(E) \to \mathbb{R}_+$ satisfying these conditions will be termed a *precapacity*. For each $X \subseteq E$ he now defines the inner γ-capacity $\gamma_*(X)$ of X by

$$\gamma_*(X) = \sup\{\, \gamma(K) : K \in \mathcal{K}(E),\ K \subseteq X \,\}$$

and the outer γ-capacity $\gamma^*(X)$ by

$$\gamma^*(X) = \inf\{\, \gamma_*(G) : G \in \mathcal{O}(E),\ X \subseteq G \,\},$$

where $\mathcal{O}(E)$ denotes the set of all open subsets of E. If γ is a precapacity, then γ^* has the properties Γ_1 and Γ_3 displayed below. After much experimentation Choquet found that if γ is also strongly subadditive, in other words if

$$\gamma(K \cup L) + \gamma(K \cap L) \leqslant \gamma(K) + \gamma(L)$$

for all $K, L \in \mathcal{K}(E)$, then γ^* satisfies property Γ_2. The three properties of γ^* to which we have referred are:

Γ_1. the map $\gamma^* : \mathfrak{P}(E) \to [-\infty, \infty]$ is isotone;
Γ_2. if (A_n) is an increasing sequence of subsets of E with union A, then $\gamma^*(A) = \sup \gamma^*(A_n)$;
Γ_3. if (K_n) is a decreasing sequence in $\mathcal{K}(E)$ with intersection K, then $\gamma^*(K) = \inf \gamma^*(K_n)$.

A function γ^* satisfying these conditions is known as a *Choquet (outer) capacity*.

Given a Choquet (outer) capacity γ^*, we say that a subset X of E is *capacitable* (with respect to γ^*) if $\gamma^*(X) = \gamma_*(X)$, where

$$\gamma_*(X) = \sup\{\, \gamma^*(K) : \mathcal{K} \ni K \subseteq X \,\},$$

and $\mathcal{K} = \mathcal{K}(E)$. Compact sets are capacitable, and it is easy to show that so are $\mathcal{K}_\sigma$-sets. Choquet defines the $\mathcal{K}$-Borel sets as those belonging to the monotone class $m(\mathcal{K})$ generated by $\mathcal{K}$. Mindful of his problem concerning Newtonian capacity, Choquet decided to investigate the question whether all $\mathcal{K}$-Borel sets are capacitable. An important step in this direction was his proof that $\mathcal{K}_{\sigma\delta}$-sets are capacitable. Seeking a more comprehensive result, Choquet was led to generalize the notion of an analytic set by defining a *$\mathcal{K}$-analytic set* in E to be any subset of E that is a continuous image of a $\mathcal{K}_{\sigma\delta}$ subset of a compact Hausdorff space. He defines $\mathcal{K}$-Souslin sets in E as those produced by applying Souslin's operation (A) to elements of $\mathcal{K}$, and he proves that a subset of E is $\mathcal{K}$-Souslin if and only if it is $\mathcal{K}$-analytic and a subset of some $\mathcal{K}_\sigma$ set. He also proves that all $\mathcal{K}$-Borel sets are $\mathcal{K}$-Souslin. Suppose then that X is a $\mathcal{K}$-Souslin subset of E. Choquet constructs an auxiliary Hausdorff space F, together with a continuous surjection $\phi : F \to E$ and a $\mathcal{K}_{\sigma\delta}$ subset Y of F such that $\phi(Y) = X$. Then $g^* = \gamma^* \circ \phi$ is a Choquet capacity on F and, consequently, the $\mathcal{K}_{\sigma\delta}$ set Y is g^*-capacitable. It follows that $X = \phi(Y)$ is γ^*-capacitable. This argument concludes the proof that all $\mathcal{K}$-Souslin sets, and *a fortiori* all $\mathcal{K}$-Borel sets, are capacitable.

It follows that to prove that Borel sets in $\mathbb{R}^3$ are capacitable with respect to Newtonian potential it is enough to show, when $E = \mathbb{R}^3$ and $\gamma^* = \text{cap}^*$, that the axioms Γ_1–Γ_3 are satisfied. The only serious difficulty concerns axiom Γ_2, for which the strong subadditivity of $\text{cap} : \mathcal{K} \to \mathbb{R}_+$ is a sufficient condition. The latter was not a known property of Newtonian potential, but Choquet duly demonstrated it, and hence concluded his proof that all Borel sets in $\mathbb{R}^3$ are capacitable. Thus, in the course of solving the capacitability problem for Newtonian potential, he created a whole new theory of capacities of great generality. He remarked later that he owed his success with the capacitability problem to the fact that, as a non-specialist in potential theory, he was not encumbered with preconceptions. In addition to its direct applications in potential theory and measure theory, the capacitability theorem has been used in probability theory to establish the measurability and stopping-time property of certain hitting times for a wide class of Markov processes (see ⟨**9**⟩). (The early evolution of the above theory was reported in a series of notes [**37**, **39**–**43**], which were followed by the magisterial article [**44**]; see also [**131**, **160**]. For capacitability in logarithmic potential theory see [**63**].)

By suitably modifying the axioms Γ_1–Γ_3, Choquet [**71**] devised in 1959 a version of capacity theory for spaces without topology. We suppose given a non-empty set E, together with a family $\mathcal{H}$ of subsets of E that is stable with respect to finite unions and countable intersections, and such that $\emptyset \in \mathcal{H}$. Then an *outer $\mathcal{H}$-capacity* (nowadays often called simply a (*Choquet*) *$\mathcal{H}$-capacity*) is defined as an isotone map $c : \mathfrak{P}(E) \to [-\infty, \infty]$ satisfying the following conditions.

C_1. For all $K \in \mathcal{H}$ we have $c(H) < \infty$.

C_2. If (A_n) is an increasing sequence of subsets of E with union A, then $c(A) = \sup c(A_n)$.

C_3. If (H_n) is a decreasing sequence of elements of $\mathcal{H}$ with intersection H, then $c(H) = \inf c(H_n)$.

A set $A \subseteq E$ is now said to be $(c, \mathcal{H})$-capacitable if

$$c(A) = \sup\{\, c(H) : H \in \mathcal{H}, H \subseteq A \,\}.$$

The $\mathcal{H}$-Souslin sets are defined as those produced by applying Souslin's operation (A) to elements of $\mathcal{H}$, and Choquet proves that, under the above conditions C_1–C_3, every $\mathcal{H}$-Souslin set is $(c, \mathcal{H})$-capacitable. Dellacherie ⟨**19**, p. 43⟩ has pointed out that Davies ⟨**18**⟩, working quite independently on a problem about Hausdorff measures, established in essence this non-topological capacitability theorem in 1952 at the moment when Choquet's theory of capacities in topological spaces was just taking definitive shape (see the Notes [**37**, **40**–**43**]). However, the argument of Davies was exclusively directed to the solution of his problem about Hausdorff measures and, as a result, its applicability in abstract capacity theory was for a long time overlooked.

Choquet's work in capacity theory led him to important discoveries in descriptive set theory [**35**, **44**], and also saw the start of his theory of integral representations in convex sets [**44**]. In his early work on capacities Choquet, as noted above, discovered that, for a strongly subadditive precapacity γ, all $\mathcal{K}_{\sigma\delta}$-sets are capacitable. It followed at once that $\mathcal{K}_{\sigma\delta\sigma}$-sets are capacitable, but he found himself unable to proceed to $\mathcal{K}_{\sigma\delta\sigma\delta}$-sets. His tentative conclusion was that further progress could depend on going beyond strong subadditivity to new inequalities. With this in mind he defines in [**35**], by transfinite induction, a class $\mathcal{K}_\alpha$ of $\mathcal{K}$-Borel sets for each countable ordinal α as follows: (i) let $\mathcal{K}_0 = \mathcal{K}$, (ii) for odd α let $\mathcal{K}_\alpha$ consist of all sets that are countable unions of sets extracted from $\bigcup_{\beta<\alpha} \mathcal{K}_\beta$, and (iii) for even $\alpha > 0$ let $\mathcal{K}_\alpha$ consist of all sets that are countable intersections of sets extracted from $\bigcup_{\beta<\alpha}\mathcal{K}_\beta$. Then, for instance, $\mathcal{K}_1=\mathcal{K}_\sigma$, $\mathcal{K}_2=\mathcal{K}_{\sigma\delta}$, $\mathcal{K}_3=\mathcal{K}_{\sigma\delta\sigma}$, and so on. Moreover, he proves that $m(\mathcal{K})=\bigcup_{\alpha<\omega_1}\mathcal{K}_\alpha$. His hope was now to find, for each $\alpha<\omega_1$, sufficient conditions for $\mathcal{K}_\alpha$-sets to be capacitable. In order to formulate his putative new inequalities, he makes the following definitions, in which $X, A_1, A_2, \dots$ are arbitrary compact subsets of E:

$$\Delta_1(X;A_1)=\gamma(X\cup A_1)-\gamma(X),$$

and, for all $n\geqslant 1$,

$$\Delta_{n+1}(X;A_1,\dots,A_{n+1})=\Delta_n(X\cup A_{n+1};A_1,\dots,A_n)-\Delta_n(X;A_1,\dots,A_n).$$

