JACOB LIONEL BAKST COOPER, 1915–1979

facing p. 429

JACOB LIONEL BAKST COOPER

Jacob Lionel Bakst Cooper, Professor of Pure Mathematics at the University of London, Chelsea College, and one of the leading analysts in Britain, died on 8 August, 1979 in London, having been unconscious since a heart operation on July 23. He had been a member of the Society since 1944, and made notable contributions to operator theory, Fourier analysis, integral transform theory and thermodynamics.

Lionel Cooper was born on December 27, 1915 in Beaufort West, Cape Province, South Africa. His father Isaiah had a farm at Nelspoort, some thirty miles away, and when he died in 1919 the family moved into Beaufort West. Shortly after Lionel's seventh birthday they moved again, this time to Cape Town, where his mother Frances (née Bakst) thought the educational opportunities would be better for him, and there he, his younger sister and Frances lived with his maternal grandparents. The home was comfortable and cultured, his grandfather being a rabbinical scholar and his grandmother widely read. Even before the family moved to Cape Town indications of Lionel's ability were apparent. He used to go out onto the roof of their house late at night and watch the stars, learning all the constellations in the Southern hemisphere; later, when a student at Oxford, he learnt about the Northern sky with the same enthusiasm.

From 1924 to 1931 he attended the South African College School in Cape Town, and when in his final year there he took the matriculation examination (the University entrance examination in those days) he attracted the attention of the principal Mathematics examiner, Professor Philip Stein of the University of Natal, who asked to meet Lionel as he had never before seen such a brilliant paper. He entered the University of Cape Town in 1932 and did outstandingly well, obtaining first class honours in all the nine courses he took and winning numerous prizes, including the Governor-General's prize for Pure Mathematics, the Darter prize for Applied Mathematics and the Bartle Frere prize for History; finally he was awarded the B.Sc. degree with first class honours in Mathematics and Physics. Besides all this he found time to take an active part in student politics, and became a socialist member of the Students' Parliament, with strong views against apartheid and Nazism. There were many German–Jewish refugees in Cape Town then, and through his friendship with one of them he became fluent in German. He had a gift for languages, and later taught himself French, Italian and Russian.

He came as a Rhodes Scholar to The Queen's College, Oxford in 1935 and read Mathematics. In this more demanding environment awards continued to come his way: in 1937 he graduated with first class honours and distinction in the theory of functions, following which he was awarded the Senior Mathematical Scholarship and the Johnson Memorial Prize of the University of Oxford, and won a research scholarship at Queen's to enable him to work under Titchmarsh; the D.Phil. degree was awarded to him in 1940 for his thesis 'Theory and applications of Fourier integrals'. His work made an immediate impact, and as early as January, 1939 he was invited to speak at Hadamard's seminar in the Collège de France.

At Oxford he had the great good fortune to meet Kathleen Dixon, who was reading History, and they were married in June, 1940. Their marriage was a long and exceptionally happy one, and it is remarkable that all four of their children (Barbara, Frances, David and Deborah) read Mathematics at University.

[BULL. LONDON MATH. SOC., 13 (1981), 429–450]

When war came in 1939, like most young men of his generation he volunteered for the armed services (the Fleet Air Arm), but was turned down because of his poor eyesight and thus decided to take a job with the Bristol Aeroplane Company, working on projects connected with the war effort. He stayed at Bristol from 1940 to 1944, when he was allowed to leave to take a Lectureship in Mathematics at Birkbeck College, London, which in those days was open for teaching only at the weekends. A part-time weekly teaching post at Imperial College for a term or so helped to lessen the impact of the considerable reduction in salary he had taken on leaving Bristol. He remained at Birkbeck until 1951, being appointed Reader in Applied Mathematics in 1948. Those early years in London were productive and rewarding for him: it was then that he wrote two of the three important papers [**6**], [**7**], [**9**] on operator theory which are cited in many books on functional analysis and which led to the award of the Junior Berwick Prize by the Society in 1949; and his growing correspondence included two letters from Albert Einstein in 1949 concerning possible logical inconsistencies in quantum mechanics.

In 1951 he was appointed Professor of Mathematics and Head of the Mathematics Department at University College, Cardiff, and for the next few years the greater part of his energy went into the considerable task of reorganising and reorienting the Department. The existing courses in Pure Mathematics there were of a classical nature, and it was entirely due to his efforts that they were improved and modernised, the end product being an attractive blend of forward-looking courses with functional analytic methods given some prominence. He brought research more to the forefront by giving advanced courses and seminars and before long the climate in the Department was transformed. A break from this activity came in 1954, when he interchanged with Philip Stein and spent some time in the University of Witwatersrand as Visiting Professor and Acting Head of Department. Back in Cardiff he played a full part in College and University affairs, being Dean of the Faculty of Science from 1956 to 1958 and a member both of the Council and Court of the Governors of the College and of the Court and Academic Board of the University of Wales; he was a prominent member of the Commission set up in the 1960's to enquire into the future organisation of the University of Wales. The links between the College and local schools were assiduously fostered by him, and he organised numerous highly successful refresher courses for teachers of mathematics.

These activities on behalf of the College and the University were not achieved without considerable personal cost, for they effectively cut off his research at its most promising stage, and from 1953 it was not until 1960 that research papers began to appear again, first one on positive-definite functions [**24**] and then a series of fundamental contributions to Fourier analysis [**27**], [**28**], [**29**]. In 1963 he made the first of many visits to Oberwolfach, and participated in a conference on Approximation Theory organised by Professor P. L. Butzer, with whom he struck up a particularly happy relationship. He spent the academic year 1964–65 in Pasadena as Visiting Professor at the California Institute of Technology, and during this period his feeling that it was time for a move came to a head and led to his resignation from his chair at Cardiff in order to accept a full professorship at the University of Toronto. He settled down quickly and easily in Canada, where his sharp intellect was much appreciated; he was Editor of the *Canadian Journal of Mathematics* from 1965 to 1967. However, the pull of England, and London in particular, was too much for him to resist, and he left Toronto in 1967 to become Professor of Pure Mathematics and Head of the Department of Mathematics at Chelsea College.

At Chelsea he made an immediate impact on his department and on the College, which was emerging from its former status as a Polytechnic; soon he was in the thick of the British mathematical scene once more. He served on the Mathematics Panel of the University Grants Committee from 1971 to 1975, and his interest in all aspects of education led him to accept membership of many committees. In 1974 he organised a highly successful Science Research Council Rencontre on Differential Equations at Chelsea, the theme being the interaction between Functional Analysis and Differential Equations. Throughout this period he maintained his contacts with Canada and made three visits there. He was also a leading participant in the triennial conferences on Functional Analysis, Approximation Theory and related topics at Oberwolfach organised by Paul Butzer from 1965 onwards, and took part in all but the first of these.

Lionel Cooper did much hard work for the Society: he was Editor of the Proceedings from 1952 to 1959 and served on the Council from 1949 to 1961. In 1959 and 1960 he took part, on behalf of the Society, in the negotiations with DSIR to arrange for the translation of the mathematical periodical *Uspekhi Matematicheskikh Nauk* from the Russian, and from 1961 to 1963 was Editor of *Russian Mathematical Surveys*, in which these translations appeared. Another considerable service which he performed for the mathematical community in this country was as one of the organisers of the Instructional Conference on Functional Analysis held in University College, London in April, 1961. This was the first of the Society's Instructional Conferences and was a great success.

As a lecturer he could be hard to follow: sometimes the sequence of ideas came too quickly for the comfort of those in the audience with less agile minds; sometimes he overestimated the background knowledge of his audience. However, many of his lectures were enormously stimulating and were full of unexpected insights into the topics being studied. To students he was invariably helpful, and the sweep of his knowledge was so great that he could be relied upon to suggest a fruitful line of attack upon a problem; he had numerous research students, notably E. Benham-Dehkordy (Iran), D. E. Davies (Farnborough), B. P. Duggal (Nairobi), R. E. Edwards (Canberra), C. E. Finol (Venezuela), G. G. Gould (Cardiff), F. Holland (Cork), M. B. Sadiq (Iran) and J. D. Stewart (McMaster). Many others benefited from his advice: Professor Butzer testifies in his very moving tribute to Lionel (*Jacob Lionel Bakst Cooper—In Memoriam*, *Functional Analysis and Approximation*, Proc. Oberwolfach Conf. 1980 dedicated to J. L. B. Cooper (P. L. Butzer, E. Görlich and B. Sz.-Nagy, Eds.) ISNM 60, Birkhäuser Basel 1981) to the great assistance he (and his students) received from him over many years. For my part I am happy to acknowledge the profound influence he had upon me, which was largely responsible for the shape my career has taken and for my interest in functional analysis and partial differential equations.

