EDWARD FOYLE COLLINGWOOD

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The London Mathematical Society owes much to its former president Sir Edward Collingwood who died in office on 25 October, 1970, at his home at Lilburn Tower, Alnwick, Northumberland, where he was born on 17 January, 1900. Collingwood's family goes back to before 1600 in Northumberland and he was descended from the third brother John of Admiral Lord Collingwood who fought at Trafalgar. Edward's mother comes from a Somerset family.

The naval tradition was strong in the family and Edward went to Osborne in 1913. Dartmouth in 1914 and a year later joined the navy as a midshipman in H.M.S. *Collingwood.* He was invalided out of the navy soon after joining, failed the medical for Woolwich and so found himself a student at Trinity College, Cambridge in the Michaelmas term of 1918.

At Lilburn there were letters from the admiral containing instructions to the tutor of his children that mathematics was to form an important part of their education and that he, the Admiral, was personally following the mathematical lessons being given to midshipmen on his ship. So it is not perhaps so surprising that Edward found himself studying Mathematics. At Cambridge he came under the influence of G. H. Hardy who was his director of studies, and first inspired him to do research in Pure Mathematics and in fact Edward joined our Society on 11 December, 1919, when he was still an undergraduate. At Cambridge he lived well but not ostentatiously but he mixed with many people from various walks of life and certainly his extremely wide interests in Biology, literature, economics and administration were fostered and encouraged at this stage.

Collingwood visited Aberystwyth in the lent term of 1922. W. H. Young who was head of the department of mathematics had invited G. Valiron to give a course of lectures in French on integral functions and Collingwood had the job of preparing a translation after each lecture, which was available in the library. Later in 1924–25 he held a Rouse–Ball Travelling studentship mainly at the Sorbonne, where he heard Borel lecture (unfortunately on probability, rather than Analysis as he says) and was well received by him in Borel's apartment. He also visited Strassbourg to see Valiron and to complete the book containing the Aberystwyth lectures. It was during this period that he started his research and there can be no doubt about the importance that these journeys had on the whole of Collingwood's later Mathematical work.

Collingwood obtained a Rayleigh prize in 1923 for an essay on "the formal factorization of an integral function of integral order", a year when Smith's prizes were awarded to Burkill and Ingham and other Rayleigh prizes to W. R. Dean, E. C. Francis, C. G. James and M. H. A. Newman, of whom James died young of his war injuries and Francis became a missionary. The others all became well known Mathematicians.

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He took his M.A. in 1925 and was made a member of the High Table at Trinity. In 1929 he took his Ph.D. with Littlewood as his supervisor and in the same year began giving lectures to the Mathematical Faculty, usually two advanced courses on Integral and Meromorphic functions in the Lent and Easter terms. He also became Steward of Trinity in 1930 and his general ability was so highly thought of that, although not a fellow, he was elected to the Council. During this period he supervised A. J. Macintyre, who later became a Senior Lecturer at Aberdeen and Professor at Cincinnati.

Dame Mary Cartwright pays tribute to the influence his lectures had on her work and so indirectly on mine and the great trouble that Collingwood took as a referee of some of her early papers. This layed the foundation of what was to become a most fruitful collaboration later on.

Even while at Cambridge Collingwood had been much involved with the family estate at Lilburn, which he took over in 1928, and the social life that went with his position there. He had for instance been a Lieutenant in the Northumberland Hussars (Yeomanry) from 1923–27 and became a J.P. in 1935 and later he was chairman of the bench for many years. In 1937 he became High Sheriff of Northumberland and his life as part of Cambridge ended, although he continued to visit his college often.

Collingwood was not able to enjoy for long a quiet life on his estate. When the war came he became a naval scientist in the Mine Sweeping division under Sir Edward Bullard. He became Chief Scientist of the Admiralty mine design department 1943 having acted as liaison officer between the US and British Navies in Washington in 1942. He was awarded the Legion of merit, degree of officer, U.S.A. in 1946. The value of his work was also recognised here and he became one of very few full Captains in the RNVR in 1944.

After the war Collingwood's sound financial sense, administrative ability and sense of public service were increasingly in demand. He made three careers, each of which would have satisfied most men, in University administration medical administration and mathematical research. In the first he was Chairman of the Council of Durham University from 1955 until his death. This period included the time when Durham and Newcastle became separate universities.

In Medicine he was Chairman of the Newcastle regional hospital Medical Board from 1953–68. He was Treasurer of the Medical Research Council from 1960–67, and Chairman of the Central Health Services Council from 1963–70.