The statement that γ is increasing and strongly subadditive can now be expressed as

$$\Delta_1\geqslant 0 \quad\text{and}\quad \Delta_2\leqslant 0.$$

Choquet conjectured that for Newtonian capacity (that is, for $\gamma=\text{cap}$) we have $(-1)^{n+1}\Delta_n\geqslant 0$ for all n. To his immense gratification, he succeeded in proving this [**42**, **44**], confiding later that this discovery was the occasion of the greatest emotion of his scientific career. However, the hope that the new inequalities would lead to further progress with the capacitablility problem was disappointed. Choquet showed, for instance, that classes in the hierarchy $(\mathcal{K}_\alpha)$ having infinite index, such as $\mathcal{K}_\omega$ and $\mathcal{K}_{\omega+1}$, could never be reached by recurrence arguments based on these inequalities, and hence that a fresh approach was called for. This we have already described above. Fortunately, the new inequalities were to bear fruit in a quite unexpected way. Choquet was reminded by them of the inequalities $(-1)^n f^{(n)}\geqslant 0$ that define the class of completely monotone functions on $(0,\infty)$. Recalling Bernstein's theorem that every such f satisfying $f(0+)=1$ has a unique representation of the form

$$f(x)=\int_0^\infty e^{-xt}\,d\mu(t),$$

where μ is a probability Radon measure on $[0,\infty)$, led Choquet to conjecture, and then to prove, that an analogous representation exists for capacities γ that satisfy $(-1)^{n+1}\Delta_n\geqslant 0$ for all n. Denote the set of such capacities, termed *alternating capacities of infinite order*, on E by $\mathfrak{A}_\infty$. Suppose that the space E is compact Hausdorff, and let $\mathcal{K}=\mathcal{K}(E)$ have the Vietoris topology, noting that $\mathcal{K}$ is also then compact Hausdorff. Let $B=\{\,\gamma\in\mathfrak{A}_\infty:\gamma(E)=1\,\}$, and for each $A\in\mathcal{K}$ let $\gamma_A:\mathcal{K}\to\{0,1\}$ be defined by

$$\gamma_A(K)=\begin{cases}1 & \text{if } K\cap A\neq\emptyset,\\ 0 & \text{if } K\cap A=\emptyset.\end{cases}$$

Then, for a certain topology, $\mathfrak{A}_\infty$ is a cone having B as compact convex base, and Choquet proves, in the notation of § 2.3, that $\partial_e B=\{\,\gamma_A : A\in\mathcal{K}\,\}$. Invoking the Krein–Milman theorem, he shows also that, if $\gamma\in B$, then there exists a $\mu\in\mathcal{M}^1_+(\mathcal{K})$ such that, in an appropriate sense, we have

$$\gamma=\int_{A\in\mathcal{K}}\gamma_A\,d\mu(A).$$

(That μ is uniquely determined here was later proved by Revuz.) This result was accompanied in [**44**] by similar representations for several other classes of set-functions, including *monotone set-functions of infinite order*, namely, those $\gamma:\mathcal{K}\to\mathbb{R}$ such that $\Delta_n\geqslant 0$ for all n. In these results we see the first steps in Choquet's great theory of integral representations in convex sets; but certain results of this part of [**44**] are significant in other ways too. For example, the above representation for an element γ of $\mathfrak{A}_\infty$ implies that $\gamma(K)=\int_{A\in\mathcal{K}}\gamma_A(K)\,d\mu(A)$ for $K\in\mathcal{K}$. In other words, we have

$$\gamma(K)=\mu(\{\,A\in\mathcal{K}: A\cap K\neq\emptyset\,\}).$$

Thus in the probability space $(\mathcal{K}, \mathcal{B}(\mathcal{K}), \mu)$, where $\mathcal{B}(\mathcal{K})$ is the σ-algebra of Borel subsets of $\mathcal{K}$, the quantity $\gamma(K)$ is the probability of the event that the random element A meets K. Thus we have here a statement about *random sets*, the study of which has subsequently developed into a large subject in which Choquet's work in capacity theory plays an indispensable role (see ⟨**39**, **43**⟩). Another product of these investigations is the *Choquet integral*, defined as follows. Let $\mathcal{F}$ be a family of subsets of a non-empty set X with $\emptyset \in \mathcal{F}$ and suppose that $\mu : \mathcal{F} \to \mathbb{R}_+$ is an isotone function such that $\mu(\emptyset) = 0$. Suppose given also a function $f : X \to \mathbb{R}_+$ such that $\{ x : f(x) \geqslant t \} \in \mathcal{F}$ for all $t > 0$. Then the Choquet integral of f with respect to μ is defined by

$$\int f\, d\mu = \int_{0+}^{\infty-} \mu(\{ x : f(x) \geqslant t \})\, dt \in [0, \infty],$$

where the integral on the right is an improper Riemann integral. In the special case where $(X, \mathcal{F}, \mu)$ is a finite measure space and $f \in L^1_+(\mu)$, it is an instance of Cavalieri's principle that the Choquet integral agrees with the usual integral; but, in general, the scope of the Choquet integral is far wider, since there is no additivity assumption concerning μ. Choquet [**44**], for instance, uses it to integrate with respect to a capacity, and his integral has become a basic tool in non-additive integration theory. In econometrics the *Choquet expected utility* is defined by means of the Choquet integral.

For further information see, in addition to the writings of Choquet already cited, ⟨**6**, **9**, **14**, **16**, **19–21**, **34**, **36**, **41**, **51**, **57**, **58**⟩.

2.3. *Integral representation theory* [**44**, **49–51**, **54**, **74**, **76**, **78**, **80–83**, **85–88**, **90**, **93**, **95**, **105**, **106**, **112**, **114**, **116**, **121**, **130**, **138**, **154**, **157–159**]

In what follows we denote by E a locally convex Hausdorff topological real vector space, by E' its dual, and by X a non-empty compact convex subset of E. The set of extreme points of X is known as the *extreme boundary* of X and is denoted by $\partial_e X$. Given a subset Y of E, we denote by $\operatorname{conv} Y$ and $\overline{\operatorname{conv}}\, Y$, respectively, the convex hull and the closed convex hull of Y. The celebrated Krein–Milman theorem states that $X = \overline{\operatorname{conv}}\, \partial_e X$. By a Hahn–Banach argument this is equivalent to the statement that every functional $f \in E'$ attains the value $\max\{ f(x) : x \in X \}$ at some point of $\partial_e X$. The latter assertion has been sharpened by Bauer ⟨**4**, **5**⟩, who proves that every upper semicontinuous convex function $g : X \to [-\infty, \infty)$ attains the value $\max\{ g(x) : x \in X \}$ at some point of $\partial_e X$.

Now consider the set $\mathcal{M}^1_+(X)$ of probability Radon measures on X. For each $\mu \in \mathcal{M}^1_+(X)$ there exists a point b_μ of X, the *barycentre* or *resultant* of μ, such that $f(b_\mu) = \int_X f(x)\, d\mu(x)$ for all $f \in E'$, and as shorthand for the preceding equation we write $b_\mu = \int_X x\, d\mu(x)$. Given a point $x \in X$ and a measure $\mu \in \mathcal{M}^1_+(X)$, we call μ a *representing measure* for x if $x = \int_X y\, d\mu(y)$, and we denote by M_x the set of all representing measures for x. Clearly $\varepsilon_x \in M_x$, and we have $M_x = \{\varepsilon_x\}$ if and only if $x \in \partial_e X$. By the Krein–Milman theorem $\operatorname{conv} \partial_e X$ is a dense subset of X, and hence every point of X can be approximated by points of X that possess (discrete) representing measures μ such that $\mu(\partial_e X) = 1$. A principal aim of Choquet theory for X is to improve on this approximation property by showing that, for every $x \in X$, there exists a representing measure μ which is in some appropriate sense carried by $\partial_e X$. If X is metrizable, the latter stipulation amounts to requiring that $\mu(\partial_e X) = 1$. If X is not metrizable, a satisfactory statement of our requirement that μ be carried by $\partial_e X$ is more difficult to formulate, because $\partial_e X$ may fail to be μ-measurable.

One simple case can be dealt with at once. The set $\mathcal{M}^1_+(X)$ is convex and is compact with respect to the weak* topology $\sigma(\mathcal{M}^1_+(X), \mathcal{C}(X))$, and the barycentre map $\mathcal{M}^1_+(X) \ni \mu \mapsto b_\mu$ is affine and continuous. It follows that

$$\{ b_\mu : \mu \in \mathcal{M}^1_+(\overline{\partial_e X}) \} = X,$$

where we have identified $\mathcal{M}_+^1(\overline{\partial_e X})$ with $\{\, \mu \in \mathcal{M}_+^1(X) : \operatorname{supp}\mu \subseteq \overline{\partial_e X}\,\}$. This shows that, for each $x \in X$, there exists a representing measure μ satisfying $\mu(\overline{\partial_e X}) = 1$. Thus, if $\partial_e X$ is a closed subset of X, then this elementary result supplies the representing measures we are looking for. With the aid of this result one can construct, for instance, a proof of the Herglotz representation $u(x) = \int_{\partial\mathbb{U}} P_y(x)\, d\mu(y)$ for a positive harmonic function u on the open unit ball $\mathbb{U}$ of $\mathbb{R}^n$ satisfying $u(0) = 1$, where $P_y(x)$ is the Poisson kernel and μ a probability measure on the boundary $\partial\mathbb{U}$ of $\mathbb{U}$ (see ⟨**46**⟩). Nevertheless, if $\partial_e X$ is not closed, a representing measure on $\overline{\partial_e X}$ is not in general helpful, because the latter set can be very big: we may even have $\overline{\partial_e X} = X$.

Next suppose for the moment that X is metrizable. Then Choquet's integral representation theorem [**49**–**51**] tells us that $\partial_e X$ is a $\mathcal{G}_\delta$ set and that every point of X has a representing measure μ such that $\mu(\partial_e X) = 1$. In addition to this representation theorem, Choquet [**50**, **51**] also obtained a uniqueness theorem. To state this we need the notion of a *cone* in E, by which we shall understand a non-empty subset C of E such that $\lambda C \subseteq C$ for all $\lambda \geqslant 0$. By a generator of C we shall mean any set of the form $\mathbb{R}_+ x$, where $0 \neq x \in C$. A subset B of a cone C is termed a base of C if $0 \notin B$ and each generator of C meets B in exactly one point. We may suppose that our compact convex set X is the base of a convex cone C in E. In the subspace $C - C$ of E the specific order is defined by declaring that $x \leqslant y$ if and only if $y - x \in C$. We say that X is a (Choquet) simplex if the subspace $C - C$ of E is a vector lattice with respect to its specific order. Choquet's uniqueness theorem for a metrizable compact convex set X states that in order that each point of X have exactly one representing measure μ satisfying $\mu(\partial_e X) = 1$ it is necessary and sufficient that X be a Choquet simplex.

