HAROLD DAVENPORT

C. A. ROGERS

Harold Davenport's father, Percy Davenport, worked in the office of Perseverance Mill, a cotton mill in Huncoat, near Accrington. At first a clerk, he became the company secretary. He married Nancy Barnes, one of the daughters of John Barnes, the owner of the mill. Their first child, Harold, was born on the 30th October, 1907, their only other child Grace was born a few years later.

When 10 or 11 years old, Harold started at Accrington Grammar School, and later he wrote that he had had on the whole a very happy and enjoyable time there. He discovered the public library, and read every work of Dickens that he could obtain. He maintained and extended his interest in the English classics throughout his life. He was much inspired by his chemistry master, Mr. Ackroyd, and by his mathematics mistress, Miss Heap, " a lady with enthusiasm for mathematics, who paid no attention —thank God—to any regular syllabus or curriculum there may have been ". He specialized in these two subjects, and in 1924 obtained scholarships from Lancashire County and from Manchester University that enabled him to spend the next three years, from the age of 16 to 19, at Manchester University.

He went to Manchester University with the intention of taking both Honours Mathematics and Honours Chemistry, but was forced to make a choice between the two. For better or worse he chose mathematics. It is clear, from what he wrote later, that he enjoyed the mathematical content of his course, but was rather too shy to take full advantage of the social and cultural opportunities. In particular, he learnt real analysis from C. Walmsley, complex analysis from L. J. Mordell, and applied mathematics from E. A. Milne. He obtained his degree in 1927 with First Class Honours.

Encouraged by Milne, he had entered for, and obtained, a Scholarship to Trinity, and now left home for Cambridge to take a second first degree; it was quite usual in those days for graduates from other universities to spend two years taking a Cambridge degree. He quickly found himself one of a group of mathematical friends, including H. S. M. Coxeter, R. E. A. Paley, D. H. Sadler and H. D. Ursell. His main recreations were walking, the cinema and the theatre. His Directors of Studies were R. H. Fowler for Applied Mathematics and S. Pollard and later A. S. Besicovitch for Pure Mathematics.

Recently Coxeter wrote in a letter to Mordell:

" When Davenport was working for the Tripos he seemed wonderfully relaxed.

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He would give me a cheerful welcome whenever I dropped in to see him in his room under the Clock in Trinity Great Court. I would find him listening to ' Scheherezade ' on the phonograph or reading Gibbon's *Decline and Fall* for the third time. He and Sadler and I often went together to a cinema or to the Festival Theatre. I once asked him how he still managed to do such a prodigious amount of mathematics. He replied ' Between midnight and 3 a.m.' His mind must have worked so rapidly that he could do in those 3 hours more than anyone else could in 6."

While at Cambridge, Davenport took some Applied Mathematics courses each year, but he was more attracted by the pure mathematics, especially the advanced lectures of J. E. Littlewood on the theory of primes and of A. S. Besicovitch on Almost Periodic Functions and on Sets of Points. He took Part II in 1929, offering both Schedule A and Schedule B, and, after writing nearly twenty three-hour papers, emerged as a Wrangler in Schedule A and with a Distinction in Schedule B.

Davenport stayed at Cambridge to work under the supervision of J. E. Littlewood. Initially Littlewood gave him a selection of problems in analysis and number theory, he evidently found the number theory problems, especially those on the distribution of quadratic residues, the more attractive. Littlewood now regards his supervision of Davenport as " nominal ", saying that Davenport thought of his own problems and that he (Littlewood) just read his work and made encouraging noises. Although he submitted his first papers [**1, 2**] in the summer of 1930 (to the *Journal* of the London Mathematical Society) and was awarded a Rayleigh Prize in 1931 and a Trinity Fellowship in 1932, he did not start his mathematical career in any very spectacular way and gave no hint of his later productivity. Littlewood recalls a conversation at the beginning of a new Lent term. He asked Davenport what he had been doing in the Vac.

" Nothing ", Davenport replied.

" That's all right, the great time for work is in the long vacation, and that's about all you get when you have a job."

Something in Davenport's reaction made Littlewood ask, " You do work in the Long Vac. ? "

" No! "

" Well, I think a young man who can't live for pleasure, *and* do a substantial job of work shows a poor mastery of the art of life."

" I'll think it over."

I myself have no doubt that the result of thinking it over was a resolution to combine a substantial job of work with a determination to find time for pleasure as well. No matter how hard he worked later he always found time to do the things that he enjoyed.