Finally Collingwood's research in Mathematics—which will be discussed in more detail later—led him to Mathematical conferences all over the world. It was here that I really got to know him. He was always surrounded by a group of eager young people discussing problems, often raised by himself, listening a lot, but saying relatively little, except in his formal lectures. He usually had a car, either his own, or on hire and was always willing to take us out on trips, where Mathematics and amusement was combined. I vividly remember a visit in which he took Ganelius and me to the zoo at San Diego from the La Jolla Conference in 1966. Afterwards Ganelius sent me a photograph of Collingwood and me standing next to a flamingo, with the words "had you forgotten what distinguished company you kept?" This occasion was just one instance of the "Collingwood Taxi Service".

As Treasurer and later President of The Society he was largely responsible for the state of relative prosperity in which we now find ourselves. He made excellent use of the Hardy bequest, which came during this period.

Collingwood was awarded the C.B.E. in 1946 and he was knighted in 1962. He was given an Hon. D.Sc. of Durham University in 1950, and the Sc.D. of Cambridge University in 1959. He was elected a Fellow of the Royal Society in 1965.

It is difficult to do justice to Edward Collingwood in a short article. The fact the he held no paid post during most of his life, and that he was unmarried, enabled him to know a large number of people from many different circles which do not usually intersect. One catches occasional glimpses of these more private contacts. When a friend was ill in St. George's Hospital, he took the trouble to speak to the Matron whom he had met on a medical committee to make quite sure that his friend was well looked after. When a quotation from Jane Austen arose in the conversation he could cap it exactly and very much to the point. He was a fine dancer. In concluding this assessment of Collingwood as a man I feel I cannot do better than to quote some words of Sir George Godber in the *Lancet*, 31 October, 1970.

"He was a friend and adviser to many of us, always sound in judgement and full of common sense. He managed to be an academic of the highest standing, but he was above all a guide to the practical solutions of problems of ordinary life. He was among the foremost mathematicians of his generation. His shrewdness, humour, unfailing understanding, and support will be deeply missed by a very wide circle of friends".

II

Collingwood's mathematical output falls naturally into three periods. In the first period from 1924–32 he published a series of nine short papers, which contain some of his most significant work mainly on integral and meromorphic functions in the plane.

He published nothing from 1932–48, when he again took up some of his early ideas which had meanwhile been improved by others notably H. Selberg. From 1948–52 he published a group of eight papers based on this inequality of Selberg [1948, 1949 and 1952, b–d].

Finally [1952e] he published a fundamental paper with M. L. Cartwright in which the theory of cluster sets was really put on the map. This subject continued to occupy him until his death and in it his contributions were fundamental. I would now like to describe the work of these three periods in more detail.

1. After he came to Cambridge, Collingwood was strongly influenced by Hardy and Littlewood. I have mentioned already his visit to Aberystwyth in 1922, when he met Valiron and how later in 1924–25 he went to the Sorbonne with a Rouse–Ball travelling studentship. At this time R. Nevanlinna had just begun to create the theory of meromorphic functions which bears his name, a creation which was perhaps

the most important single event in function theory in the present century. If $f(z)$ is meromorphic in the plane, i.e. regular except for poles we write

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(r\exp(i\theta))|\, d\theta,$$

$$N(r,f) = \int_0^r n(t,f)\frac{dt}{t}, \quad f(0) \neq \infty,$$

where $n(t,f)$ is the number of poles of $f(z)$ in $|z| \leqslant t$. Then the first fundamental theorem states that

$$T(r,f) = m(r,f)+N(r,f) = m\left(r, \frac{1}{f-a}\right)+N\left(r, \frac{1}{f-a}\right)+O(1), \qquad (1)$$

as $r \to \infty$ for any fixed finite a. The second fundamental theorem states that for any $p \geqslant 2$ distinct finite values of a

$$\sum_{\nu=1}^{p} m\left(r, \frac{1}{f-a}\right) \leqslant 2T(r,f)-N_1(r)+S(r), \qquad (2)$$

where

$$N_1(r) = N(r,f)-N(r,f')+N\left(r, \frac{1}{f'}\right),$$

counts the multiple values of $f(z)$, including multiple poles and $S(r)$ is in general a small error term. In this connection I should now like to quote from a letter of Professor Nevanlinna's, which I have translated from the German.