After Choquet had obtained his existence and uniqueness theorems for metrizable X, the subject underwent a succession of improvements. Hervé ⟨**32**⟩ and Bonsall ⟨**11**⟩ found extremely simple proofs of the existence theorem. Bishop and de Leeuw ⟨**8**⟩ obtained an existence theorem for non-metrizable X by studying the maximal measures with respect to a certain order relation in $\mathcal{M}_+^1(X)$. Bauer ⟨**4**⟩ studied boundaries for function spaces and threw light on the uniqueness question. Mokobodzki ⟨**42**⟩ characterized maximal measures. Choquet and Meyer [**78**, **82**, **93**] and ⟨**40**⟩ unified these developments, made further improvements, and extended the uniqueness theorem to non-metrizable X. Their paper [**93**] is the definitive account of the existence and uniqueness theorems for general compact convex X. These can be very briefly summarized as follows.

Denote by $S = S(X)$ the convex cone of real continuous convex functions on X. For each $f \in \mathcal{C}(X)$ we denote by $\hat{f}$ the concave upper semicontinuous function on X defined by

$$\hat{f} = \inf\{\, g : f \leqslant g \in -S \,\}.$$

The map $f \mapsto \hat{f}$ is isotone and sublinear. For each $f \in S$ we define the *border set* B_f by

$$B_f = \{\, x \in X : f(x) = \hat{f}(x) \,\}.$$

Each border set is a $\mathcal{G}_\delta$, and moreover

$$\partial_e X = \bigcap_{f \in \mathcal{S}} B_f.$$

Choquet [**78**], modifying a construction in ⟨**8**⟩, defines an order relation in $\mathcal{M}_+^1(X)$ by writing $\mu \preceq \nu$ whenever $\mu(f) \leqslant \nu(f)$ for all $f \in S$. Then, for all $x \in X$, we have $\varepsilon_x \preceq \mu$ if and only if $\mu \in M_x$; and $b_\mu = b_\nu$ whenever $\mu \preceq \nu$. The relation $\preceq$ is an inductive partial order, and hence every $\mu \in \mathcal{M}_+^1(X)$ is majorized by a maximal element of $\mathcal{M}_+^1(X)$. Applying this to each Dirac measure ε_a, where $a \in X$, we obtain the integral representation theorem of Choquet, Bishop, and de Leeuw for general compact convex X: every point $a \in X$ has a representing measure μ that is a maximal element of $\mathcal{M}_+^1(X)$. Moreover, μ is maximal if and only if $\mu(B_f) = 1$ for all $f \in S$. The Choquet–Meyer uniqueness theorem states that in order that each point of X should have exactly one representing measure μ satisfying $\mu(B_f) = 1$ for all $f \in S$ it is necessary and sufficient that X should be a Choquet simplex.

A sufficient condition for a measure $\mu \in \mathcal{M}_+^1(X)$ to be maximal is that we have $\mu(G) = 1$ for each open set G that contains $\partial_e X$. The converse statement is in general untrue. A necessary condition for the maximality of μ is that we have $\mu(A) = 1$ for each $\mathcal{K}$-Souslin set A containing $\partial_e X$; this condition is, however, not sufficient. In particular, if μ is maximal, then $\mu(B) = 1$ for every Baire subset B of X that contains $\partial_e X$. If X is metrizable and $\mu \in \mathcal{M}_+^1(X)$, then μ is maximal if and only if $\mu(\partial_e X) = 1$.

In 1959 a problem in potential theory, studied in collaboration with J. Deny (see the discussion of [**77**] in § 2.4 below), drew Choquet's attention to the importance of certain cones without compact base and to the need to extend his integral representation theory to such cones. This need was in due course satisfied by the generalizations of his theory to *well-capped* cones and *weakly complete* cones, which we now describe. (For his comments on the history of these developments, see the Postface in ⟨**6**⟩ and also ⟨**12**⟩.)

Let C be a convex cone in a Hausdorff locally convex space E. A *cap* of C is by definition a non-empty compact convex subset K of C such that the set $C \setminus K$ is convex. A cap K for the cone C is termed a *universal* cap if $C = \bigcup_{\lambda=0}^{\infty} \lambda K$. The *roof* K_1 of a cap K is the set of $x \in K$ such that, for all $\rho > 1$, we have $\rho x \notin K$; the roof is a convex $\mathcal{G}_\delta$ set, not in general compact, and $\partial_e K = \partial_e K_1 \cup \{0\}$. If C has a universal cap K, then K_1 is a base for C. For a convex cone with a universal cap K it is easy to extend the previous representation theorem to the roof of K as follows: if $x \in K_1$, then there exists in $\mathcal{M}_+^1(K)$ a maximal representing measure μ for x and we have $\mu(K_1) = 1$, and if K is metrizable we have $\mu(\partial_e K_1) = 1$. More generally, the cone C is said to be *well-capped* (bien coiffé) if it is the union of its caps. If C is well-capped and K is one of its caps, then $C_K = \bigcup_{\lambda=0}^{\infty} \lambda K$ is a convex subcone of C having K as a universal cap. Since each element of C belongs to such a subcone, we deduce from the theory of universal caps a representation theorem for well-capped cones. If the specific order associated with a well-capped cone C is a lattice order, then all the caps of C are Choquet simplexes and hence, for $x \in C$, the Choquet–Meyer uniqueness theorem applies to each cap that contains x.

Many classical representation theorems can be proved by the use of the above theory. Examples include the Bochner–Weil representation of continuous positive-definite functions on a locally compact abelian group, and the Bernstein representation of completely monotone functions on $(0, \infty)$, to name only two.

By a weakly complete cone in E we shall mean a cone that is complete in the uniformity defined by the pseudo-metrics $(x, y) \mapsto |f(x) - f(y)|$ for E, where $f \in E'$. By a proper cone we shall understand a cone C such that $C \cap (-C) = \{0\}$. We denote by $\mathcal{S}$ the set of all weakly complete proper convex cones in E. Denote by $h(E)$ the vector sublattice of $\mathbb{R}^E$ generated by E', and by $s(E)$ the convex cone consisting of all functions on E of the form $\max(l_1, l_2, \ldots, l_n)$, where $n \geqslant 1$ and $l_r \in E'$ for $r = 1, 2, \ldots, n$. Thus $E' \subseteq s(E) \subseteq h(E)$ and $h(E) = s(E) - s(E)$. Choquet defines a *conical measure* in E to be a positive linear map $\mu : h(E) \to \mathbb{R}$, and denotes by $M^+(E)$ the set of all conical measures on E. Given a closed cone C in E, one says that μ is carried by C if $\mu(f) = 0$ for all $f \in h(E)$ such that $f|_C = 0$. We denote by $M^+(C)$ the set of elements of $M^+(E)$ that are carried by C. The *resultant* r_μ of μ is the restriction $\mu|_{E'}$ of μ to E', and is an element of the algebraic dual $(E')^*$ of E'. If μ is carried by a weakly complete convex cone C, then $r_\mu \in C$. Moreover, if $r_\mu \in K$, where K is a cap of C, then there exists a Radon measure $\theta \in \mathcal{M}_+(K)$ such that $\theta(K) \leqslant 1$ and $\mu(f) = \theta(f|_K)$ for all $f \in h(E)$, and we say that μ is *localized* on K. A partial order is defined in $M^+(E)$ by writing $\mu \preceq \nu$ whenever $\mu(f) \leqslant \nu(f)$ for all $f \in s(E)$, which, incidentally, implies that $r_\mu = r_\nu$. For $C \in \mathcal{S}$ Choquet shows that the order $\preceq$ is inductive on $M^+(C)$, and hence that each ρ in $M^+(C)$ is majorized by some maximal element μ of $M^+(C)$. Hence we have the following representation theorem. For each $x \in C$ there exists a maximal measure μ of $M^+(C)$ such that $r_\mu = x$. Moreover, there corresponds to each $x \in C$ a unique maximal measure μ of $M^+(C)$ satisfying $r_\mu = x$ if and only if $C - C$ is a vector lattice with respect to its specific order.

The foregoing theory has achieved canonical status in functional analysis and has found applications in, for instance, ergodic theory, operator algebras, function algebras, stochastic processes, random sets, potential theory, statistical mechanics, and harmonic analysis (see [**44**, **95**, **114**, **154**], and also § 2.11 and ⟨**1**, **2**, **5**, **6**, **8**, **10**, **22**, **26**, **31**, **33**, **44**, **46**, **47**, **52**⟩).

2.4. *Potential theory [**19**, **38**, **45**, **47**, **52**, **55–59**, **64**, **66**, **72**, **73**, **77**, **79**, **100**, **109**, **133**, **142**, **144**]*

In [**19**], Choquet and Deny investigate characterizations of harmonic functions in $\mathbb{R}^2$ in the spirit of the Gauss characterization of them in terms of mean values on circles. We say that a real continuous function f on a domain $D \in \mathbb{R}^2$ is *associated* with a signed Radon measure μ on the unit disc B if, for each similarity transformation ϕ of the plane such that $B \subseteq \phi(D)$, we have $\mu(f \circ \phi) = 0$. The authors characterize the measures μ such that the f associated with μ are precisely the harmonic functions. Further study of the condition $\mu(f \circ \phi) = 0$ leads them to a characterization of polyharmonic functions (solutions of $\Delta^n f = 0$).