Outside mathematics, Lionel had many interests. He loved music and the theatre, enjoyed poetry and was exceptionally widely read; the force of his intellect was undeniable. A delightful account is given by Alan Hill of this side of Lionel (A. J. W. Hill, *A Testimony from a Friend*, published with Professor Butzer's tribute mentioned above). When younger he was a keen squash player, and after giving this up he continued to swim, to play tennis with great determination and to walk, especially in the Lake District which he loved and knew so well. In public he appeared reserved or even shy, and there was certainly nothing at all ostentatious in his make-up; underneath the reserve, however, there was a quiet self-confidence which enabled him to assess problems dispassionately and on their merits. It took time to know him, but those who became close to him found him absolutely reliable, a tower of strength in difficult times and always a marvellous companion. He was an excellent after-dinner speaker, with a real flair for story-telling; this never failed to surprise those who had seen him on less festive occasions. But he was at his very best with his family. Wherever they were, Kathleen and Lionel had always gone to great pains to entertain students, faculty and visitors, and one's problems were soon placed in perspective in the warm family atmosphere which they created at such gatherings. Some of my happiest moments have been spent with them and their children, with conversation swirling about, everyone talking at once, all shades of opinion being projected; yet the overwhelming impression produced was that of a family deeply united and in harmony. Seldom have I seen so remarkable a family, with such deep affection for one another.

In 1978 Lionel became alarmingly breathless while walking in the Lake District. After a variety of checks it was discovered that he had a heart defect and that surgery was essential. He faced up to this calmly, telling me wryly that the medical problems involved were interesting, but perhaps a little too interesting. Having waited until all his examination duties were over he went into hospital, there being every reason to believe that he would make a full recovery. In the event he suffered a massive haemorrhage shortly after the operation and never recovered from this.

Lionel Cooper's mathematical qualities are intertwined with his personal ones; those who knew him best admired his ability to get to the roots of a problem, his sense of justice and humanity, his calmness and his energy and determination in overcoming difficulties. He was the most modest of men, entirely free of self-importance. Britain has lost a distinguished analyst who did much to further the cause of his subject. It was a privilege to know him.

Contributions to Mathematics

Lionel Cooper worked on diverse but inter-related areas of mathematics, and an account of his work will now be given by mathematicians working in the different fields:

B. Sz.-Nagy—Operator Theory,

P. L. Butzer and R. J. Nessel—Transform Theory.

The section on Thermodynamics is adapted from a manuscript which J. Serrin has kindly given me.

His work on Functional Analysis, on Differential Equations and on miscellaneous topics I have summarised myself. Items occurring in the lists of additional references in the various sections are marked by ⟨ ⟩, while Cooper's papers will throughout by denoted by [ ].

J. L. B. Cooper and Operator Theory

B. Sz.-Nagy

One of the main areas of functional analysis to which Cooper contributed with particularly original ideas is the theory of (linear) operators in (real or complex) Hilbert space.

In the first period of the theory, namely the first quarter of this century, continuous—and therefore bounded—operators only were considered. The main result obtained at that time was the spectral resolution theorem for bounded symmetric operators and the corresponding spectral multiplicity theorem. But with the advent of quantum mechanics, in the late 1920's, it became evident that a limitation to *bounded* symmetric operators did not comply with the mathematical needs of that theory. This stimulated a rapid development of the theory of non-necessarily bounded operators, in particular by J. von Neumann, M. H. Stone and F. Riesz. The first problem was to single out those operators A which correspond to the needs of quantum mechanics. It was accepted that one of those needs was that the operator A be symmetric on its (linear) domain of definition, not necessarily filling the whole Hilbert space, and that it should have a 'spectral resolution',

$$A = \int_{\mathbb{R}} \lambda dE_{\lambda},$$

where E is an orthogonal projection-valued Borel measure on $\mathbb{R}$, the vectors x of the domain of definition of A being exactly those for which the numerical integral

$$\int_{\mathbb{R}} \lambda^2 d(E_{\lambda}x, x)$$

converges, its value then being (Ax, x).

It was J. von Neumann who, following a suggestion of E. Schmidt, first proved that the symmetric operators having the above property are exactly the self-adjoint operators, i.e. those for which $A = A^*$. Symmetry only implies that $A \subset A^*$, and while every (densely defined) symmetric operator B has at least one maximal symmetric extension, it can happen that none of these extensions is self-adjoint.

For the spectral resolution of a self-adjoint operator there exists nowadays numerous proofs. The original one, by J. von Neumann, reduces the problem *via* the use of 'Cayley transforms' to the (relatively simple) problem of the spectral resolution of unitary operators. Namely, von Neumann observed that the problem of finding all symmetric extensions of a symmetric operator A is equivalent to that of finding all isometric extensions of the (isometric) Cayley transform V of A, where $V = (A - iI)(A + iI)^{-1}$, and that A is self-adjoint if and only if V is unitary (that is, an isometric map of the Hilbert space onto itself). Since a unitary V is known to have a spectral resolution $V = \int_0^{2\pi} e^{it} dE_t^{(V)}$, one infers, *via* the inverse Cayley transform of V, that is, $A = i(I + V)(I - V)^{-1}$, that A also has a spectral resolution, namely of the form

$$(*) \qquad A = \int_{-\infty}^{\infty} \lambda dE_{\lambda}^{(A)},$$

the spectral (i.e. orthogonal projection-valued) measures $E^{(A)}$ and $E^{(V)}$ being simply related.

Other proofs of the same result, given by M. H. Stone, F. Riesz, etc., used a variety of different techniques, but none of these entered into a discussion of whether the existence of a spectral resolution (*) is the only motivation, from the point of view of physics, of the exceptional role of self-adjoint operators among other types of operators, or least among the symmetric operators.

In his paper [6], Cooper starts with the objection that the self-adjointness condition "does not convey any direct physical notion at all" and he proposes a different approach. Instead of taking, as a point of departure, the Schrödinger eigenvalue equation (which leads, mathematically, to the spectral resolution problem), he considers the time-dependent Schrödinger equation

$$(\substack{*\\*}) \qquad \frac{1}{i}\frac{d\psi}{dt} = H\psi\,, \qquad \psi(0) = \phi$$

for an arbitrary (closed) symmetric operator H, with domain of definition D in a Hilbert space R and subject to the following condition:

(a) for every element ϕ of a linear sub-manifold D' of D, dense in D, $(\substack{*\\*})$ has a solution $\psi(t)$ on the whole real line $(-\infty < t < \infty)$.

Then he shows in a direct way, partly by a method due to Titchmarsh, but using the real-variable theory of Fourier–Stieltjes integrals instead of Laplace integrals, that the operator H has a spectral resolution (*). He also shows that, conversely, every self-adjoint operator H satisfies condition (a).

In such a way his method also yields the von Neumann theorem of spectral resolution of self-adjoint operators. This way may appear somewhat lengthier, but is undeniably of great interest, both from the physical and mathematical points of view.

Still more important is that the method also applies to symmetric operators of more general type, as shown by Cooper in [9]. Indeed, he proves there (Theorem I) that equation $(\substack{*\\*})$ has a solution $\psi(t)$ for every maximal symmetric operator H and for every $\phi \in D$, either for all $t \geqslant 0$ or for all $t \leqslant 0$. In the self-adjoint case, $U(t)\phi = \psi(t)$ defines the usual Stone group $U(t) = \exp(iHt)(-\infty < t < \infty)$, and this leads again to spectral theory. For maximal symmetric operators, $U(t)\phi = \psi(t)$ defines only a semigroup of isometries $\{U(t)\}_{t \geqslant 0}$ or $\{U(t)\}_{t \leqslant 0}$, according to the above cases. Investigation of these semigroups led Cooper to all the essential results on maximal symmetric operators, obtained originally by von Neumann by means of the Cayley transform, and he points out that here the physical interpretation of the quantum mechanical requirement that the operator H in Schrödinger's equation be self-adjoint is immediately seen to be equivalent to the physical statement that the system has both a complete past $(-\infty < t \leqslant 0)$ and a complete future $(0 \leqslant t < \infty)$.

These investigations were continued by Cooper in [7]. Whereas the essential result of the preceding paper was to show that a maximal symmetric operator is the infinitesimal generator of a semigroup of isometric operators, this paper handles the converse problem and shows that every one-parameter semigroup $\{U(t)\}_{t \geqslant 0}$ of isometric operators, depending in a measurable way on t, and with $U(0) = I$, leads to a maximal symmetric operator H as infinitesimal generator, that is, for which $-i\dfrac{d}{dt}U(t) = H\,U(t)$ $(t \geqslant 0)$. One obtains H by forming the integral $A = \displaystyle\int_0^\infty U(t)e^{-t}dt$, and then the graph of H is the set of pairs $\{iAh, Ah-h\}$, h

running through the elements of the underlying Hilbert space. The same ideas were further developed in his subsequent paper [**17**].

These papers of Cooper in operator theory can be regarded as pioneering pieces of work. Although strongly attached to the basic work of F. Riesz, J. von Neumann, M. H. Stone and others, his work was among the first which—still remaining close to problems posed by physics—pointed out the necessity and fruitfulness of investigations of non-self-adjoint and non-unitary operators. His work had notable influence on the later development of operator theory.

It is not the place here to examine the whole *oeuvre* of J. L. B. Cooper in functional analysis, its broad spectrum and originality, but his scientific strength and devotion is perhaps truly and convincingly mirrored by the very valuable piece of work which he did in operator theory.