"I had in (1923) extended Picard's theorem by obtaining the inequality (2) for an integral function and $p = 2$. In summer 1924 I received a letter from Littlewood in which he asked me why I had not used the same method to prove the result for p (finite) values. I had missed this possibility since I had been so fascinated by the number 2 of Picard—values through Picard's theorem, so that the extension from 2 to p had not occurred to me. In the autumn of 1924 it was pointed out to me that Collingwood, who was at that time completely unknown to me, had seen the same possibility of obtaining (2) for general p and had published it in a C. R. note [1924a]. I regard this as possibly his greatest achievement."

However Collingwood's achievement did not rest there. He also discussed values a for which

$$\overline{\lim_{r\to\infty}} \frac{N(r, 1/(f-a))}{T(r)} \leqslant \theta < 1. \qquad (3)$$

and noted that in view of (2) there can be at most $1/(1-\theta)$ such values for fixed θ· Values satisfying (3) were later called defective or deficient (defekt) by Nevanlinna. But this notion was first introduced by Collingwood. In [1924, b] Collingwood proved that a defective value is either asymptotic or a limit of algebraic critical points and raised the question of whether a defective value is necessarily asymptotic. This question was raised also by Nevanlinna and was answered in the negative finally by Arakelyan even for integral functions of finite order, which may have countably many defective values but only finitely many asymptotic values.

Collingwood also pointed out in [1924b] that values a may be replaced by polynomials $a(z)$ in (2) by a suitable use of the qth derivative.

The rest of Collingwood's early papers although also interesting cannot compare with [1924a and b] which had such a far reaching effect on the subsequent theory of meromorphic functions. Some of the results were not published in full since they were overtaken by other authors notably Nevanlinna himself. Thus in [1930] Collingwood states that

$$m\left(r, \frac{1}{f-a}\right) = O(\log r), \tag{4}$$

when f is an integral function of finite order and a lies outside a set of linear measure zero. This result had meanwhile been published by Nevanlinna in his Borel tract in a slightly extended form but it remains to-day the strongest result that is known to be true outside a small set as far as I am aware, and that is what Collingwood aimed to do in proving (4). In [1932] there is a rather pretty proof of the convexity theorems for the maximum modulus and pth means.

2. Collingwood published no mathematics between 1932 and 1948, although he did valuable work on various forms of magnetic and pressure mines during the war. In the meantime Collingwood's result in [1924b] had been notably sharpened by Teichmüller and Selberg†, who proved that it is sufficient for a to be defective that $f(z)$ is at most p-valent in each island $|f(z)-a| < \sigma$, when σ and p are constant. Collingwood generalised this result by allowing σ to tend to zero and p to tend to infinity with the distance of the island from the origin. It seems that he was still hoping in this way to prove his old conjecture that at least for integral functions of finite order defective values are asymptotic, so that such a function can have only a finite number of such values. The general view is probably that this group of papers is less effective than the other two. However Collingwood made propaganda at various times for Selberg's fine result and the seed fell on good ground in the form of Weitsman who recently used a refinement of this result to prove‡ that for meromorphic functions of finite order

$$\Sigma\delta(a, f)^{\frac{1}{3}} < \infty,$$

where

$$\delta(a, f) = 1 - \overline{\lim}\, \frac{N(r, 1/f-a)}{T(r, f)}.$$

† Eine Ungleichung der Potentialtheorie und ihre Anwendung in der Theorie der meromorphen Funktionen, *Comment. Math. Helv.*, 18 (1946), 309–326.

‡ *Acta Mathematica*, 128 (1972) 41–52.

This result, which is best possible, had been conjectured for about 10 years and would probably not have been proved so soon without the discussions between Weitsman and Collingwood in June 1970. This illustrates the way in which Collingwood inspired other mathematicians right up to his death.

3. Finally we come to the theory of Cluster sets, which is probably the subject for which Collingwood is known best. Suppose that $f(z)$ is a function in $|z| < 1$ with values on the Riemann sphere. We say that w belongs to the *cluster set* C of $f(z)$ if there exists a sequence z_n, such that

$$|z_n| \to 1, \quad \text{and} \quad f(z_n) \to w.$$

If in addition

$$z_n \to \xi = \exp(i\theta),$$

we say that $w \in C(f, \xi)$. If further z_n lies on a set such as a radius ρ or a Stolz angle Δ, i.e. a triangle lying except for one vertex ξ in $|z| < 1$, the corresponding cluster set is denoted by $C_\rho(f, \xi)$, $C_\Delta(f, \xi)$ etc. A point such that $C_\Delta(f, \xi)$ reduces to a single point for every Δ at ξ is called a *Fatou point*.