Let $\mathcal{P}_n$ be the space of real homogeneous polynomial functions of degree n on $\mathbb{R}^m$. In the space of all real polynomial functions on $\mathbb{R}^m$ define an inner product by writing $(p, q) = \int pq\, d\mu$, where μ is the rotation-invariant probability measure on the unit sphere S^{m-1}. Let $r^2 = \sum_{i=1}^m x_i^2$. Then Brelot and Choquet [**45**] show that, for $n \geqslant 2$, the set $\mathcal{H}_n$ of all homogeneous harmonic polynomials of degree n is the orthogonal complement of $r^2\mathcal{P}_{n-2}$ in $\mathcal{P}_n$. Among applications of this is the theorem that a domain D in $\mathbb{R}^m$ is ellipsoidal if and only if, for each polynomial p on $\mathbb{R}^m$, there exists a harmonic polynomial on $\mathbb{R}^m$ that coincides with p on the boundary of D.

In [**38**], Brelot and Choquet show how classical potential theory in open subsets of $\mathbb{R}^n$ can be extended to the structures they call *$\mathcal{E}$-spaces*. We define an *$\mathcal{E}$-manifold* as a connected Hausdorff space $\mathcal{M}$ equipped with an n-dimensional atlas of local charts, where $n \geqslant 2$, such that the chart-change maps are isometries if $n \geqslant 3$, or conformal (directly or not) if $n = 2$. (The $\mathcal{E}$-spaces of [**38**] are more general than $\mathcal{E}$-manifolds in that the authors allow the chart-parameter space to be the Alexandroff compactification of $\mathbb{R}^n$ rather than $\mathbb{R}^n$; for the sake of simplicity we confine attention here to $\mathcal{E}$-manifolds.) Such manifolds are metrizable, locally compact, and σ-compact, and much of classical potential theory can be extended to them. For example, the Dirichlet problem for relatively compact domains is treated here by the Perron–Wiener–Brelot method. The authors are principally concerned with *Green spaces*, namely those *$\mathcal{E}$-spaces* that carry a Green function, and for such spaces they treat inter alia the Dirichlet problem for domains that are not relatively compact. Assume that the $\mathcal{E}$-manifold $\mathcal{L}$ has a Green function, denoting it by G_x when the point x is taken as pole. Then a maximal trajectory orthogonal to the level sets of G_x and issuing from x is called a *Green line* from x, it being assumed that $\operatorname{grad} G_x \neq 0$ along the trajectory. A Green line from x is said to be *regular* if the infimum of G_x along the line is zero. Each Green line from x is tangent at x to a half-line issuing from x and vice versa, and the half-lines corresponding to non-regular Green lines form a set of solid-angle measure zero. The *Green measure* $g_x(A)$ of a Borel subset A of the level set $G_x = \lambda$ is defined, for sufficiently large λ, as the normalized solid-angle measure of the set of initial directions of the Green lines from x that meet the set A, and is equal to the harmonic measure of A relative to the domain $G_x > \lambda$. If $n > 2$ and the volume of $\mathcal{L}$ is finite, then almost all Green lines are of finite length.

Let E be a topological space. A function $f : \mathfrak{P}(E) \to [0, \infty]$ is said to be *stabilizable* if it is isotone and such that, for each $X \subseteq E$ and each $\varepsilon > 0$, there exists a partition of X into two sets X_1 and X_2 satisfying $f(\overline{X_1}) \leqslant f(X)$ and $f(X_2) < \varepsilon$. Choquet [**72**] investigates the properties of such maps and shows in particular that, when E is a Green space, the exterior

Green capacity cap* is stabilizable. He does this by proving that the set of points of E at which X is thin is contained in an open set G such that $\mathrm{cap}^*(X \cap G) < \varepsilon$. Another finding is that there exists a partition of X into two sets, X_1 and X_2, such that $\mathrm{cap}^*(\overline{X_1}) = \mathrm{cap}^*(X_1)$ and $\mathrm{cap}^*(X_2) < \varepsilon$. The results of [**72**] contributed to the understanding of the relation between fine continuity and quasi-continuity in potential theory (see ⟨**15**⟩).

Suppose that A is a $\mathcal{G}_\delta$-set of zero capacity in $\mathbb{R}^m$ $(m \geqslant 3)$. Then Deny ⟨**23**⟩ shows that there exists a positive measure μ on $\mathbb{R}^m$ whose potential U^μ satisfies $A = \{\, x \in \mathbb{R}^m : U^\mu(x) = \infty \,\}$. Choquet [**73**] sharpens this result by showing that one can impose on μ the condition that $\mu(\complement A) = 0$. After proving this for Newtonian potential, Choquet notes that the result also holds for the potentials associated with certain other kernels.

Choquet devoted a considerable effort in a number of papers to the study of the potential theory associated with a given kernel, with the aim of understanding how the regularity properties (usually known as *principles*) of the kernel are interrelated and how they affect the potential theory. This resulted in a large number of definitions, results, and examples, of which we can only give a representative sample here. Suppose that E is a locally compact Hausdorff space. By a *kernel* for E Choquet means a lower semicontinuous map $G : E \times E \to [0, \infty]$. Let $\mathcal{M}_+(E)$ denote the set of positive Radon measures on E. For $\mu \in \mathcal{M}_+(E)$ the *potential* $G\mu : E \to [0, \infty]$ of μ (relative to G) is defined by

$$G\mu(x) = \int_E G(x, y)\, d\mu(y).$$

Given a kernel G, one defines the transpose kernel $\check{G}$ via $\check{G}(x, y) = G(y, x)$.

The simplest situation is that in which the space E contains only finitely many points. This is studied in depth in [**47**, **55**] by Choquet and Deny. Here, of course, μ can be identified with a vector in $\mathbb{R}^n$, where n is the number of points in E, and $\mu \mapsto G\mu$ with a linear endomorphism of $\mathbb{R}^n$. The authors hoped that the study of potential theory in this simple setting would clarify the basic principles and their interrelations, and also lead to new principles as well as, especially, to an improved understanding of the general case. Their work reveals hitherto unsuspected links between familiar principles: domination, balayage, equilibrium, maximum, energy, lower bound, as well as revealing new principles. Completely new is the notion of duality in potential theory: the properties of G and $\check{G}$ exhibit a duality. For example, the principles of balayage and of domination constitute a dual pair; if G is symmetric, it follows that these two principles are equivalent properties and they imply that G is of positive type (positive-definite). A special study is made of degenerate kernels (those G such that $\det G = 0$).

The kernel G is said to be *regular* if it satisfies the following *continuity principle*: whenever $\mu \in \mathcal{M}_+(E)$ is compactly supported and such that the restriction of $G\mu$ to $\operatorname{supp}\mu$ is finite and continuous, it follows that $G\mu$ is finite and continuous on E. The *dilated maximum principle on compact sets* states that, for each compact set K, there exists a constant $\lambda = \lambda(K) \geqslant 1$ such that, for all $\mu \in \mathcal{M}_+(E)$ such that $\operatorname{supp}\mu \subseteq K$, we have $\sup_K G\mu \leqslant \lambda \sup_L G\mu$, where $L = \operatorname{supp}\mu$. In [**52**] it is shown that G is regular if the dilated maximum principle on compact sets is satisfied, but that the converse is only true under certain conditions. This work is continued in [**58**], where, for simplicity, it is assumed that the space E is compact (see also [**59**]). We can also define the (outer) *G-capacity* of subsets of E by the same procedure as for Newtonian capacity, the kernel G now taking the role formerly played by the Newtonian kernel. The kernel G is said to be *very regular* if it is finite and continuous away from the diagonal Δ of $E \times E$, infinite on Δ, and such that G and $\check{G}$ are both regular. If G is regular and also finite and continuous away from Δ, then, for each $\varepsilon > 0$ and each $\mu \in \mathcal{M}_+(E)$, there exists an open set O of $\check{G}$-capacity $< \varepsilon$ such that the restriction of $G\mu$ to $\complement O$ is finite and continuous. Inequalities relating G-capacity and $\check{G}$-capacity are obtained, and it is shown that if G is very regular, then a set is of zero G-capacity if and only if it is of zero $\check{G}$-capacity.

Suppose now that the kernel G is symmetric. The *mutual energy* of $\mu, \nu \in \mathcal{M}_+(E)$ is, by definition, $[\mu, \nu] = \int G\mu \, d\nu$; the energy of μ is $[\mu, \mu]$. The kernel G is said to satisfy the *domination principle* if $G\mu \leqslant G\nu$ everywhere whenever μ and ν are compactly supported elements of $\mathcal{M}_+(E)$ such that $[\mu, \mu] < \infty$ and $G\mu \leqslant G\nu$ on $\operatorname{supp}\mu$. Choquet [**64**] shows that if G satisfies the domination principle, then G is of positive type (which means that $[\mu, \nu]^2 \leqslant [\mu, \mu][\nu, \nu]$ for all $\mu, \nu \in \mathcal{M}_+(E)$). On the other hand, G is said to satisfy the k-*dilated maximum principle* if $G\mu \leqslant k$ for all compactly supported $\mu \in \mathcal{M}_+(E)$ such that $G\mu \leqslant 1$ on $\operatorname{supp}\mu$. The latter condition is shown to imply that $[\mu, \nu]^2 \leqslant k^2[\mu, \mu][\nu, \nu]$ for all $\mu, \nu \in \mathcal{M}_+(E)$.

Let (μ_i) be a family of positive Radon measures on $\mathbb{R}^n$. Then a theorem of Cartan and Brelot states that there exists a measure $\mu \geqslant 0$ whose potential is equal, save on a set of zero exterior capacity, to the infimum of the potentials of the μ_i. Taking as context a locally compact space equipped with a kernel, Brelot and Choquet [**57**] obtain a much more general result. Their proof makes use of a topological lemma of Choquet concerning lower semicontinuous functions.