During the tenure of his Trinity Fellowship he was invited by H. Hasse to stay with him in Marburg, mainly so that Hasse could improve his English. Later Davenport wrote " I learnt a lot from him, though nothing like as much as I should have done had I been more receptive (less pigheaded). In return he took an interest in quadratic residues and was able to go further than I had done."

Davenport returned to Cambridge and started to write a crescendo of papers. While much influenced by Hardy and Littlewood, he mainly worked by himself, or in the later years with Heilbronn. His interests at this time clearly centred on the theory of exponential sums.

From 1933 onwards a succession of mathematicians arrived in Cambridge from Germany, driven out by the actions and threats of the Nazi party. Davenport, with his ready sympathy, and his knowledge of German, did much to help them in all sorts of practical matters. In particular, he came to know R. Rado, K. A. Hirsch, R. Courant, A. Walfisz, Miss O. Taussky, H. Kober, and K. Mahler in this way.

On the expiry of his Trinity Fellowship in 1937, Davenport was appointed by Mordell to an Assistant Lectureship in Manchester. Under Mordell's influence Davenport acquired a lasting interest in the Geometry of Numbers and in Diophantine Approximation. Mordell also recruited K. Mahler, P. Erdős, and for a short while B. Segre, forming a concentration of talent that can seldom have been equalled. Davenport, Erdős, Ko, Mahler and Žilinskas found time to play regular bridge. Mahler, who had only recently taken up the game, was prone to miss the best play, and the others and Mahler himself were soon describing poor play as being O.M. (or more correctly $O(M)$). Mahler long remained unaware that the Landau notation was in use and that this stood for "Order of Mahler".

At this time and later Davenport often worked with Erdős. Erdős never missed an opportunity of seeking out Davenport to interest him in his number theoretical problems and to work with him.

In 1938 Davenport received the Cambridge Sc.D.; I am told that both examiners referred independently to the "grace" of his work. In 1940, while still an Assistant Lecturer at Manchester, he was elected a Fellow of the Royal Society for his distinguished contributions to the Theory of Numbers. In 1941 he was awarded the Adams prize of the University of Cambridge for essays on Waring's Problem and on the Geometry of Numbers. Then in October 1941 he joined the University College of North Wales as Professor of Mathematics in succession to Professor W. E. H. Berwick.

While at Bangor, Davenport met Miss Anne Lofthouse who was on the staff of the modern languages department. They were married in 1944. Although she never attempted to understand mathematics, Anne took a great interest in Harold's mathematical work and mathematical friends, and sustained him in all he did. They welcomed very many mathematical visitors to their London flat and later to their Cambridge house.

The mathematics students and some of the mathematics staff of University College, London had been evacuated from London to Bangor. The mathematics honours students of the two University Colleges were taught together by the two staffs. In 1945, University College London ill repaid the hospitality they had received from the University College of North Wales by taking Davenport to London to succeed G. B. Jeffery as Astor Professor of Mathematics. Initially H. S. W. Massey was head of the Department of Mathematics with Davenport in charge of the pure side of the department. Later, in 1950, when Massey became Quain Professor of Physics, Davenport became head of the department with W. R. Dean in charge of the applied side under Davenport. Both arrangements worked well, the two professors working closely together. Davenport undertook considerable administrative work for the University of London, being Chairman of the Board of Studies, of its Higher Degrees Sub-Committee and of the Board of Examiners for the B.Sc. (Special) Degree (perhaps not all at once). He was glad to be relieved of this work when he moved to Cambridge, but regretted the loss of the secretarial assistance he had had in London. In his day-to-day administration of the Department he was much helped by H. Kestelman, whom he held in high regard, and to whom he acknowledged a debt of gratitude.

On coming to London, Davenport undertook the supervision of the first of a long line of research students, and started his Number Theory Seminars. J. H. H. Chalk (who later took a second Ph.D. under the supervision of L. J. Mordell at Cambridge) was one of Davenport's first students. Amongst the early attenders at the seminar was Freeman J. Dyson, and Dyson's remarkable proof (Dyson, 1948a, b) of Minkowski's conjecture for the product of 4 non-homogeneous linear forms was an immediate reaction to one of Davenport's seminars, the first draft, 60 pages long, being written in three or four days. Although I, myself, was working under the supervision of R. G. Cooke and L. S. Bosanquet, I attended Davenport's lectures and seminars, and I am proud to claim to be one of Davenport's students. He gave me inspiration and unlimited help and friendship. At this time Davenport worked mainly on the Geometry of Numbers and on Diophantine Approximation; he also acquired a lasting interest in problems of packing and covering. It was this last interest that spurred me to some of my most satisfying work.