The paper by Collingwood and Cartwright [1952e] probably represents the beginning of the subject. It is too large to summarise here but we may permit ourselves to quote one result on which the authors lectured at the International Congress in Havard [1952a].

THEOREM 1. *If f is meromorphic in $|z| < 1$ and omits 3 values near $\xi = \exp(i\theta_0)$, then ξ is a limit of Fatou points.*

This theory of cluster sets and related matters occupied Collingwood until his death. He collaborated with a number of authors (Cartwright, Lohwater and Piranian) and wrote a book with Lohwater which is a standard text book on the subject [1966c]. It would not be worthwhile to quote in detail results from all these papers since in many cases Collingwood and his collaborators improved their results considerably between 1952 and 1967. In one case [1956b and 1957a] Collingwood and Lohwater used the new technique to extend some of the results of the second group to functions in the unit disk. However I would like to mention one series of results which refers to functions in the unit disk which need not even be continuous and yet helped to solve an outstanding problem in the theory of conformal mapping.

We refer back to the general definitions above. We also need the left and right boundary cluster sets

$$C_{Bl}(f, \xi) = \bigcap_{\eta > 0} \overline{\bigcup_{\theta_0 - \eta < \theta < \theta_0} C(f, \exp(i\theta))},$$

$$C_{Br}(f, \xi) = \bigcap_{\eta > 0} \overline{\bigcup_{\theta_0 < \theta < \theta_0 + \eta} C(f, \exp(i\theta))},$$

and the boundary cluster set

$$C_B(f, \xi) = C_{Bl}(f, \xi) \bigcup C_{Br}(f, \xi).$$

Here $\bar{E}$ denotes the closure of the set E.

It was discovered by Collingwood and his co-workers that in certain senses to be made more precise below, the various cluster sets obtained in this way are the same for most values of θ. There is also a tendency for cluster sets to be total, i.e. to consist of the whole closed sphere, which corresponds to Weierstrass's theorem on isolated essential singularities, or else to reduce to a single point, in which case f has a limit as $z \to \xi$, radially, in angles, or globally as the case may be.

We now come to measures of smallness for infinite sets E on $|z| = 1$. The most stringent measure is for such a set to be countable. Another criterion is for E to have *measure zero*, and a third for E to be a set of *first category*. To explain this latter condition, we say that a set E is nondense if every arc γ of $|z| = 1$ contains a subarc γ', which does not meet E. A set of first category is one which is the union of a sequence of nondense sets. Other sets are said to be of second Category. The complement of a set of first Category is called *residual*. No arc can be of first Category so that a residual set has non-countably many points on every arc. In many ways a set of first category may be considered small, and a residual set large, but this criterion does not agree well with that of measure. A residual set may have measure zero and a set of first category measure 2π, so that $|z| = 1$ can be decomposed into the union of two sets, each of which is "small" in one of these two senses. A countable set of course has zero measure and first category so that it is small in both senses.

We are now in a position to state a number of Collingwood's striking results.

THEOREM 2 [1960c]. *Let f be an arbitrary function in $|z| < 1$ with values on the closed sphere. Then*

(a) *The left and right boundary cluster sets are equal to each other and to the cluster set for all ξ on $|\xi| = 1$ outside a countable set.*

(*b*) *For all ξ on a residual set we have $C_\Delta(f, \xi) = C(f, \xi)$ for all Stolz angles Δ; i.e. all angular cluster sets at ξ are equal to the cluster set.*

THEOREM 3 [1958c]. *Let λ be an arc lying in $|z| < 1$ except for one end point at $z = 1$. Let $\lambda(\theta)$ be the path obtained by rotating λ through an angle θ around the origin. Then if f is continuous in $|z| < 1$, we have for a residual set of values $\xi = \exp(i\theta)$, $C_{\lambda(\theta)}(f, \xi) = C(f, \xi)$, where $C_{\lambda(\theta)}(f, \xi)$ denotes the cluster set of f as $z \to \exp(i\theta)$ on $\lambda(\theta)$.*

In particular the radial cluster set $C_\rho(f, \xi) = C(f, \xi)$ for a residual set of values ξ.

These results are best possible, as is shown by two examples. Let $\Pi(z)$ be a Blaschke product whose zeros z_n have every point on $|z| = 1$ as a limit-point. Then 0 belongs to the cluster set for every ξ on $|\xi| = 1$. But $\Pi(z)$ has angular limits of modulus 1 on a set E of measure 2π on $|\xi| = 1$, so that the radial cluster sets $C_\rho(f, \xi)$ and angular cluster sets $C_\Delta(f, \xi)$ do not contain zero for ξ on E. Thus in this case the "exceptional" set in Theorem 2(b) and Theorem 3 has measure 2π.