Cooper's Work in Transform Theory

P. L. Butzer and R. J. Nessel

1. *Introduction*

Transform theory is one of the central parts of the work of J. L. B. Cooper. His first paper submitted for publication, on 18 May, 1938, dealt with this fundamental area of mathematics, and he returned to this topic again and again, writing at least twenty papers on it. Cooper did not really treat transform theory from the (operational) point of view of the applied mathematician—a field for which Britain is especially well known—but always in the spirit of the pure mathematician. Indeed, Cooper was a doctoral student of E. C. Titchmarsh (1899–1963) at Oxford, and therefore Titchmarsh's classical treatise on Fourier Integrals of 1937 (cf. ⟨**21**⟩) was no doubt his initial guiding star. Titchmarsh in turn was G. H. Hardy's (1877–1947) first student, and it was Hardy and J. E. Littlewood (1885–1977) who built up the great British school of analysis, second to none in the world. So Cooper had the fortune to grow up in this great tradition. On the other hand, already Cooper's early work on transform theory is permeated with the ideas of the theory of linear operators, just evolving at the time (in fact, he often cites Banach's book ⟨**1**⟩). However, all this does not mean that he was not interested in the applications. On the contrary, his work is indeed applicable. See, for example, the account by D. E. Edmunds of Cooper's work on differential equations.

The wide area of transform theory in which Cooper did important research may be classified into three fields: his work on representation and uniqueness of various integral transforms, on representation and approximation, and his work on linear transformations obeying appropriate functional equations. These contributions are not so much concerned with some concrete property of any particular transform but mostly with problems pertaining to the basic underlying structures. In many cases the situation is as follows: Cooper picks up a certain result and generalizes it from a given situation to one which is general enough to treat the sharpness of the results deduced, perhaps in the sense that the conditions involved are best possible or even that inverse assertions hold, thus extending the specific point of departure to a necessary and sufficient setting. An interesting feature is that he rather quickly turns running through the elements of the underlying Hilbert space. The same ideas were further developed in his subsequent paper [**17**].

These papers of Cooper in operator theory can be regarded as pioneering pieces of work. Although strongly attached to the basic work of F. Riesz, J. von Neumann, M. H. Stone and others, his work was among the first which—still remaining close to problems posed by physics—pointed out the necessity and fruitfulness of investigations of non-self-adjoint and non-unitary operators. His work had notable influence on the later development of operator theory.

It is not the place here to examine the whole *oeuvre* of J. L. B. Cooper in functional analysis, its broad spectrum and originality, but his scientific strength and devotion is perhaps truly and convincingly mirrored by the very valuable piece of work which he did in operator theory.

Cooper's Work in Transform Theory

P. L. Butzer and R. J. Nessel

1. *Introduction*

Transform theory is one of the central parts of the work of J. L. B. Cooper. His first paper submitted for publication, on 18 May, 1938, dealt with this fundamental area of mathematics, and he returned to this topic again and again, writing at least twenty papers on it. Cooper did not really treat transform theory from the (operational) point of view of the applied mathematician—a field for which Britain is especially well known—but always in the spirit of the pure mathematician. Indeed, Cooper was a doctoral student of E. C. Titchmarsh (1899–1963) at Oxford, and therefore Titchmarsh's classical treatise on Fourier Integrals of 1937 (cf. ⟨**21**⟩) was no doubt his initial guiding star. Titchmarsh in turn was G. H. Hardy's (1877–1947) first student, and it was Hardy and J. E. Littlewood (1885–1977) who built up the great British school of analysis, second to none in the world. So Cooper had the fortune to grow up in this great tradition. On the other hand, already Cooper's early work on transform theory is permeated with the ideas of the theory of linear operators, just evolving at the time (in fact, he often cites Banach's book ⟨**1**⟩). However, all this does not mean that he was not interested in the applications. On the contrary, his work is indeed applicable. See, for example, the account by D. E. Edmunds of Cooper's work on differential equations.

The wide area of transform theory in which Cooper did important research may be classified into three fields: his work on representation and uniqueness of various integral transforms, on representation and approximation, and his work on linear transformations obeying appropriate functional equations. These contributions are not so much concerned with some concrete property of any particular transform but mostly with problems pertaining to the basic underlying structures. In many cases the situation is as follows: Cooper picks up a certain result and generalizes it from a given situation to one which is general enough to treat the sharpness of the results deduced, perhaps in the sense that the conditions involved are best possible or even that inverse assertions hold, thus extending the specific point of departure to a necessary and sufficient setting. An interesting feature is that he rather quickly turns to an operator-theoretical view of the problems, as indicated. In this regard see also the appreciation of Cooper's work on operator theory given by B. Sz.-Nagy.

Let us conclude these preliminary remarks with some terminology used throughout. With $\mathbb{R}$ the set of real numbers, let $L^p(\mathbb{R})$, $1 \leqslant p < \infty$, denote the space of measurable functions, p-th power integrable over $\mathbb{R}$ with $\|f\|_p^p := \int_{-\infty}^{\infty} |f(u)|^p du$, and BV $(\mathbb{R})$ the space of functions which are of bounded variation on $\mathbb{R}$. Let $L^1_{\text{loc}}(\mathbb{R})$ or $\text{BV}_{\text{loc}}(\mathbb{R})$ be those sets of functions which are either locally Lebesgue integrable on $\mathbb{R}$, that is, on every compact subinterval of $\mathbb{R}$, or locally of bounded variation on $\mathbb{R}$. If $f \in L^1(\mathbb{R})$ and $g \in \text{BV}(\mathbb{R})$, the Fourier and Fourier–Stieltjes transforms are given by

$$f^{\wedge}(v) := \int_{-\infty}^{\infty} e^{-ivx} f(x)dx\,, \quad g^{\vee}(v) := \int_{-\infty}^{\infty} e^{-ivx} dg(x)\,,$$

respectively, whereas the conjugate function is defined by

$$f^{\sim}(x) := \frac{1}{\pi} \lim_{\varepsilon \to 0+} \int_{\varepsilon}^{\infty} \frac{f(x+u)-f(x-u)}{u} du\,.$$

2. *Representation and uniqueness*

Cooper's first paper [1] was published in 1938, actually two years before receiving his doctorate. He considers the integral equation

$$(1) \qquad \int_{x}^{\infty} k(y-x)f(y)dy = 0\,,$$

suggested to him by Titchmarsh who treated (⟨19⟩) the system of linear equations $\sum_{n=1}^{\infty} a_n x_{n+m} = 0$. Obviously, if $k^{\wedge}$ has an n-fold zero at $-\alpha$, then $P(y)e^{i\alpha y}$ is a solution of (1) for any algebraic polynomial $P(y)$ of degree $(n-1)$. The interesting point is that under suitable conditions all solutions are of this type. For example, Cooper is able to prove the following result *via* complex Fourier transform methods: *If* $e^{at}k(t)$, $e^{-bt}f(t) \in L^2(0, \infty)$ *for some* $a > b > 0$, *and* $e^{ct}f(t) \in L^2(-\infty, 0)$ *for some real* c, *then* f *satisfying* (1) *has the form* $f(y) = \sum P(y)e^{i\alpha y}$ *where the sum ranges over the zeros of* $k^{\wedge}$ *for which* $-c < \text{Im}(-\alpha) < b$, $P(y)$ *being a polynomial of degree one less than the order of the zero.* Further results in connection with different order conditions upon the functions are derived by means of a theorem of N. Wiener. Finally, there are a couple of theorems concerned with positive non-increasing kernels k. For example, *if* $k(t)$ *is a non-increasing function of* t, *and* $k(t) > 0$ *for* $t > 0$, *then* (1) *has no solution* f *not identically zero* (a.e.) *such that* $\lim_{t\to\infty} f(t) = 0$ *or* $\int_0^{\infty} f(t)dt$ *exists.* Finally it is shown that results of this type cannot be true without the assumption of k being non-increasing.

Cooper's second paper [**2**], indeed received for publication even earlier than [**1**], follows a suggestion of L. S. Bosanquet and extends the results of ⟨**5**⟩ concerning the absolute Cesàro summability of Fourier series to Fourier integrals—in fact, Cooper uses Riesz summability. The results correlate the absolute Riesz summability of the Fourier inversion integral of a function with the bounded variation of some fractional mean of the function. To indicate just his first assertion, following Cooper's terminology, the integral $\int_0^\infty g(x)dx$ is summable $|C,\alpha|$, $\alpha \geqslant 0$, if

$$I_\alpha(\lambda) := \int_0^\lambda \left(1-\frac{u}{\lambda}\right)^\alpha g(u)du$$

is of bounded variation over $[0,\infty)$. With $\phi(t) := f(t+x)+f(t-x)$ let $\psi(u) := \int_0^\infty \phi(t)\cos ut\,dt$. *If* $\phi(t) \in \mathrm{BV}[0,\infty)$ *tends to zero as* $t \to \infty$, *and if* $\psi(u) \in L^1_{\mathrm{loc}}[0,\infty)$, *then* $\int_0^\infty \psi(u)du$ *is summable* $|C,\alpha|$ *for any* $\alpha > 0$. Again it is shown that the results obtained are best possible as regards the values of α for which they hold. Cooper also indicates corresponding results for more general methods of summability.