In the first part of [**66**] Choquet studies conditions for a kernel under which the convergence of a sequence of measures implies the convergence of the corresponding sequence of potentials. Next, a kernel G is said to satisfy the *equilibrium principle* if, for each compact set K, there exists a measure $\mu \geqslant 0$ with $\operatorname{supp}\mu \subseteq K$ such that $G\mu \leqslant 1$ everywhere, and $G\mu = 1$ on K, save on a set of zero outer $\check{G}$-capacity. In the second part of this paper it is shown that if G and $\check{G}$ are regular and satisfy the equilibrium principle, then the (outer) capacities associated with G and $\check{G}$ are equal and satisfy the fundamental axioms Γ_1–Γ_3 of § 2.2.

Now let $\mathcal{C}_c = \mathcal{C}_c(E)$ and $\mathcal{M}_c = \mathcal{M}_c(E)$ denote the subspaces of $\mathcal{C} = \mathcal{C}(E)$ and $\mathcal{M} = \mathcal{M}(E)$, respectively, consisting of their compactly supported elements. The bilinear form $\langle f, \mu\rangle = \int f \, d\mu$ puts $\mathcal{C}_c$ in duality with $\mathcal{M}$ and $\mathcal{C}$ in duality with $\mathcal{M}_c$. A positive linear map $T : \mathcal{M}_c \to \mathcal{M}$ that is weakly continuous with respect to the above dualities is called by Choquet and Deny [**56**] a *diffusion kernel*. Its transpose T^* is a weakly continuous positive linear map $\mathcal{C}_c \to \mathcal{C}$. A diffusion kernel T is said to satisfy the *balayage principle* if, for each relatively compact open subset W of E and each $\mu \in \mathcal{M}_c^+$, there exists $\mu' \in \mathcal{M}_c^+$ such that $\operatorname{supp}\mu' \subseteq \overline{W}$, $T\mu' \leqslant T\mu$, with $T\mu' = T\mu$ on Borel subsets of W. The transpose T^* satisfies the *domination principle* if, for all $f, g \in \mathcal{C}_c$ such that $T^*f \leqslant T^*g$ on $\operatorname{supp} f$, we have $T^*f \leqslant T^*g$ everywhere. It is proved in [**56**] that T satisfies the balayage principle if and only if T^* satisfies the domination principle. In particular, a positive measure κ on a locally compact abelian group defines a diffusion kernel T_κ by the formula $T_\kappa\mu = \kappa \star \mu$ and a transposed diffusion kernel T'_κ by $T'_\kappa f = \kappa \star f$. It follows from the preceding theorem that T_κ satisfies the balayage principle if and only if T'_κ satisfies the domination principle.

Turning to the study of convolution operators on a locally compact abelian group G, Choquet and Deny [**79**, **133**] say that a positive measure κ on G satisfies the balayage principle on relatively compact open sets if T_κ satisfies the balayage principle enunciated in the preceding paragraph, and they denote the corresponding set of measures by $\mathcal{B}(G)$. The balayage principle for open sets is defined in the same way, except that W now runs through all the open sets and not merely the relatively compact ones; the corresponding set of measures is denoted by $\mathcal{B}_0(G)$. The main objective here is to characterize the elements of $\mathcal{B}_0(G)$. A *pseudo-period* of a measure N is an $x \in G$ such that $N \star \varepsilon_x$ is proportional to N. A *perfect* kernel can be defined as a measure of the form $\int_0^\infty \mu_t \, dt$, where $(\mu_t)_{t \geqslant 0}$ is a vaguely continuous convolution semigroup of measures with $\mu_0 = \varepsilon_0$. The set P of pseudo-periods of a measure N is a closed subgroup of G and N can be written as fK, where f is an exponential (that is, $f(x+y) = f(x)f(y)$ for all $x, y \in G$) and K is a measure admitting P as period group; the map $G \to G/P$ conveys K to a measure on G/P denoted by K/P. The authors prove that (i) if P is compact, then $N \in \mathcal{B}_0(G)$ if and only if K/P is perfect and (ii) if P is not compact, then $N \in \mathcal{B}_0(G)$ if and only if $K/P = a\sum_{n=0}^\infty \sigma^n$ in the vague topology, where a is a positive constant, σ a measure possessing convolution powers σ^n of all orders, and $\sigma^0 = \varepsilon_0$. An important tool in this work is the inequality

$$N_*(X)N_*(Y) \leqslant N_*(X-Y)N_*(X+Y)$$

for all $N \in \mathcal{B}(G)$ and $X, Y \subseteq G$, where N_* denotes inner N-measure.

In [**77**] (see also ⟨**6**, **24**⟩) Choquet and Deny study the convolution equation $\mu \star \sigma = \mu$, where μ and σ are Radon measures on a locally compact abelian group G. Here σ is a given positive measure, and the object is to investigate the measures μ that solve the equation. A signed measure μ is said to be *bounded* if, for each continuous real function ϕ on G, the convolution $\mu \star \phi$ is a bounded function. If $\sigma(G) \neq 1$, then no non-zero bounded μ satisfies the equation; but, if $\sigma(G) = 1$, it is proved that the bounded solutions are the periodic bounded measures whose group of periods contains the support of σ.

To simplify the discussion we shall now suppose that the topology of G has a countable base and that the subgroup of G generated by the support of σ is dense in G. The solutions $\mu \geqslant 0$ form a convex cone $\mathcal{H}$, termed the cone of σ-*harmonic* measures. If $\mu, \nu \in \mathcal{H}$, we write $\nu \preceq \mu$ to signify that $\mu - \nu \in \mathcal{H}$. The relation $\preceq$ is a lattice order for $\mathcal{H}$. The set $\partial\mathcal{H}$ of *extreme* elements of $\mathcal{H}$ is defined as the set of non-zero $\mu \in \mathcal{H}$ such that whenever $\mathcal{H} \ni \nu \preceq \mu$ it follows that $\nu = c\mu$ for some $c \in \mathbb{R}_+$. A real exponential f on G is said to be σ-harmonic if the measure $f(x)\omega(dx)$ belongs to $\mathcal{H}$, where ω denotes Haar measure on G, and the σ-harmonic exponentials form a locally compact space when given the topology of uniform convergence on compact subsets of G . The elements of $\partial\mathcal{H}$ are shown to be precisely the measures of the form $cf(x)\omega(dx)$, where $c \in \mathbb{R}_+$ and f is a σ-harmonic exponential. Using Choquet's integral representation theory, the authors show that each σ-harmonic μ has a density h_μ with respect to Haar measure given by the formula

$$h_\mu(x) = \int f(x)\,\pi(df),$$

where π is a positive Radon measure on the space of σ-harmonic exponentials. Moreover, this representation is unique.

This work has found applications in probability theory (see, for example, ⟨**7**, **25**, **48**, **59**⟩). It has also been extended to certain other types of topological group and has been developed in other ways (see, for instance, ⟨**17**⟩ and the bibliography therein).

Let B_n be the closed ball of radius n centred at 0 in $\mathbb{R}^\tau$. A measure $\mu \geqslant 0$ on $\mathbb{R}^\tau$ is said to have superexponential growth if $\limsup_{n\to\infty} n^{-1} \log \mu(B_n) = \infty$. Continuing the study of convolution operators, Choquet [**142**] shows that if N belongs to $\mathcal{B}(\mathbb{R}^\tau)$ and is of superexponential growth, then $\operatorname{supp} N$ is contained in a closed half-space of $\mathbb{R}^\tau$. A function called a *growth indicator* for a positive measure is defined in [**144**] and used to prove that if $\lambda \in \mathcal{B}(\mathbb{R}^\tau)$, then either λ is of superexponential growth or λ is the product of an exponential function by a bounded element of $\mathcal{B}(\mathbb{R}^\tau)$.

Getoor ⟨**27**⟩ has proved that, for every Radon measure $\mu \geqslant 0$ on $\mathbb{R}^p$ $(p \geqslant 3)$ that vanishes on polar sets (that is, sets of zero Newtonian capacity), there exists a least finely closed set that carries μ. Choquet [**100**] gives an elegant short and non-probabilistic proof of a more general result that, in particular, allows Getoor's theorem to be extended to the axiomatic potential theories of Brelot and Bauer.

Keldych ⟨**35**⟩ proves that if Ω is a bounded domain in $\mathbb{R}^p$ $(p \geqslant 3)$, then there exists a countable set D of irregular boundary points for the Dirichlet problem for Ω such that, if $f \in C(\partial\Omega)$ and if the Perron–Wiener–Brelot function H_f is continuous at each point of D, then H_f is continuous at every point of $\partial\Omega$. Choquet [**109**] gives three proofs of this theorem, of which two are valid in a very general topological setting. The third proof, more explicit about D, involves the fine topology and establishes a connection between the behaviour of the Green function and that of H_f for $f \in C(\partial\Omega)$.

2.5. *Linear functional analysis* [46, 61, 81, 94, 102, 104–106, 116, 119, 122, 125, 126, 130, 132, 135, 138]

In [46, 104] simple proofs are given for two versions of the minimax theorem of the theory of games. We are given compact convex subsets X and Y, respectively, of two real topological vector spaces L and M, say. In [46] the payoff function is the restriction to $X \times Y$ of a continuous bilinear function on $L \times M$. In [104] the spaces L and M are finite-dimensional and the payoff function $f(x, y)$ is defined and continuous on $X \times Y$, convex in x, and concave in y; for this case an extremely elementary proof is given, together with another that uses the Brouwer fixed-point theorem.