Again let z_n be a sequence of points as above, with the additional proviso that no radius contains more than one of the points z_n. Let $f(z) = 0$, $z \neq z_n$, $f(z_n) = 1$, $n = 1, 2, \ldots$. Then $f(z)$ is upper semi-continuous in $|z| < 1$, but the radial cluster set is 0, while the cluster set consists of the two points $\{0, 1\}$ for every ξ on $|\xi| = 1$.

Thus if continuity is replaced by semi-continuity in Theorem 3 the corresponding set of ξ may be empty.

It should be said that Theorem 2b was also discovered independently by Dolzhenko in the Soviet Union and by Erdös and Piranian at about the same time. Collingwood also acknowledges [1960c] his debt to W. H. Young, who to some extent anticipated Theorem 2 about half a century earlier. Collingwood writes "Perhaps for lack of a suitable terminology and notation to give point to the ideas, Young's theorems attracted little notice, and so far as I can discover, have not hitherto been mentioned by writers on complex function theory." We are indeed indebted to Collingwood and his collaborators for seeing what striking consequences these sort of results can have in function theory. Here are some examples.

Suppose that f is meromorphic in $|z| < 1$ and has radial limit zero at every point of some arc γ of $|z| = 1$. Then it follows from Theorems 3 that f tends to zero as $z \to \xi_0$ in any manner for some ξ_0 on $|z| = 1$, so that f is bounded in a neighbourhood of ξ_0. It then follows easily from classical theorems that $f(z) \equiv 0$. This conclusion contained in [1954c, Theorem 1] extends an old theorem of Privalov from regular to meromorphic functions.

Finally we give some applications to conformal mapping. Suppose that $f(z)$ is regular and univalent in $|z| < 1$, so that $f(z)$ maps $|z| < 1$ onto a simply connected domain D. In this case the points ξ on $|\xi| = 1$ correspond to prime ends in D, and the theory of cluster sets can greatly illuminate the study of these prime ends.

It turns out that the radial cluster set corresponds to the set of principal points ξ, i.e. the points near which every curve must go which goes into the prime end (p). In addition there are left and right subsidiary points (l, r) which a curve going into the prime end may approach. Now $p \cup l$, $p \cup r$ correspond to the so called left and right cluster sets (which contain the left and right boundary cluster sets defined above).

It now follows from Theorem 2, that $l = r$ outside a countable set [1961c, Theorem 2]. The authors also show that any countable set on $|z| = 1$, can be the set of asymmetric prime ends, where $l \neq r$.

An equally interesting application can be made of Theorem 3. This shows at once that prime ends without subsidiary points correspond to a residual set on $|z| = 1$. In particular such prime ends are everywhere dense in the boundary. This result [1956a, Theorem 4] solved an old problem of Caratheodory. Some examples of domains each of whose prime ends contains either one principal point and some subsidiary points or several principal points and no subsidiary points are given in [1959b].

In the last few years Collingwood became interested in Tsuji functions [1964a, 1966a, 1968a]. These are functions meromorphic in the unit disk, with the property that they map each circle $|z| = r$ onto a curve $\gamma(r)$ on the Riemann sphere, whose length is bounded for varying r. A point ξ on $|z| = 1$ is called a Julia point if $f(z)$ assumes all values with at most two exceptions infinitely often in every Stolz angle Δ with vertex at ξ. In [1964a] the authors show among other things that meromorphic Tsuji functions exist for which every point on $|\xi| = 1$ is a Julia point. They also raised a number of interesting questions which provided me with material for a couple of subsequent papers. Then in [1968a] Collingwood showed that for a Tsuji function $f(z)$ almost all points of $|z| = 1$ are either Fatou-points (so that $f(z)$ has a limit in every angle) or Julia points. The result extends for this class of functions an older result of Plessner.

Collingwood thought long and deeply about problems and was a master in the art of obtaining highly significant results by a series of apparently small steps. Results such as Theorems 2 and 3 above with their consequences will surely give him a permanent place in mathematics.

The Mathematical part of this obituary is essentially the same as one being published in the biographical memoirs of the Royal Society. However, the biographical part is a shortened version of the corresponding part written for the Royal Society by Dame Mary Cartwright. I am most grateful for being allowed to use this material. Among Sir Edward's other friends, I should like to express my particular indebtedness to Professor A. J. Lohwater, Professor R. Nevanlinna, Sir George Godber and Mr. John Buckingham. W.K.H.

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