In his next publication [**5**] concerned with integral transforms, Cooper turns to the important question of uniqueness of trigonometrical integrals. The new idea he introduces into the subject is the following: What happens when the relevant integrals are only convergent in mean? His main result in this connection reads as follows: *If* $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ *and if over any finite interval of* t

$$\lim_{\lambda\to\infty} \int_{-\infty}^{\infty} dt \left| \int_{-\lambda}^{\lambda} \left(1-\frac{|u|}{\lambda}\right)^n f(u)e^{-iut}du - F(t)\right| = 0,$$

then for almost all u

$$f(u) = \frac{1}{2\pi}\lim_{\lambda\to\infty}\int_{-\lambda}^{\lambda}\left(1-\frac{|t|}{\lambda}\right)^{n+3} F(t)e^{iut}dt.$$

This extends results of Macphail–Titchmarsh ⟨**14**⟩ in as much as uniform convergence is now replaced by mean convergence. (Offord's work ⟨**16**⟩ is of a different, more pointwise nature.) Again the sharpness of the conditions and results deduced is considered to some extent. Finally, Cooper treats corresponding problems for a general class of summability kernels; in this case the results are largely based upon some growth condition at infinity such as $(1+|u|^n)^{-1}f(u) \in L^1(\mathbb{R})$. In [**14**] he observes that the results of [**5**] hold true, almost without change in the proofs, if convergence in mean is replaced by weak convergence.

The discussion of problems connected with functions satisfying a growth condition of the previous algebraic type is continued in [**11**] with representation theorems of the

Poisson–Stieltjes type for functions analytic in a half-plane. Cooper's main result states: *Let* $f(w) := p(u, v) + iq(u, v)$ *be an analytic function of* $w = u + iv$ *for* $v > 0$. *Suppose that* $M(v) := \sup\{|f(u+iv)|; u \in \mathbb{R}\} < \infty$ *for* $v > 0$, $\lim_{v\to\infty} M(v) = 0$, *and that there exists* α, $0 \leqslant \alpha \leqslant 2$, *such that for all* $v > 0$

$$\int_{-\infty}^{\infty} (1+|u|)^{-\alpha} |p(u, v)| du \leqslant N(v)$$

where $N(v)$ *is bounded for* $0 < v \leqslant v_0$. *Then there exists a function* $\rho \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ *such that*

$$\int_{-\infty}^{\infty} \frac{|d\rho(x)|}{(1+|x|)^{\alpha}} \leqslant \limsup_{v\to 0} N(v),$$

$$\rho(u_2) - \rho(u_1) = \lim_{v\to 0} \int_{u_1}^{u_2} p(u, v) du.$$

If $\alpha \leqslant 1$, *then for all* $v > 0$

$$f(w) = (1/\pi i) \int_{-\infty}^{\infty} (x-w)^{-1} d\rho(x),$$

and if $\alpha \leqslant 2$,

$$p(u, v) = (1/\pi) \int_{-\infty}^{\infty} [(x-u)^2 + v^2]^{-1} v d\rho(x).$$

A converse assertion is valid as well. This in particular extends pertinent work of Titchmarsh ⟨**20**⟩ (who gave the result for $\alpha = 0$). A simple example of a function covered by Cooper's theorem but not by the earlier results on the subject is $f(w) = e^{iw}/iw$; it satisfies the conditions stated above for any $\alpha > 0$.

The representation theorems of [**11**] are used in Cooper's treatment [**13**] on Fourier–Stieltjes integrals—in fact, both publications were received and read at the same date (May 20 and June 19, 1947, respectively). Up to that time Fourier–Stieltjes integrals, important as a generalization of both Fourier series and integrals, were considered for functions which are of bounded variation over the whole real axis. But this only covers the case of absolutely convergent Fourier series and integrals. Therefore in [**13**] Cooper discusses a wider class of Fourier–Stieltjes integrals, sufficiently general to include Fourier series and integrals of all L^p-classes. These integrals are of the type $f(x) \sim \int_{-\infty}^{\infty} e^{-itx} d\rho(t)$, where $\rho \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R})$ is such that for some $\alpha \geqslant 0$

$$\int_{-\infty}^{\infty} (1+|t|)^{-\alpha} |d\rho(t)| < \infty.$$

The method employed by Cooper to make these ideas precise depends heavily on his representation theorem derived in [11]. Applications are given to positive definite functions, generalizing classical work of S. Bochner (1932). Again a large portion of the paper is devoted to extensions to rather general classes of summability kernels, as well as to a discussion of the sharpness of the conditions and results deduced.

Let us conclude this section with the remark that the topics considered by Cooper in this first group of contributions to transform theory were indeed central to the subject and of great interest at the time. This is in particular revealed by the independent investigations of F. Wolf ⟨**22**⟩ and recent work of F. Holland ⟨**11**⟩.

3. *Representation and approximation*

In 1959 the senior author of this review was stuck on some basic problems in Fourier transform theory, the solutions of which were needed to solve problems in trigonometric approximation theory. Several of the most prominent experts of the field were asked, but it was Cooper who communicated a complete solution within a few weeks. One of these problems was conjectured to have the following solution (cf. ⟨**6**⟩; p. 407): *For* $f \in L^1(\mathbb{R})$ *and* $g \in \mathrm{BV}(\mathbb{R})$ *one has*

$$|v| f^{\wedge}(v) = g^{\vee}(v) \Leftrightarrow f^{\sim} \in \mathrm{BV}(\mathbb{R}). \tag{2}$$

Cooper established this conjecture in [**27**] and, moreover, extended the matter to fractional derivatives of the conjugate function, i.e., to the case $|v|^{\gamma} f^{\wedge}(v) = g^{\vee}(v)$, $\gamma > 0$. He also considered counterparts for the spaces $L^p(\mathbb{R})$, $1 < p \leqslant 2$. This material was taken up in detail in ⟨**8**⟩; pp. 194, 214, 224, 406, and ⟨**7**⟩; p. 247.

Since that time there arose a close connection between Cooper and the Aachen group on approximation theory. One of its concrete common points of interest was concerned with extensions of a classical representation theorem due to H. Cramér ⟨**9**⟩ (see also Thm. 5, case $\alpha = 0$, of [**13**]; p. 274). It states that for a continuous function F to be representable as a Fourier–Stieltjes integral, thus $F(v) = g^{\vee}(v)$ for some $g \in \mathrm{BV}(\mathbb{R})$, it is necessary and sufficient that the first Cesàro means of the Fourier inversion integral be bounded in $L^1(\mathbb{R})$; thus for $p = 1$

$$\left\| \int_{-\rho}^{\rho} \left(1 - \frac{|v|}{\rho}\right) F(v) e^{ixv} dv \right\|_p = O(1) \qquad (\rho \to \infty). \tag{3}$$

In connection with saturation theory (see below) the question came up whether there are counterparts of Cramér's theorem for $1 < p \leqslant 2$. This problem was solved by Cooper in [**29**]; he showed that (cf. ⟨**8**⟩; p. 237) *for* $F \in L^1_{\mathrm{loc}}(\mathbb{R})$ *and* $1 < p \leqslant 2$ *condition* (3) *is necessary and sufficient for the existence of some* $g \in L^p(\mathbb{R})$ *such that* $F(v) = g^{\wedge}(v)$ a.e.

As already mentioned, Cooper's solution of the preceding problems in harmonic analysis paved the way for basic saturation results in approximation theory. Let us explain the situation in connection with the particular example of the Cauchy–Poisson process

$$P(f, x, y) := \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(x-u)}{y^2 + u^2} du \quad (x \in \mathbb{R},\ y > 0).$$

It solves Dirichlet's problem for the upper half-plane

$$\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2} = 0, \quad w(x,0) = f(x)$$

in the sense that the boundary value $f \in L^1(\mathbb{R})$ is attained strongly in the metric of $L^1(\mathbb{R})$, that is,

$$\lim_{y \to 0+} \|P(f,x,y) - f(x)\|_1 = 0.$$

Concerning rates of convergence, these are limited according to the saturation phenomenon occurring. Indeed, if $f \in L^1(\mathbb{R})$ is such that

$$\|P(f,x,y) - f(x)\|_1 = o(y) \quad (y \to 0+),$$

then necessarily $f(x) = 0$ a.e. In order to characterize the saturation class

$$S := \{f \in L^1(\mathbb{R}) : \|P(f,x,y) - f(x)\|_1 = O(y), y \to 0+\},$$

one may proceed *via* the Fourier transform method (cf. ⟨**6**⟩) to show that $f \in S$ if and only if there exists some $g \in \mathrm{BV}(\mathbb{R})$ such that $|v| f^\wedge(v) = g^\vee(v)$ for all $v \in \mathbb{R}$. Here representation theorems of the Cramér–Cooper type were needed. Instead of this characterization in the Fourier transform state, one would of course like to proceed further and to derive characterizations in terms of the original function spaces. Here (2) meets the needs, showing that $f \in S$ if and only if $f^\sim \in \mathrm{BV}(\mathbb{R})$. For further details, however, we have to refer to the relevant treatments in textbooks on the subject, e.g. ⟨**8**⟩, ⟨**15**⟩.