Suppose that X is a Hausdorff space, that S is a family of pairs (x, σ), where $x \in X$ and σ is a compactly supported positive Radon measure on X that does not charge x, and that T is a family of compactly supported positive Radon measures on X. Let E be the set of all real continuous functions f on X such that $\int f\, d\sigma \leqslant f(x)$ for all $(x, \sigma) \in S$ and $\int f\, d\tau \leqslant 0$ for all $\tau \in T$. Then E is a convex cone that contains 0, is closed with respect to uniform convergence on compact sets, and is a lower semilattice, that is to say, we have $f \wedge g \in E$ for all $f, g \in E$. The principal theorem of Choquet and Deny in [61] is the converse: any such cone of real continuous functions on X can be characterized by a set of inequalities of the above type. A special case is the classical characterization of the cone of all continuous superharmonic functions on a domain in $\mathbb{R}^n$. If the cone E is locally compact, then the supports of the measures in S and T can be taken to be finite sets, and the above characterization is then seen to be a generalization of the classical Harnack inequalities.

Suppose that E is a real or complex normed vector space, that G is a linear subspace of E, and that $x \in E \setminus G$, and suppose that there exists in G a best approximation of g_0 to x, that is to say, we suppose that $\|x - g_0\| = \inf_{g \in G} \|x - g\|$. Then there exists an element f_0 of the dual space E^* such that (i) f_0 is an extreme point of the unit ball of E^*, (ii) $\Re f_0(g_0) \geqslant 0$, and (iii) $f_0(x - g_0) = \|x - g_0\|$. This theorem was proved by Singer ⟨56⟩ for those normed vector spaces he called *Choquet spaces*, which include in particular all separable normed vector spaces. However, in [94] Choquet shows that the theorem is true, without restriction, in all normed vector spaces.

Let K be a compact convex subset of a locally convex space E. In [102] Choquet, Corson, and Klee show that if E is of finite dimension $d \geqslant 2$, then the set $\exp K$ of exposed points of K is the union of a $\mathcal{G}_\delta$ set, an $\mathcal{F}_\sigma$ set, and $d - 2$ other sets, each of which is the intersection of a $\mathcal{G}_\delta$ set and an $\mathcal{F}_\sigma$ set. Further results in this vein are obtained in a more general setting that replaces K by a compact metric space and linear functionals by upper semicontinuous functions. In the final section of the paper the authors construct, in an infinite-dimensional E, a compact convex K that has no algebraic exposed points, and hence for which $\exp K = \emptyset$.

In [125] Choquet studies the properties of certain subspaces of a complete metrizable topological vector space E. For instance, suppose that H is the vector subspace of E generated by a set X that is either an analytic subset of E or is of the form $\bigcup X_n$ for some finite or infinite sequence (X_n) of closed vector subspaces of E. Then it is proved that either H is closed or H is of the first category in itself and the algebraic dimension of the quotient space E/H is uncountable.

By means of a counterexample Choquet [138] shows that two weakly complete cones in a locally convex space that intersect only in their common vertex cannot, in general, be separated by a closed hyperplane.

Let H be a separating vector subspace of $\mathcal{C}(X, \mathbb{C})$, where X is a compact Hausdorff space, let B be the unit ball of H', and suppose that $0 \neq l \in H'$. The evaluation map identifies X with a subset of B and hence $\mathcal{M}^1_+(X)$ with a subset of $\mathcal{M}^1_+(B)$. Hustad, Hirsberg, Fuhr, and Phelps have shown that there exists a complex Radon measure μ on X such that (i) $l(f) = \int f\, d\mu$ for all $f \in H$, (ii) $\|\mu\| = \|l\|$, and (iii) $|\mu|/\|\mu\|$ is a maximal element of $\mathcal{M}^1_+(B)$ (see § 2.3). Fuhr and

Phelps also prove a uniqueness theorem for the case when H contains the constant functions. In [**130**] Choquet summarizes a new treatment of these topics and, in particular, obtains a uniqueness result for the case when H is not assumed to contain the constant functions. (For more details, see also ⟨**45**⟩.)

Let X be a locally compact Hausdorff space, let H be a linear subspace of $\mathcal{C}(X,\mathbb{R})$, and let $H^+ = \mathcal{C}^+(X,\mathbb{R}) \cap H$. Then H is said by Choquet to be an *adapted* subspace if (i) for each $x \in X$ there exists $h \in H^+$ such that $h(x) > 0$; (ii) $H = H^+ - H^+$; and (iii) for each $f \in H^+$ there exists $g \in H^+$ such that, for every constant $k > 0$, there exists a compact set K such that $g(x) \geqslant kf(x)$ outside K. Choquet [**81**] proves that, if ϕ is a positive linear functional on an adapted subspace H of $\mathcal{C}(X,\mathbb{R})$, then there exists a positive Radon measure μ on X such that $\phi(f) = \int f\,d\mu$ for all $f \in H$. Noting that the real polynomial functions on $\mathbb{R}$ form an adapted subspace of $\mathcal{C}(\mathbb{R},\mathbb{R})$, he shows that this theorem provides an elegant treatment of the Hamburger moment problem.

In [**119**, **122**, **126**, **132**] Choquet undertakes an investigation of the problem of representing positive linear functionals on a real function space by means of integrals. Direct use of integrals is not always possible. For example, suppose that Q is the space of real quadratic functions $q(t) = a_0 + a_1 t + a_2 t^2$ on $\mathbb{R}$ and that $\phi : Q \to \mathbb{R}$ is defined by $\phi(q) = a_2$. Then $\phi(q) = \varepsilon_\infty(\tilde{q})$, where $\tilde{q}$ is the continuous extension to $\overline{\mathbb{R}} = [-\infty,\infty]$ of the quotient $q(t)/(t^2+1)$. This construction is systematized here in a very general setting. Given a real function space V, Choquet seeks to represent the general positive linear functional on V in terms of quotients integrated with respect to a Radon measure on a compact space associated with V. For this purpose he introduces *sub-Stonian* spaces. For Choquet a sub-Stonian space is a compact Hausdorff space E in which we have $\overline{A} \cap \overline{B} = \emptyset$ whenever A and B are disjoint open $\mathcal{F}_\sigma$-subsets of E. (Some authors use a wider definition.) Choquet points out that a space is sub-Stonian if and only if it is a compact Hausdorff F-space in the sense of ⟨**28**⟩. Given a compact Hausdorff space E, let $\mathcal{D}(E)$ denote the set of all those $f \in \mathcal{C}(E,\overline{\mathbb{R}})$ that are infinite only on a nowhere dense subset of E. Then E is sub-Stonian if and only if $\mathcal{D}(E)$ is an algebra such that, for all $f, g \in \mathcal{D}(E)$, the quotient f/g, defined on $\operatorname{supp} g$, belongs to $\mathcal{D}(\operatorname{supp} g)$. For such E Choquet studies linear functionals on subspaces of $\mathcal{D}(E)$. He motivates this as follows. Let $\mathcal{S}$ be a σ-algebra of subsets of a set I and let $\mathcal{N}$ be a σ-ideal of $\mathcal{S}$. Then $M(\mathcal{S},\mathcal{N})$ is defined as the space of equivalence classes (modulo $\mathcal{N}$) of real $\mathcal{S}$-measurable functions on I, and it is proved that there exists a sub-Stonian space E and an isotone algebra-isomorphism of $M(\mathcal{S},\mathcal{N})$ onto $\mathcal{D}(E)$.

Now let V be a linear subspace of $\mathcal{D}(E)$ such that $V = V^+ - V^+$, where E is a sub-Stonian space. Suppose that $f \in \mathcal{D}^+(E)$ and that μ is a positive Radon measure on $\operatorname{supp} f$ such that $g/f \in L^1(\mu)$ for all $g \in V$. Then the map $g \mapsto \mu(g/f)$ is a positive linear form on V, denoted by $[f,\mu]$ and called in [**132**] a *submeasure* (though this term is used differently by many authors). The submeasure $[f,\mu]$ is *proper* if $f \in V^+$ and is a *subvaluation* if $\mu = \varepsilon_a$ for some $a \in E$. There is an extensive investigation here of positive linear forms in terms of these concepts. Among many results it is shown, for instance, that every positive linear form can be expressed as a sum of proper submeasures, and that every extreme positive form is a proper subvaluation.

Let E be a weak topological vector space. We denote by $b(E)$ the vector sublattice of $\mathbb{R}^E$ generated by the real affine continuous functions on E, and by $b^*(E)$ the algebraic dual of $b(E)$. The *affine measures* on E are defined by Choquet [**105**, **106**, **116**] to be the elements of the space $b^*_+(E)$ of positive linear forms on $b(E)$. He shows that, with respect to the topology $\sigma(b^*(E), b(E))$, the set $B = \{\mu \in b^*_+(E) : \mu(1) = 1\}$ is a Choquet simplex that has closed extreme boundary $\partial_e(B)$, and he characterizes the latter in various ways. Given a continuous affine map ϕ of E into another weak space F together with $\mu \in b^*_+(E)$, he defines $\phi(\mu) : b(F) \to \mathbb{R}$ by the formula $\phi(\mu)(g) = \mu(g \circ \phi)$, and we have $\phi(\mu) \in b^*_+(F)$. Moreover, if $\phi(\mu)$ is a Radon measure on F for every finite-dimensional F and every continuous affine $\phi : E \to F$, then μ is called a *cylindrical measure*; several characterizations of cylindrical measures are given. We say that $\mu \in b^*_+(E)$ has a resultant if $\mu(|l|) := \sup\{\,\mu(f) : b^+(E) \ni f \leqslant |l|\,\} < \infty$

for all $l \in E'$. Every affine measure that has a resultant is cylindrical, but the converse is false. A bijection is established between certain conical measures on $E \times \mathbb{R}$ and the affine measures on E that have resultants. A special study is made of the affine measures μ on a pre-Hilbert space H that are invariant under all isometries of H. Denote by X the set of the invariant μ that also satisfy $\mu(1) = 1$. Then X is a Choquet simplex that has $\partial_e X$ homeomorphic with $[0, \infty]$ via a map $t \mapsto \mu_t$ with $\mu_0 = \varepsilon_0$ and $\mu_\infty = \lim_{t\to\infty} \mu_t$. For $0 < t < \infty$, we see that μ_t is a Wiener–Gauss measure if $\dim H = \infty$ and is the normalized solid-angle measure on the sphere $S(0, t)$ if $\dim H < \infty$. Each $\mu \in X$ has a unique representation of the form

$$\mu = a\mu_0 + b\mu_\infty + \int \mu_t \, d\pi(t),$$

where $a, b \in \mathbb{R}_+$ and π is a positive measure on $(0, \infty)$.