Davenport spent the year 1947–48 visiting Stanford University in California. Although he acquired a taste for the American way of life, and made life-long friends of Pólya and of Szegö, it is scarcely possible to detect any influence of this visit on his work. It is perhaps surprising that essentially the only joint work we did, was done by correspondence during this period. Although we often discussed problems of mutual interest at other times, when solutions came they were due to one or the other (usually Davenport). He was an exceedingly rapid worker and I could not keep up with him.

In 1946 K. F. Roth came from Cambridge, after a year's schoolmastering at Gordonstoun, to research under the supervision of T. Estermann. On the completion of his thesis, he joined the staff at University College. Naturally he attended Davenport's lectures and seminars. He soon turned from problems of the Hardy–Littlewood type to problems of a more diverse nature, allowing freer scope for his remarkable originality, and he received much encouragement from Davenport.

Roth often told of an amusing incident at the Amsterdam Conference in 1954. Roth had given a twenty-minute talk about his work on irregularities of distribution (Roth, 1954), giving a concise but essentially complete proof of his main result, but leaving himself no time to discuss its significance. Davenport found Roth afterwards, and explained to him what he had done wrong in preparing his talk, and exactly how he should have presented his material. Roth escaped from Davenport only to meet Mordell, who congratulated Roth on his talk, saying how much he had enjoyed it, and how much Roth's style of lecturing had reminded him of Davenport's. It need hardly be said that by the next international conference in 1958, Roth had become a brilliant lecturer. Sometime after this Amsterdam conference Davenport initiated a "teaching seminar" where the participants were to study Siegel's and Dyson's work on the Thue–Siegel theorem, and to explain it to each other. One of the major assignments fell to Roth, and he soon obtained a deep understanding of the method and went on to obtain his spectacular improvement for which he received a Field's Medal in 1958. Davenport was very proud of Roth's brilliance and held him in high regard and affection.

In 1950, shortly after his return from India, G. L. Watson sent his elementary proof of the seven cube theorem to Davenport. Davenport at once gave Watson all possible encouragement and in a few years Watson had an established reputation as a mathematician and a position at University College.

D. A. Burgess worked under Davenport's supervision from 1956 to 1958. His work on the distribution of quadratic and higher residues to a large prime modulus (Burgess, 1957) was the first major advance in some of these problems since the original work of Vinogradov in 1918. Burgess's acknowledgment "I take this opportunity of thanking Professor Davenport for much valuable advice, and also for preparing the final draft of the paper" is remarkably frank (much franker than some other acknowledgments that have been written to Davenport): but Davenport made it clear that the credit for the advance was due to Burgess.

While in London, Davenport did a great deal of work for the London Mathematical Society. He was an ordinary member of Council from 1944 to 1947, then Librarian from 1950 to 1957 and finally President from 1957 to 1959. In particular, it was largely through the joint efforts of Davenport and W. B. Pennington that the Society obtained an interest-free loan from the Nuffield Foundation and embarked on its ambitious and highly profitable reprint programme.

By 1953, Davenport was concerned with the increasing delays in publication. He initiated a proposal for the publication of a journal by the University of London. Although the mathematicians at the other London Colleges were anxious to help, they could find no way of offering any financial support. It was only at University College that such support could be found, and so the new journal was published from the Mathematics Department of the College. Davenport consulted his classical colleagues and was delighted to find that, not only was the title *Mathematika* a good Greek word, but it was also the title the Greeks might well have given to a mathematical journal, had they published one. Of course he retained his editorship and his interest in *Mathematika* throughout his life.

In 1956 he obtained his result [**112**] on indefinite quadratic forms in many variables, showing that if a quadratic form in n variables is of signature $(r, n-r)$ and

$$n \geqslant 185, \quad r \geqslant 37, \quad n-r \geqslant 37,$$

then the quadratic form takes arbitrarily small values, non-trivially. This was a quite new departure, combining a re-modelled form of the Hardy–Littlewood method with ideas from the Geometry of Numbers. It marked a quite sudden change in Davenport's interests; he turned from the Geometry of Numbers to other problems in Number Theory. In particular, his follow-up work on cubic forms brought him into contact with D. J. Lewis and B. J. Birch who independently obtained results overlapping substantially with Davenport's; this led to most fruitful collaborations between Davenport and Lewis and Davenport and Birch.