In his original contribution [**29**], Cooper in fact considered the representation problem in much more general terms. As is well known, every $f \in L^p(\mathbb{R})$, $1 \leqslant p \leqslant 2$, has Fourier transform $f^\wedge \in L^{p'}(\mathbb{R})$, $1/p + 1/p' = 1$, and the set of Fourier transforms of L^p forms a proper subset in $L^{p'}$. The problem of characterizing the L^p-transforms in $L^{p'}$ has attracted the attention of many mathematicians, so that there have been developed a variety of necessary and/or sufficient conditions guaranteeing, for a given $F \in L^{p'}$, the existence of some $f \in L^p$ such that $F(v) = f^\wedge(v)$ a.e. In his paper [**29**] Cooper discusses a very general class of criteria. To this end he starts off with integral transforms of the type

$$f_\rho(u) := \int_{-\infty}^{\infty} k(u,v,\rho) F(v)\,dv,$$

where $k(u,v,\rho)$ is a certain kernel, and ρ is a positive parameter tending to infinity, for example. Then Cooper examines the relations between the following propositions:

(A) The set of functions $f_\rho(u)$ is bounded in $L^p(\mathbb{R})$ for $\rho > 0$,

(B) F is the Fourier transform of some $f \in L^p(\mathbb{R})$,

(C) $\lim_{\rho \to \infty} f_\rho = f$ in some appropriate sense.

Cooper derives various theorems unifying and generalizing all the representation theories scattered in the literature. Let us just mention: *Let* $k(v, \rho)$ *tend to* 1 *as* $\rho \to \infty$, *boundedly over every finite interval. Let* $F \in L^1_{\mathrm{loc}}(\mathbb{R})$ *such that* $k(v, \rho)F(v) \in L^1(\mathbb{R})$ *for all* $\rho > 0$. *Let*

$$f_\rho(u) = \int_{-\infty}^{\infty} e^{iuv} k(v, \rho) F(v)\, dv \tag{4}$$

be bounded in $L^p(\mathbb{R})$. *Then* $F = f^\wedge$, *where* $f \in L^p(\mathbb{R})$ *is the weak limit in* $L^p(\mathbb{R})$ *of* $\{f_\rho\}$. This theorem applies in particular to the Cesàro kernel, establishing the sufficiency of the Cooper criterion mentioned (cf. (3)).

In [**29**] Cooper also considered corresponding problems for Fourier series and coefficients. In fact, in his lecture [**28**] at the 1963 Oberwolfach Conference he not only outlined the main features of his general approach in [**29**], but he also added details concerning counterparts for functions defined on arbitrary locally compact abelian groups. The methods were further refined and developed in [**30**], [**32**] in connection with representation theories for Laplace transforms. In comparison with the Fourier transform situation sketched above there are two main differences: In the one-sided Laplace transform case one is essentially dealing with functions (or distributions) with support confined to $[0, \infty)$, and the Laplace transform is an analytic function. Indeed, as a consequence of assumptions of analyticity and of restrictions on the growth at infinity it is possible to prove Laplace representation theorems by considering the corresponding norms of the integral transforms (cf. (3, 4))

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} k(v, \rho) F(c+iv) e^{i(c+iv)}\, dv$$

only over $(0, \infty)$, their natural domain of definition in the one-sided Laplace case. This in fact extends results derived in ⟨**2**⟩, ⟨**4**⟩. Again such representation theorems have important applications to saturation theory, see ⟨**3**⟩.

4. *Group representations*

Cooper's contributions [**36**], [**37**], [**41**], [**42**], [**44**] to transform theory during the last decade are concerned with linear transformations which obey functional equations induced by certain group representations. The motivation underlying these studies was expressed very pointedly by Cooper himself in the first sentence of [**42**]: "The integral transforms important in analysis are those which have particularly simple behaviour when the spaces on which the transformed functions are defined are subjected to certain transformation groups." In a review (MR **50** # 14079) of [**42**], R. A. Askey (Madison, Wis.) then continued: "This is the first sentence in the paper under review, and there is no disputing this fact. The cottage industry which adds one more parameter to an already published transform should take note of this sentence and refrain from further sterile generalization." Let us follow the introduction of [**37**] for a short outline of the problems Cooper is interested in; in fact, we may be very brief since the matter is already treated in textbooks on the subject, see e.g. G. O. Okikiolu ⟨**17**⟩, Chapter 8.

Let $A(X)$ be a space of functions on an interval $X \subset \mathbb{R}$. A group of transformations $\{W(\alpha) : \alpha \in \mathbb{R}\}$ of $A(X)$ into itself is called an appropriate (one parameter) group if each $W(\alpha)$ is of the form $W(\alpha)f(x) = Q(x, \alpha)f(V(\alpha)x)$, where

$$W(\alpha+\beta) = W(\alpha)W(\beta), \quad W(0) = I, \tag{5}$$

$V(\alpha)$ is a group of transformations of X onto itself satisfying (5), and $Q(x, \alpha)$ and $V(\alpha)x$ are continuous functions of α for each $x \in X$. A transformation $T : A(X) \to B(Y)$, with range $B(Y)$ defined analogously to $A(X)$, is said to obey an appropriate functional equation if there are appropriate groups $\{W(\alpha)\}$ and $\{W^*(\alpha)\}$ on A and B, respectively, such that $TW(\alpha) = W^*(\alpha)T$ for all $\alpha \in \mathbb{R}$. This general approach subsumes practically all the classical integral transforms, e.g., the Fourier, Hilbert, or Mellin transform, fractional integrals. Particular functional equations of the present type had already been studied, e.g., by M. Plancherel ⟨**18**⟩ and H. Kober ⟨**12**⟩, ⟨**13**⟩; Kober had actually attributed some results and proofs of ⟨**13**⟩ to Cooper. He himself considered those functional equations from a general point of view. For example, Cooper is interested in classifying appropriate functional equations. One of his main results is that under quite general assumptions about the topologies of A and B the possible equations can be reduced to four canonical forms, roughly speaking to

$$[Tf(x+\alpha)](u) = [Tf(x)](u+\alpha),$$

$$[Tf(x+\alpha)](u) = e^{\alpha\phi(u)}[Tf(x)](u),$$

$$[Te^{\alpha\psi(x)}f(x)](u) = [Tf(x)](u+\alpha),$$

$$[Te^{\alpha\psi(x)}f(x)](u) = e^{\alpha\phi(u)}[Tf(x)](u).$$

Cooper then turns to questions concerning the solvability and the explicit evaluation of solutions of these equations. In his subsequent lectures and contributions [**36**], [**41**], [**42**], [**44**] to transform theory, he further refined and extended this approach to appropriate functional equations. In fact, his work in this regard heavily influenced that of a number of mathematicians, notably that of Duggal (cf. ⟨**10**⟩) and Okikiolu (cf. ⟨**17**⟩ and the literature cited).

Let us finally add that Cooper also contributed an article to the *Handbook of physics* [**34**]. It deals with linear spaces, Hilbert spaces, linear functionals and operators, integral equations (Hilbert–Schmidt theory), integral transforms (including Fourier and Laplace transforms, and applications to differential and integral equations), distributions. He managed to cover this large amount of material, the selection of which is excellent, in an expert fashion in about six printed pages. This article, or any direct discussion with Cooper on a specific mathematical problem—both authors had the pleasure to have had many such discussions with him since 1963—readily reveal his great strength: He always had a terrific amount of material at his disposal, the broad scope of his knowledge ranged from pure to applied mathematics and theoretical physics; he was indeed a great scholar with a sharp and penetrating intellect.

5. *List of additional references*

⟨1⟩ S. Banach, *Théorie des opérations linéaires* (Warsaw, 1932; reprinted by: Chelsea Publ., New York, 1955).

⟨2⟩ H. Berens and P. L. Butzer, "Über die Darstellung holomorpher Funktionen durch Laplace- und Laplace–Stieltjes-Integrale", *Math. Z.*, 81 (1963), 124–134.

⟨3⟩ H. Berens and P. L. Butzer, "On the best approximation for singular integrals by Laplace-transform methods", in: *On approximation theory* (ed. P. L. Butzer and J. Korevaar, ISNM 5, Birkhäuser Verlag, Basel, Stuttgart, 1964), 24–42.

⟨4⟩ H. Berens and P. L. Butzer, "Über die Darstellung vektorwertiger holomorpher Funktionen durch Laplace-Integrale", *Math. Ann.*, 158 (1965), 269–283.

⟨5⟩ L. S. Bosanquet, "The absolute Cesàro summability of a Fourier series", *Proc. London Math. Soc.*, (2), 41 (1936), 517–528.

⟨6⟩ P. L. Butzer, "Fourier-transform methods in the theory of approximation", *Arch. Rational Mech. Anal.*, 5 (1960), 390–415.

⟨7⟩ P. L. Butzer and H. Berens, *Semi-groups of operators and approximation* (Springer-Verlag, Berlin, Heidelberg, New York, 1967).

⟨8⟩ P. L. Butzer and R. J. Nessel, *Fourier analysis and approximation, I: One-dimensional theory* (Birkhäuser Verlag, Basel, Stuttgart, and Academic Press, London, New York, 1971).

⟨9⟩ H. Cramér, "On the representation of a function by certain Fourier integrals", *Trans. Amer. Math. Soc.*, 46 (1939), 191–201.

⟨10⟩ B. P. Duggal, "Functional equations and linear transformations, I: Solvability on L_p spaces; II: Classes G_λ and I_μ; IIIA: Permutability and inversion; IIIB: Factorization; IIIC: Permutability; IV: Interpolation", *J. Indian Math. Soc.*, 38 (1974), 71–97; 99–120; *Period. Math. Hungar.*, 9 (1978), 93–107; 10 (1979), 177–191; 285–292; *Math. Ann.*, 237 (1978), 277–285.