In the final part of [**106**, **116**], E is a complete weak space, and for each conical measure μ on E the set of all resultants of the conical measures $\nu \leqslant \mu$ is denoted by K_μ. The K_μ are symmetric compact convex sets, and their translates (which generalize the zonohedra of Coxeter) are known as *zonoforms*. It is proved that a symmetric compact convex set is a zonoform if and only if the gauge of its polar set is a function of negative type.

Suppose that X is a compact Hausdorff space and that T is a positive linear operator in $\mathcal{C}(X)$. Then Choquet and Foiaş [**135**] prove that (i) if $\inf\{T^n 1 : n \geqslant 1\} < 1$, then $T^n 1$ tends to 0 uniformly as $n \to \infty$ and (ii) if $\sup\{T^n 1 : n \geqslant 1\} > 1$, then $T^n 1$ tends to ∞ uniformly as $n \to \infty$. Several applications are made to the study of the limiting behaviour of $(\sum_{r=0}^{n-1} T^r f)/n$ and of $(T^n f)^{1/n}$. For a sequel to this, see $\langle$**3**$\rangle$.

2.6. *Set theory* [**103**, **107**, **108**]

A cardinal number $\mathfrak{n}$ is said to be 2-*measurable* if, for each set A of cardinality $\mathfrak{n}$, there exists a countably additive measure μ on the power set of A taking only the values 0 and 1 and such that $\mu(\{x\}) = 0$ for all $x \in A$ and $\mu(A) = 1$. Cardinals that lack this property are termed *moderate* (modéré) by Choquet [**103**]. Given a set I, he shows that the positive cone $\mathbb{R}_+^{(I)}$ of the direct sum $\mathbb{R}^{(I)}$ is complete with respect to the weak topology $\sigma(\mathbb{R}^{(I)}, \mathbb{R}^I)$ if and only if the set I is of moderate cardinality. He also considers the cone of compactly supported positive Radon measures on a locally compact Hausdorff space E and shows that it is complete with respect to the weak topology of the duality with the space $\mathcal{C}(E)$ of all real continuous functions on E if and only if E is real-compact.

After the notions of *filter* and *ultrafilter* were introduced by Henri Cartan in 1937 they became standard currency, especially for Bourbaki, whose treatment of them was based on the axiom of choice. Choquet's papers [**107**, **108**] are an investigation, based on the continuum hypothesis, of the existence and classification of ultrafilters on the set of natural numbers $\mathbb{N}$ that have special properties of interest in analysis.

Choquet's work on $\mathcal{K}$-analytic sets has been described above in § 2.2. See also § 2.11.

2.7. *Measure theory* [**27**, **28**, **118**, **120**]

Suppose that λ is one-dimensional Hausdorff measure in the plane. Choquet [**27**] shows that the continuum hypothesis implies that there exists a set S such that (i) $\lambda(S) = \infty$, (ii) if $T \subseteq S$, then T is λ-measurable and $\lambda(T) = 0$ or $\lambda(T) = \infty$, and (iii) every analytic subset of S is countable. In [**28**] he considers the question whether there exists a set E in the plane such that (a) $\complement E \cap \{\, (x, y) : y = c \,\}$ is of linear measure zero for each real number c and (b) $\lambda(E \cap L) = 0$ for each λ-measurable set L that intersects every line $y = c$ in a set of linear measure zero. Assuming again the continuum hypothesis, he shows (1) that such a set E does not exist and (2) that existence can, however, be proved if one modifies the question by restricting the sets L to be Borel, analytic, projective, or of finite λ-measure.

In [**120**] Choquet shows that the continuum hypothesis implies the existence of a function $f:[0,1]\to[0,1]$ the graph Γ of which is a universally measurable subset of $[0,1]^2$ satisfying $\mu(\Gamma)=0$ for every diffuse Radon measure μ on $[0,1]^2$. He deduces that the projection onto $[0,1]$ of a universally measurable subset of $[0,1]^2$ need not be a universally measurable subset of $[0,1]$, and that a subset of $[0,1]^2$ can be universally measurable without being universally capacitable.

Next, let E be a compact Hausdorff space, let $N\subseteq E$, and let $M\subseteq\mathcal{M}(E)$. The set N is said to be *M-negligible* if $|\mu|(N)=0$, for all $\mu\in M$, and *uniformly M-negligible* if, for each $\varepsilon>0$, we can find an open set U containing N and such that $|\mu|(U)<\varepsilon$ for all $\mu\in M$. Choquet [**118**] asks the following question: if M is a $\sigma(\mathcal{M}(E),\mathcal{C}(E))$-compact subset of $\mathcal{M}_+(E)$ and N is M-negligible, does it follow that N is uniformly M-negligible? The answer is shown to be affirmative (i) if N is a $\mathcal{K}_\delta$-set (theorem of Prohorov), or (ii) if E is metrizable, M is countable, and N is a $\mathcal{G}_\delta$-set. However, for $E=[0,1]^2$ a counterexample is obtained via the construction in [**120**] (which depends on the continuum hypothesis). Choquet also provides a counterexample for a non-metrizable E, with N a $\mathcal{G}_\delta$-set.

2.8. *General topology* [**29**, **31**, **36**, **62**, **111**, **112**]

The article [**31**] consists of three parts. The first is a detailed study of binary relations, including semicontinuous relations, between two topological spaces. This part also includes a definition and investigation of the expressions $\liminf_{\mathcal{F}}\mathcal{X}$, $\limsup_{\mathcal{F}}\mathcal{X}$, and $\lim_{\mathcal{F}}\mathcal{X}$, where $\mathcal{X}=(X_i)_{i\in I}$ is a family of subsets of a topological space and $\mathcal{F}$ is a filter on the index set I. For example, $\limsup_{\mathcal{F}}\mathcal{X}=\bigcap_{J\in\mathcal{F}}\overline{\bigcup_{i\in J}X_i}$. A tool used in the discussion is the notion of a *grill*, for which see also [**29**].

In the second part of [**31**] Choquet studies a type of relation $R(\mathcal{F},m)$, termed the relation of *pseudo-convergence*, between the filters $\mathcal{F}$ and the points m in a given set E. If $\mathcal{F}$ and m satisfy $R(\mathcal{F},m)$, then we say that $\mathcal{F}$ is *pseudo-convergent* to the *pseudo-limit* m. These considerations are applied, in particular, to the space 2^E of all non-empty closed subsets of a topological space E. Given a filter $\mathcal{F}$ in 2^E and a non-empty closed set X in E, the relation $R(\mathcal{F},X)$ is defined to mean that $X=\lim_{\mathcal{F}}\mathcal{Y}$, where $\mathcal{Y}=(Y)_{Y\in 2^E}$. This formulation generalizes the treatment, when E is a metric space, of convergence in 2^E by means of the Hausdorff metric (see ⟨**37**⟩). There now exists an extensive literature on *convergence spaces*, for which these investigations can be seen as a precursor.

The third part of [**31**] consists of a very general account of Choquet's contingent–paratingent theorem, which has already been discussed above.

In [**36**] Choquet examines four different ways in which one might conceivably generalize Baire's category theorem, and shows that none of them is possible.

By modifying the Banach–Mazur topological game, Choquet [**111**] defines a new class of Baire spaces that is large enough to include the standard examples of Baire spaces, but restricted enough to be stable with respect to numerous operations (for example, the formation of products, whether countable or not). Given a topological space E, two players α and β choose alternately a non-empty open subset of E in such a way that the resulting sequence (G_n) of open sets is decreasing, β having the first move. The player α is then declared the winner if $\bigcap_n G_n\neq\emptyset$; otherwise β wins. It is assumed that each player remembers only his or her opponent's last move. The space E is termed *α-favourable* if α has a winning strategy. It is proved that every α-favourable space is a Baire space. In a modification of the game, α is required at each step to choose an open set that contains a point prescribed by β; the space E is *strongly α-favourable* if α has a winning strategy for this modified game. Choquet [**112**] proves, for instance, that the extreme boundary $\partial_e(X)$ of a compact convex set X is strongly α-favourable. In his earlier paper [**62**] Choquet defines another class of Baire spaces that is stable with respect to numerous operations, namely the *siftable* spaces (*espaces tamisable*).

However, the latter are always α-favourable. For variants and more information see [**62**, **111**, **112**] and ⟨**34**⟩.

2.9. *Late work [**141**, **143**, **145–149**, **151**, **152**, **156**, **161–163**, **169**]*

We confine attention here to late papers that have not already been considered above.

Let g be a C^∞ Riemannian metric on $\mathbb{R}^3$ and let g_0 be the Euclidean metric. In [**141**] the authors prove that if g/g_0 tends to 1 at infinity, then each Killing vector on $\mathbb{R}^3$ associated with g and tending to zero at infinity is identically zero.

In [**143**] Choquet is interested in a class T of C^1-manifolds equipped with continuous C^0 metrics and having a certain good geodesic behaviour. These are 'variétés tendues', for whose existence see [**18**]. He shows that a number of classical results that hold for smooth Riemannian manifolds extend to the class T.

In the 1980s Choquet wrote eight papers [**145–149**, **151**, **152**, **156**] investigating the distribution modulo 1 of real sequences of the form $(k\theta^n)_{n=1}^\infty$, where $k > 0$ and $\theta > 1$. It is not possible to summarize these papers here, since they are themselves summaries, but some brief indications can be given. He is mainly interested in the case where θ is rational, and especially that in which $\theta = 3/2$. For $t \in \mathbb{R}$ let $\hat{t}$ denote the image of t in the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Choquet remarks that, although the sequence $(\widehat{k(3/2)^n})$ is uniformly distributed for almost every k, no explicit value of k is known for which this is true. To study such sequences he proposes to examine them in a wider setting, in order to be able to make use of a variety of mathematical tools.