⟨11⟩ F. Holland, "On the representation of functions as Fourier transforms of unbounded measures", *Proc. London Math. Soc.*, (3), 30 (1975), 347–365.

⟨12⟩ H. Kober, "On certain linear operations and relations between them", *Proc. London Math. Soc.*, (3), 11 (1961), 434–456.

⟨13⟩ H. Kober, "On functional equations and bounded linear transformations", *Proc. London Math. Soc.*, (3), 14 (1964), 495–519.

⟨14⟩ M. S. Macphail and E. C. Titchmarsh, "The summability of Fourier's integral", *J. London Math. Soc.*, 11 (1936), 313–318.

⟨15⟩ R. G. Mamedov, *Approximation of functions by linear operators* (Azerbaijani) (Baku, 1966).

⟨16⟩ A. C. Offord, "On the uniqueness of the representation of a function by a trigonometric integral", *Proc. London Math. Soc.*, (2), 42 (1937), 422–480.

⟨17⟩ G. O. Okikiolu, *Aspects of the theory of bounded integral operators in L^p-spaces* (Academic Press, London, New York, 1971).

⟨18⟩ M. Plancherel, "Quelques remarques à propos d'une note de G. H. Hardy: The resultant of two Fourier kernels", *Proc. Cambridge Phil. Soc.*, 33 (1937), 413–418.

⟨19⟩ E. C. Titchmarsh, "A system of linear equations with an infinity of unknowns", *Proc. Cambridge Phil. Soc.*, 22 (1924), 282–286.

⟨20⟩ E. C. Titchmarsh, "On expansions in eigenfunctions (V)", *Quart. J. Math. Oxford Ser.*, 12 (1941), 89–107.

⟨21⟩ E. C. Titchmarsh, *Introduction to the theory of Fourier integrals* (Clarendon Press, Oxford, 1948 (2nd edit.)).

⟨22⟩ F. Wolf, "Contributions to a theory of summability of trigonometric integrals", *Univ. California Publ. Math. (N.S.)*, 1 (1947), 159–227.

Functional Analysis

While not of so pioneering a character as his work on operator theory, Cooper's research in this area nevertheless contains much of interest. The paper **[12]**, for example, is concerned with conditions under which it can be shown that if a family of operators on a space converges everywhere, then it converges uniformly; such results essentially apply only when the space involved is a completely additive Boolean ring, or when a reduction to this case is possible. With a view to the unification of various theorems from different branches of analysis he introduces the concept of a *mesh*: a filter $\mathscr{M}$ on a complete Boolean ring is called a mesh if it has a base $\mathscr{B}$ such that if $X \in B \in \mathscr{B}$ and $Y \subset X$, then $Y \in \mathscr{B}$. His main result is the following: *Let* $(L_i)_{i \in I}$ *be a* *family of completely additive set functions such that $L_i(X)$ is defined for all elements X of a complete Boolean ring K and is absolutely continuous with respect to a mesh $\mathcal{M}$ on K; for each X in K let $L_i(X)$ tend to a limit according to a filter $\mathcal{F}$ on I having a countable base. Then given any $\delta > 0$, there exist $M(\delta) \in \mathcal{M}$ and $F(\delta) \in \mathcal{F}$ such that $|L_i|(X) < \delta$ if $X \in M(\delta)$ and $i \in F(\delta)$.*

Theorems due to Vitali ⟨**32**⟩, Lebesgue ⟨**29**⟩ and Hahn ⟨**26**⟩ are special cases of this; and from it also follows the result (which generalises a theorem of Steinhaus (cf. ⟨**24**⟩; p. 392)) that a matrix which sums all convergent sequences of 0's and 1's to their actual limits cannot sum all sequences of 0's and 1's to a limit.

In [**19**], meshes and the topologies defined by them are examined in detail, the results obtained being subsequently applied in [**21**] and [**31**] to Cooper's theory of co-ordinated linear spaces, which are generalisations of the Köthe spaces (cf. ⟨**27**⟩, ⟨**28**⟩ and ⟨**25**⟩) and are characterised by the existence in them of a Boolean algebra of projections and by a dual defined not in terms of continuity but rather by the fact that the elements of the dual induce completely additive set functions on this algebra.

Finally, in [**50**] he turned to spaces with indefinite scalar products. These have applications in quantum field theory (cf. ⟨**30**⟩), and were first studied by Pontryagin ⟨**31**⟩, although the spaces he introduced, subsequently called after him, were restricted in that there was an upper bound to the dimension of the subspaces on which the scalar product was negative definite; a comprehensive account of the basic theory is given in ⟨**23**⟩. Cooper observed that there was no adequate theory of unbounded hermitian operators in spaces other than Pontryagin spaces, and his paper is devoted to this gap. In Hilbert space, his paper [**9**] had shown that a solution of the Schrödinger equation $\frac{1}{i}\frac{d\psi}{dt} = A\psi$ valid for all time exists for arbitrary initial values only when A is self-adjoint, mere symmetry not being enough; it turns out that in Krein spaces (cf. ⟨**23**⟩) even self-adjointness is not adequate for the existence of a global solution of the Schrödinger equation, or equivalently, for the existence of the exponential map $\exp(itA)$. The main achievement of his paper is the identification of a class of maps, called by him fully self-adjoint, for which existence is assured.

Additional references

⟨**23**⟩ J. Bognár, *Indefinite inner product spaces* (Springer-Verlag, Berlin-Heidelberg-New York, 1974).
⟨**24**⟩ P. Dienes, *The Taylor Series* (Oxford, 1930).
⟨**25**⟩ J. Dieudonné, "Sur les espaces de Köthe", *J. Analyse Math.* 1 (1951), 81–115.
⟨**26**⟩ H. Hahn, "Über Folgen linearer Operationen", *Monatsh. für Math. und Physik* 32 (1922), 3–88.
⟨**27**⟩ G. Köthe, "Neubegründung der Theorie der vollkommenen Räume", *Math. Nachr.* 4 (1951), 70–80.
⟨**28**⟩ G. Köthe, *Topologische lineare Räume, Vol. I* (Springer, Berlin-Göttingen-Heidelberg, 1960).
⟨**29**⟩ H. Lebesgue, "Sur les intégrales singulières", *Ann. de Toulouse* (3), 1 (1909), 25–117.
⟨**30**⟩ K. L. Nagy, *State vector spaces with indefinite metric i.. quantum field theory* (Noordhoff Groningen and Akadémiai Kiadó Budapest, 1966).
⟨**31**⟩ L. S. Pontryagin, "Hermitian operators in spaces with indefinite metric", *Izv. Akad. Nauk SSSR, Ser. Mat.* 8 (1944), 243–280.
⟨**32**⟩ G. Vitali, "Sull' integrazione per serie", *Rend. del Circolo Mat. di Palermo* 23 (1907), 137–155.

Differential Equations

Differential equations meant a good deal to Cooper: they coloured his approach to operator theory, and he was always on the look out for applications of his work, particularly that involving transform theory, to them.

His first paper [3] on differential equations dealt with a problem originally studied by A. Weinstein ⟨36⟩, ⟨37⟩ in the context of hydrodynamics: given a real number k, the question was that of finding a function u which was harmonic in the infinite strip $\{(x, y): x \in \mathbb{R}, 0 < y < 1\}$ in the plane and satisfied the boundary conditions $u(x, 0) = 0, u_y(x, 1) = ku(x, 1)$ for all $x \in \mathbb{R}$. By using truncated Fourier transforms he was able to establish the existence of a solution (in an explicit form) without the conditions on growth at infinity which earlier authors, notably S. Bochner ⟨33⟩; p. 167, had imposed. Truncated Fourier transforms were also used in [15], where he studied the uniqueness of solutions of the heat equation $u_{xx} = u_t$ for $x \in \mathbb{R}$, $0 < t \leqslant T$, T being arbitrary. What is proved is that a solution u of this equation is identically zero if

(i) $u, u_t \in L^1([a, b] \times [\delta, T])$ for all real a and b, and all $\delta \in (0, T)$;

(ii) for some constant c there exists sequences (x_n), (x'_n), tending to ∞ and $-\infty$ respectively, such that $|u(x, t)| < e^{cx^2}$ for all $t \in (0, T)$ and any x in either sequence;

(iii) $\lim_{t \to 0} \int_a^b u(x, t) g(x) dx = 0$ for all $a, b \in \mathbb{R}$ and all functions g which are bounded and Lebesgue integrable on (a, b).

This result improves earlier work of Titchmarsh ⟨34⟩; pp. 281–283, and Tychonoff ⟨35⟩, largely on account of the comparatively weak local condition (iii) which, as Cooper points out, is a natural one on physical grounds. The same technique is used in [16] to solve the initial-value problem for the wave equation in any number n of space dimensions, use being made of spherical Abelian means to extend the validity of the inversion formula. Here his method enables him to give a set of conditions on the initial data which are transmitted to the corresponding solution. More precisely, he shows that if the solution v is such that $f(., 0)$ and $f_t(., 0)$ possess space derivatives of all orders up to and including those of order $s+1$ and s respectively, each of these derivatives being in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$, then the same is true of $f(., t_0)$ and $f_t(., t_0)$ for any $t_0 > 0$. Yet another application of Fourier transform methods occurs in [8], where he analysed the propagation of waves in an elastic rod, estimated the allowable velocities of propagation and established the dispersive nature of both transverse and longitudinal waves.