To this end he begins by introducing some basic notions concerning morphisms of compact groups and certain operations on sets or measures associated with families of morphisms. Suppose now that θ is an irreducible rational p/q. The maps $x \mapsto px$ and $x \mapsto qx$ are morphisms of $\mathbb{T}$ and a subset X of $\mathbb{T}$ is said to be θ-stable if $qX = pX$; θ-stable measures are defined similarly. A sequence (x_n) of points of $\mathbb{T}$ indexed by an interval I of $\mathbb{Z}$ is called a θ-chain if $qx_{n+1} = px_n$ whenever $n, (n+1) \in I$. A finite θ-chain $(x_r)_{r=0}^n$ such that $x_0 = x_n$ is called a θ-cycle. Two basic algorithms, θ-series and recursive θ-games, are defined and applied to the study of θ-chains and θ-stable closed sets. Using these algorithms Choquet obtains effective procedures for constructing numbers k such that the set of numbers $x_n = k(3/2)^n$ has interesting properties. Denoting by $\|x\|$ the distance of x from the nearest integer he finds, for instance, values of k for which $\|k(3/2)^n\| \leqslant 1/3$ for all $n \in \mathbb{N}$, or for which $\|k(3/2)^n\| \geqslant 1/19$ for all $n \in \mathbb{N}$, or for which the sequence $(\widehat{k(3/2)^n})$ is dense in $\mathbb{T}$. He also proves that there exist $2^{\aleph_0}$ numbers $k > 0$ such that we have $\|k(3/2)^n\| \leqslant 3^{-1}.2^{1-an}$ for infinitely many values of n, where $a = ((\log 3/2)/\log 2)^2$.

Next, he identifies various faces and extreme elements of the cone of (3/2)-stable measures on $\mathbb{T}$. Investigating the θ-stable closed subsets of $\mathbb{T}$, he confirms a conjecture of Mendès–France by showing that if k and θ are rational, with θ and θ^{-1} not integers, then $\limsup \psi(k\theta^n) = \infty$, where for a positive rational x the symbol $\psi(x)$ denotes the length of the regular continued fraction expansion of x. Writing $X_k = \{\, \widehat{k(3/2)^n} : n = 1, 2, \ldots \}$, Choquet shows in a further examination of this set that there are at most countably many values of k for which $\overline{X}_k$ is countable and has a second derived set $\overline{X}_k''$ that is finite and non-empty.

Further results concern the Hausdorff dimension of various sets, including nowhere dense θ-stable closed subsets of $\mathbb{T}$ and the set of $k > 0$ such that $\overline{X}_k \neq \mathbb{T}$. The wealth of pairs (k, x) such that $\|k\theta^n - x\|$ gets arbitrarily small is demonstrated, and there is a detailed study of θ-cycles.

In the final note of the series the set of those $k > 0$, such that $\|k(3/2)^n\| \leqslant 1/3$, for all n is characterized as a certain set of effectively constructible numbers, and some related questions are raised.

Most of the many results in these papers are stated without proof. The papers also contain many examples and a considerable number of open problems and conjectures.

Several geometrical theorems concerning the global attractors associated with a diffeomorphism of a manifold are announced in [**163**]. Properties of the stable and unstable manifolds and of the set of homoclinic points are studied. The relation between global and pointwise attraction is illuminated by a noteworthy application of a version of the contingent–paratingent theorem (see § 2.1). Suppose, for instance that V is a connected finite-dimensional differentiable manifold and that $f \in \mathrm{Diff}(\mathrm{V})$, and let K be a global attractor for f with a basin of attraction B. For $x \in B$ we denote by $\omega(x)$ the subset of K defined by

$$\omega(x) = \bigcap_{p \in \mathbb{N}} \overline{\{\, f^n(x) : n \geqslant p \,\}}.$$

Then there exists a dense $\mathcal{G}_\delta$ subset R of B such that $\omega(x) = K$ for all $x \in R$. (See also [**161**, **162**].)

Let D denote the open unit disc in $\mathbb{R}^2$ and let C be its boundary. Let $\mathcal{H}$ be the set of all harmonic homeomorphisms of D onto itself, and let $\mathcal{M}$ be the set of harmonic extensions to $\overline{D}$ of the continuous surjections $g : C \to C$ that are such that $g^{-1}(\theta)$ is connected for each $\theta \in C$. Then in [**169**] it is proved that $\mathcal{H} = \mathcal{M}|_D$.

2.10. *Essays and addresses* [**75**, **84**, **127**, **131**, **134**, **137**, **155**, **160**, **165**, **166**, **168**, **171**, **172**, **174**]

A number of publications on a variety of subjects should be noted here, including two articles on modern mathematics and education [**75**, **127**], the article 'L'analyse et Bourbaki' [**84**], two papers on mathematical creativity [**166**, **171**], and one on the continuous versus the discrete [**168**]. Choquet also composed eulogies for Brelot [**134**, **165**] and Deny [**155**] and wrote a piece [**174**] to mark the centenary of the Lebesgue integral (see also [**172**]). His remarkable accounts [**131**, **137**, **160**] of his own mathematical development and work should not be overlooked. (Further valuable material about his life and work is to be found in his interview ⟨**53**⟩ for *Hommes de Science*.)

2.11. *Books [**97**, **98**, **101**, **111–113**, **115**, **117**, **173**]*

In writing his book [**98**] on the teaching of geometry, Choquet was concerned with the instruction of young people in the foundations of elementary geometry. He presents an axiomatic treatment based on a small number of axioms that formalize various intuitive notions derived from everyday experience, such as lines, parallelism, orthogonality, and distance. In the space of less than 140 pages he establishes the essentials of Euclidean plane geometry, and gives clear accounts of a number of difficult concepts such as angles and their measurement, and orientation. The exposition is thoroughly modern and gives due attention to relevant tools from modern algebra. The book has been published in at least eight languages.

Choquet's 1950s reform of analysis teaching in the Sorbonne has been recounted above in § 1. Notes for his course were written up shortly after the lectures and were made available in hectographed form by the Centre de Documentation Universitaire. Revised and corrected versions of these notes were subsequently published in more permanent form as [**97**, **115**, **173**]. The course consisted of three parts: Algebra, Topology, Integration and Differential Calculus. Part II of the course was the subject matter of [**97**] (in English as [**101**], and revised in [**115**]), which consists of three chapters: Topological spaces and metric spaces, Numerical functions, Topological vector spaces. Little previous knowledge is required of the reader apart from elementary calculus and simple facts about vector spaces. The material, which is oriented towards applications in analysis, is carefully motivated, well illustrated, and very clearly expounded. The judicious choice of topics makes this a valuable resource for the young analyst. In [173] many chapters from the other two parts of the course have been brought together, some material (in closed brackets below) from a more advanced course has been included, and there is additional material from lectures in the École Polytechnique. The section headings are as follows: Algèbre des ensembles, Algèbre, Nombres complexes et nombres réels, Algèbre linéaire, Équations différentielles, [Structures Topologiques, Structures uniformes, Espaces de fonctions,] Intégration, Analyse de Fourier. Unfortunately, the Sorbonne chapters on differential calculus and integration could not be included, but happily a record of the École Polytechnique course on the latter topic has survived and is included here. Although this course of analysis was inaugurated more than 50 years ago, the content is still notable for its remarkable modernity. Given that the standard fare had hitherto been based on such works as the texts of Goursat and Valiron, it is small wonder that Choquet's course precipitated a revolution in the teaching of analysis in France. However, the decision to publish this work now is not just an act of piety: it can be read with real profit by today's students.

The three-volume work [111–113] is an expanded and revised account of a lecture-course in analysis given by Choquet at Princeton in the 1967 fall term. His aim was to present the analytical tools and methods that he had found the most useful in potential theory, probability theory, and harmonic analysis, and the result is a particularly original and valuable compendium of topics. Volume I treats general topology for analysis (including some descriptive set theory), Radon measures and capacity theory, and topological vector spaces. Volume II opens with further material on topological vector spaces, followed by integral representation theory and some applications thereof, and concludes with a chapter on adapted spaces and the representation of positive linear forms. Volume III is concerned with conical measures, affine measures, and functions of negative type. Many examples and problems are provided. Something that gives these volumes special weight and interest is the fact that Choquet has made substantial personal contributions to many of the topics expounded here. In particular, it is noteworthy that in 1967 the central theorems of his integral representation theory (see § 2.3) had only recently achieved their definitive shape. Although very good concise introductions to that subject had appeared in ⟨41, 44⟩, Choquet's authoritative treatment of it in Volume II was most timely, and it remains a definitive reference.

The lecture-course [117] is an account of some set-theoretic and topological tools in analysis. It treats ordinals, Baire spaces, Borel, analytic, and $\mathcal{K}$-analytic sets, capacities, classification of functions, primitives and derivatives, set-valued functions, contingents and paratingents. The text is of particular interest as an introduction to several of Choquet's major research contributions.

Acknowledgements. The task of writing this obituary has been greatly assisted by Choquet's own remarkable 1974 account [131] of his education and work, by the record ⟨53⟩ of his interview for the book *Hommes de Science*, and by his description in [160] of the genesis of the capacitability theorem. I have also drawn on material about him in a variety of other sources: see ⟨29, 38, 49, 50, 54⟩ and especially ⟨60, 61⟩. In addition, I am very grateful to Richard Becker, Jacques Deny, Gilles Godefroy, and Ivan Netuka for invaluable comments and advice. The photograph has been reproduced, by kind permission, from the Archives of the Mathematisches Forschungsinstitut Oberwolfach.

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