His final paper [45] on differential equations is concerned with the solution of constant coefficient equations by means of Laplace transforms: he shows that solutions obtained in this way are weak solutions, in a certain sense, and gives conditions for these weak solutions to be classical ones.

Additional references

⟨33⟩ S. Bochner, *Vorlesungen über Fouriersche Integrale* (Leipzig, 1932).
⟨34⟩ E. C. Titchmarsh, *Fourier integrals* (Oxford, 1937).
⟨35⟩ A. Tychonoff, *Rec. Math. (Mat. Sbornik)* 42 (1935), 199–216.
⟨36⟩ A. Weinstein, *Rend. Accad. Lincei, Roma* (6) 5 (1927), 259–265.
⟨37⟩ A. Weinstein, *Comptes rendus, Paris* 184 (1927), 497–499.

Thermodynamics

Lionel Cooper's lifelong interest in fundamental questions of applied mathematics showed itself in his research particularly clearly in his papers devoted to the foundations of thermodynamics. He was a pioneer of the modern trend toward strict rigour in this traditionally non-rigorous field, his paper **[33]** setting a standard of historical criticism and precise formulation which can hardly be surpassed. In this work the axiomatic structure developed by Carathéodory in his celebrated paper ⟨**38**⟩ was subjected to the sharpest scrutiny—revealing perhaps for the first time its genuine limitations (and some of its logical errors)—and then extended to a broader and more satisfying formulation. His later work **[35]**, **[46]** consolidated these developments and so gave us in some sense an ultimate analysis and delineation of Carathéodory's structure. Perhaps most brilliant is Cooper's treatment of irreversible processes, which Carathéodory could handle in only the most limited degree, together with his remarkable discovery that Carathéodory's axiom scheme required further augmentation to prove that the absolute temperature scale was indeed a homeomorphism on the hotness manifold. Carathéodory's error lay in his identification of absolute temperature with the reciprocal of an integrating factor of the differential form representing the work done in a small reversible (quasistatic) change of state. For this to be possible one would have to prove that this reciprocal was a monotonic function of empirical temperature, and this does not follow from his basic hypothesis (Carathéodory's law) as this hypothesis would be fulfilled if the differential form were already exact, in which case the integrating factor would be constant.

While more recent research on thermodynamical foundations has returned to the ideas of Carnot, Kelvin and Clausius, it was Cooper's work in essence which led to the critical re-evaluation of Carathéodory's concepts and to the modern standards of rigour which at last have penetrated the miasmic traditional presentations.

Additional reference

⟨**38**⟩ C. Carathéodory, "Untersuchungen über die Grundlagen der Thermodynamik", *Math. Ann.* (1909), 355–386.

Miscellaneous

Cooper's remaining papers cover a variety of topics, illustrating the breadth of his interests. For example, in **[4]** he used Laplace transform methods to give simple derivations of identities and asymptotic expansions for the Fermi–Dirac functions of quantum statistics, while **[10]** provides a criterion for the convergence of the usual relaxation process for natural frequency equations. His interest in historical matters is illustrated by **[20]**, in which he gives a delightful account of the controversy surrounding Heaviside and the origins of the operational calculus; and from his inaugural address in London **[38]** we can gain some idea of his skill as a speaker to general audiences.

Perhaps the deepest work, however, of that considered in this section, occurs in **[24]**, where he extends Bochner's celebrated result ⟨**39**⟩ about the representation of positive definite functions by monotonic non-decreasing functions. Let us recall that a function $f: \mathbb{R} \to \mathbb{C}$ is called positive-definite if

$$\sum_{i,j=1}^{n} f(x_i - x_j)\xi_i\bar{\xi}_j \geqslant 0 \quad \text{for all } n \in \mathbb{N}\text{, for all } x_1, x_2, \ldots, x_n \in \mathbb{R}$$
$$\text{and all } \xi_1, \xi_2, \ldots, \xi_n \in \mathbb{C}.$$

Bochner's result is that a function f may be represented in the form $f(x) = \int_{-\infty}^{\infty} e^{ixy} dw(y)$, where w is a non-decreasing function (the representation being unique under suitable normalisation), if and only if f is continuous and positive-definite. Other authors have replaced the defining inequality by its integral analogue:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)\phi(x)\overline{\phi(y)}\,dx\,dy \geqslant 0$$

for all functions in, say, $C_0(\mathbb{R})$ (the space of all continuous functions with compact support). In fact, if f is continuous, these integral and sum definitions of positive-definiteness coincide; differences appear when requirements of continuity or boundedness are abandoned.

In [**24**] Cooper calls a function f positive-definite for a set J of functions from $\mathbb{R}$ to $\mathbb{C}$ if the integral above exists in the Lebesgue sense and is non-negative for all ϕ in J; the class of all such functions f will be denoted by $P(J)$. It emerges that $P(L^1(\mathbb{R}))$ is nothing more than (modulo sets of measure zero) the family of all continuous functions which are positive-definite in the sense used by Bochner, while $P(C_0(\mathbb{R}))$ is a much larger family, containing unbounded functions. Cooper proved that $P(C_0(\mathbb{R})) = P(L_0^p(\mathbb{R}))$ for all $p \geqslant 2$, where $L_0^p(\mathbb{R})$ is the set of all elements of $L^p(\mathbb{R})$ with compact support; while if $1 \leqslant p \leqslant 2$ and $q = \frac{1}{2}p/(p-1)$, any element of $P(L_0^2(\mathbb{R}))$ which is in $L_{\text{loc}}^q(\mathbb{R})$ belongs to $P(L_0^p(\mathbb{R}))$. His main result is that if $f \in P(C_0(\mathbb{R}))$, then there is a non-decreasing function ρ, with $\rho(u) = o(u)$ as $u \to \pm\infty$, such that in the $(C, 1)$ sense,

$$2\pi f(x) = \int_{-\infty}^{\infty} e^{-iux} d\rho(u) \quad \text{for almost all } x \text{ in } \mathbb{R},$$

and

$$\iint f(x-y)\phi(x)\overline{\psi(y)}\,dx\,dy = \int \Phi(u)\Psi(u)\,d\rho(u) \quad \text{for all } \phi,\ \psi \text{ in } L_0^2(\mathbb{R}),$$

Φ and Ψ being their Fourier transforms. The analytic properties of positive-definite functions were examined in detail by F. Holland in ⟨**40**⟩, while Cooper returned to the topic of positive-definiteness in [**43**]. A survey of the main developments in this area is given in ⟨**41**⟩.

The last paper which he wrote, [**51**], published posthumously, dealt with the closeness by which functions of several variables, invariant under some group of transformations, may be approximated in L^2 by functions which are products of functions of the individual variables. Like much else in his work, this problem arose out of a quantum mechanical question.

Additional references

⟨39⟩ S. Bochner, *Vorlesungen über Fouriersche Integrale* (Leipzig, 1932).
⟨40⟩ F. Holland, "Contributions to harmonic analysis", Ph.D. thesis, University of Wales, 1964.
⟨41⟩ J. D. Stewart, "Positive definite functions and generalisations, an historical survey", *Rocky Mountain J. of Math.* 6 (1976), 409–430.

Acknowledgments

The reader will share my gratitude to P. L. Butzer, R. J. Nessel, J. Serrin and B. Sz.-Nagy for their contributions to this notice. I have also had great assistance from many others, notably F. F. Bonsall, R. H. Dyer and F. Smithies; to all of these I am much indebted.

Above all my thanks are due to Mrs. Kathleen Cooper who, even at a most distressing time for her, was prepared to give me the detailed information which has enabled me to write this notice.

Publications of J. L. B. Cooper

This list contains, where possible, bibliographical details of the reviews of the papers in *Mathematical Reviews* (MR) and *Zentralblatt* (Zbl.), including the name of the reviewers.

1. "An integral equation", *Quart. J. Math.* 9 (1938), 263–273; Zbl. **20,** 131 (T. H. Hildebrandt).
2. "The absolute Cesàro summability of Fourier integrals", *Proc. London Math. Soc.* (2), 45 (1939), 425–439; MR **1,** 51 (O. Szász); Zbl. **61,** 139 (L. Cesari).
3. "A mixed boundary value problem", *J. London Math. Soc.* 14 (1939), 124–128; Zbl. **21,** 130 (Wegner).
4. "The Fermi–Dirac functions", *Phil. Mag.* (7) 30 (1940), 187–189; MR **2,** 96 (J. L. Barnes); Zbl. **26,** 315 (V. Stachó).
5. "The uniqueness of trigonometrical integrals", *Proc. London Math. Soc.* (2), 48 (1944), 292–309; MR **6,** 126 (R. P. Boas); Zbl. **60,** 257 (G. Doetsch).
6. "The spectral analysis of self-adjoint operators", *Quart J. Math.* 16 (1945), 31–48; MR **7,** 125 (F. J. Murray); Zbl. **61,** 261 (H. O. Cordes).
7. "One-parameter semigroups of isometric operators in Hilbert space", *Ann. of Math.* (2), 48 (1947), 827–842; MR **10,** 257 (F. J. Murray); Zbl. **29,** 141 (B. Sz.-Nagy).
8. "The propagation of elastic waves in a rod", *Phil. Mag.* (7) 38 (1947), 1–22; MR **9,** 123 (G. F. Carrier); Zbl. **34,** 411 (A. Schoch).
9. "Symmetric operators in Hilbert space", *Proc. London Math. Soc.* (2), 50 (1948), 11–55; MR **9,** 446 (F. J. Murray); Zbl. **30,** 161 (B. Sz.-Nagy).
10. "The solution of natural frequency equations by relaxation methods", *Quart. Appl. Math.* 6 (1948), 179–183; MR **10,** 70 (E. Bodewig); Zbl. **33,** 287 (G. Olsson).
11. "Functions analytic in a half-plane", *J. London Math. Soc.* 23 (1948), 84–92; MR **10,** 185 (M. Heins); Zbl. **31,** 159 (G. Doetsch).
12. "Convergence of families of completely additive set functions", *Quart. J. Math.* 20 (1949), 8–21; MR **11,** 239 (A. Rosenthal); Zbl. **32,** 272 (J. Dieudonné).
13. "Fourier–Stieltjes integrals", *Proc. London Math. Soc.* (2), 51 (1950), 265–284; MR **12,** 496 (F. Wolf); Zbl. **35,** 31 (G. Doetsch).
14. "The uniqueness of trigonometrical integrals", *J. London Math. Soc.* 25 (1950), 61–63; MR **12,** 496 (F. Wolf); Zbl. **36,** 352 (G. Doetsch).
15. "The uniqueness of the solution of the equation of heat conduction", *J. London Math. Soc.* 25 (1950), 173–180; MR **12,** 104 (F. G. Dressel); Zbl. **37,** 346 (Bödewadt).

16. "The application of multiple Fourier transforms to the solution of partial differential equations", *Quart. J. Math.* (2), 1 (1950), 122–135; MR **12,** 184 (F. John); Zbl. **39,** 330 (J. Deny).
17. "The characterisation of quantum-mechanical operators", *Proc. Camb. Phil. Soc.* 46 (1950), 614–619; MR **12,** 508 (I. E. Segal); Zbl. **38,** 72 (B. Sz.-Nagy).
18. "The paradox of separated systems in quantum theory", *Proc. Camb. Phil. Soc.* 46 (1950), 620–625; MR **12,** 377 (N. Rosen); Zbl. **41,** 329 (C. F. v. Weizsächer).
19. "Topologies in rings of sets", *Proc. London Math. Soc.* (2), 52 (1951), 220–240; MR **12,** 728 (S. B. Meyers); Zbl. **54,** 70 (K. Krickeberg).
20. "Heaviside and the operational calculus", *Math. Gaz.* 36 (1952), 5–19; MR **13,** 612.
21. "Coordinated linear spaces", *Proc. London Math. Soc.* (3), 3 (1953), 305–327; MR **15,** 132 (J. Dieudonné); Zbl. **50,** 333 (G. Köthe).
22. "Mathematical monsters", *Math. Gaz.* 38 (1954), 258–268.
23. "Functional analysis", *Math. Gaz.* 43 (1959), 102–109; MR **22** # 6989.
24. "Positive definite functions of a real variable", *Proc. London Math. Soc.* (3), 10 (1960), 53–66; MR **22** # 6981 (S. Bochner).
25. "The main lines of mathematics", Advancement of Science 17 (1961), 505–510. Reprinted with revisions, Smithsonian Institution Report for 1961 (1962), 323–335; Zbl. **91,** 1.
26. Prof. E. C. Titchmarsh, F.R.S., *Nature* 198 (1963), 1039; Zbl. **107,** 248.
27. "Some problems in the theory of Fourier transforms", *Arch. Rational Mech. Anal.* 14 (1963), 213–216; MR **27** # 6084 (P. L. Butzer); Zbl. **122,** 344 (H. Delavault).
28. "Umkehrformeln für Fourier-Transformationen, Approximations- und Interpolationstheorie", *Approximationstheorie*, Proc. Oberwolfach Conference 1963, (ISNM 5, Birkhaüser, Basel, 1964), pp. 60–71; MR **31** # 6094 (A. E. Danese); Zbl. **135,** 336 (T. H. Ganelius).
29. "Fourier transforms and inversion formulae for L^p functions", *Proc. London Math. Soc.* (3), 14 (1964), 271–298; MR **28** # 3293 (P. G. Rooney); Zbl. **123,** 87 (G. Doetsch).
30. "The representation of functions as Laplace transforms", *Math. Ann.* 159 (1965), 223–233; MR **31** # 5041 (P. G. Rooney); Zbl. **134,** 104 (G. Doetsch).
31. "On a generalisation of the Köthe coordinated spaces", *Math. Ann.* 162 (1966), 351–363; MR **32** # 8111 (K. Sundaresan); Zbl. **137,** 314 (G. Goes).
32. "Laplace transformations of distributions", *Canad. J. Math.* 18 (1966), 1325–1332; MR **34** #3228 (S. Łojasiewicz); Zbl. **166,** 400 (G. Doetsch).
33. "The foundations of thermodynamics", *J. Math. Anal. Appl.* 17 (1967), 172–193; MR **36** # 2360 (H. A. Buchdahl); Zbl. **163,** 230 (H.-G. Schöpf).
34. "Linear spaces and operators; integral equations", *Handbook of physics* (Ed. E. U. Condon and Hugh Odishaw, 2nd ed., McGraw Hill 1967), Part 1, Chapter 6, pp. 83–89.
35. *k-fold preordered sets*, Proc. 1967 Pasadena International Symp. on Applications of Model Theory to Algebra, Analysis and Probability, California, 1969, pp. 300–307; MR **54** # 180; Zbl. **211,** 15 (R. Telgársky).
36. "Linear transformations subject to functional equations induced by group representations", *Abstract spaces and approximation theory*, Proc. Oberwolfach Conference 1968 (Eds. P. L. Butzer and B. Sz.-Nagy, ISNM 10, Birkhaüser, Basel, 1969), pp. 248–257; MR **41** # 4299 (G. O. Okikiolu); Zbl. **186,** 440 (Autorreferat).
37. "Functional equations for linear transformations", *Proc. London Math. Soc.* (3), 20 (1970), 1–32; MR **40** # 7870 (G. O. Okikiolu); Zbl. **187,** 79 (Autorreferat).
38. *Mathematics among the sciences*, Inaugural lecture, Chelsea College, University of London, 1970.
39. *Integral transforms*, Proc. Iranian Math. Conf., Shiraz (1970), 9–18.
40. *Appropriate groups and integral transforms*, Proc. Iranian Math. Conf., Shiraz (1970), 19–24.
41. *Linear transformations and group representations*, Proc. International Symp. Operator Theory 1970, *Indiana Univ. Math. J.* 20 (1971), 883–885; MR **53** # 8964 (G. O. Okikiolu); Zbl. **249,** 43018 (Y. Asoo).
42. *Group representations and integral transforms*, Proc. International Conference on Hilbert space operators and operator algebras, Tihany, 1970, pp. 107–111, Colloq. Math. Soc. János Bolyai 5 (North-Holland, Amsterdam, 1972); MR **50** # 14079 (R. A. Askey); Zbl. **254,** 44001 (Autorreferat).
43. "Subdefinite functions", *Linear operators and approximation*, Proc. Oberwolfach Conference 1971 (Birkhaüser, Basel, 1972), pp. 175–177; MR **51** #11018 (J. D. Stewart); Zbl. **252,** 42024 (Autorreferat).
44. "Translation invariant transformations of integration spaces", *Acta Sci. Math. (Szeged)* 34 (1973), 35–52; MR **47** #7507 (G. O. Okikiolu); Zbl. **262,** 47027 (Autorreferat).
45. "Fourier transform methods for solution of differential equations", *Linear operators and approximation, II*, Proc. Oberwolfach Conference 1974 (ISNM 25, Birkhaüser, Basel 1974), pp. 443–460; MR **52** # 6082 (J. Kučera).
46. "State spaces in thermodynamics", *Bull. Inst. Math. and Appl.* 10 (1974), 122–125.
47. Philip Stein, *Bull. London Math. Soc.* (2), 7 (1975), 321–322, MR **51** # 12463.
48. *Transform methods in the solution of differential equations*, Proc. 7th Nat. Math. Conference, Tabriz 1976, (1977), 21–34.

49. *Linear transformations intertwining with group representations*, Proc. 7th Nat. Math. Conference, Tabriz 1976, (1977), 35–46.

50. "The exponential map for symmetric operators in spaces with an indefinite scalar product", *Linear spaces and approximation*, Proc. Oberwolfach Conference 1977 (Eds. P. L. Butzer, B. Sz.-Nagy, ISNM 40, Birkhaüser, Basel, 1978), pp. 91–99.

51. "Approximation by products to functions with group symmetry", *J. London Math. Soc.* 21 (1980).

February 1981.
University of Sussex,
Brighton.

D. E. EDMUNDS