In 1958, Davenport moved to Cambridge to take up the Rouse Ball Professorship. He now had less administration to do and his mathematical output increased. He also found it easier to spend a term or two away visiting the United States or, for one period, in Germany to take up his appointment as the visiting Gauss Professor der Mathematik an der Akademie der Wissenschaften zu Göttingen. Visits to Ann Arbor and return visits by Lewis to Cambridge enabled the collaboration of Davenport and Lewis to flourish. A visit to Boulder for a couple of months in 1968, produced a flurry of work with W. Schmidt on problems of Diophantine Approximation.

A. Schinzel visited Cambridge for the session 1960–61. He shared Davenport's interest in the properties of Polynomials, and they were soon working together. They clearly enjoyed collaboration and did so whenever they could meet.

In 1963 and 1964 E. Bombieri spent some time with Davenport in Cambridge. In 1965 Davenport visited Milan for a month. He started work with Bombieri on the distribution of prime numbeıs, trying to show that the difference between consecutive prime numbers is sometimes rather smaller than its usual value. They found that they needed information about primes lying in arithmetic progressions. Davenport told Bombieri of Roth's large sieve result (Roth, 1965); a result that was not in a form that could be used in their problem. When Davenport returned from a four-day holiday that he had taken with his wife visiting Florence and Venice, they were met by Bombieri at the railway station. Bombieri explained that he had not been to bed for four days and produced a manuscript with his version of the large sieve and its application to the distribution of primes in arithmetic progressions (Bombieri, 1965).

At Cambridge Davenport was never without a research student. One was J. H. Conway who has done some quite remarkable work. Another was A. Baker. According to Davenport, Baker worked largely independently, just giving him completed manuscripts to read. It may be my fancy, but my impression is that

Baker was much more influenced by the writings of Mahler and others than by his contacts with Davenport. Davenport was very pleased to know that Baker had been put forward to the Committee of the International Congress of Mathematicians 1970 for consideration of the award of a Field's Medal; he would have been delighted had he known of the subsequent award. Davenport also thought very highly of one of his last students H. L. Montgomery.

Davenport was naturally one of the members of the committee set up by the London Mathematical Society to edit the collected works of G. H. Hardy. His notes on the first part of the first volume provide a model for such work. He, with assistance from his wife, did a very great deal of the general editorial work and proof-reading needed for the subsequent volumes.

After a lifetime's heavy smoking (despite various attempts to give it up) Davenport had to have a lung removed in January 1969. His condition improved but then deteriorated and he died on the 9th June. During his last few months he was not well enough to undertake mathematical work, and spent some time writing notes on his early life and on the first half of his mathematical work. These notes have been a most valuable help in writing this memoir. He leaves his widow, Anne, with their children James and Richard.

The reading of such a brief account of a man's life will bring him clearly to the minds of those who knew him well; but it leaves unsaid much that must be said to give others a true picture of him. Although mathematics was Davenport's dominant interest, he always found time for other things. He used to go on most energetic walking holidays, sometimes with Heilbronn. He read and re-read the works of Dickens, Johnson, Boswell, Trollope, Austen, Gibbon, Lewis Carroll and Wodehouse. In particular he read *The Decline and Fall* a dozen times. A few years ago he achieved a boyhood ambition of making a working pendulum clock using only standard Meccano parts. His children came late in his life, but he always gave them his time and attention, and derived much joy from them.

Davenport always looked on mathematics as a human activity. The problems were there; the task was to solve them or to help and encourage others to solve them. The measure of his greatness is the extent to which he succeeded in this task. While his own work can be surveyed, the extent to which he helped others can only be guessed; he was probably responsible for encouraging work at least as extensive as his own. But he always regarded mathematicians as people, not as abstractions. He made his collaborators and colleagues his friends, and gave them generously of his humour and wisdom. He made a practice of writing helpful letters to all who approached him on mathematical matters whether they were professionals, students, amateurs or even cranks. By correspondence and by direct contact he stimulated and encouraged many mathematicians to do much of their best mathematics. Those of his research students who could find their own problems were guided to the relevant literature and were encouraged and helped to solve them; others were found interesting problems within the bounds of their capabilities. At least two of the mathematicians

Davenport helped have resolved to try always to be as kind and helpful to young people as Davenport had been to them.

Lewis writes of Davenport's life at Cambridge:

" Davenport used to sit from 10 to 12 most mornings drinking coffee and talking to his students and colleagues, including the many post-doctoral visitors who appeared at Cambridge each year. A pad of paper was readily at hand. The students always knew where he could be found and that he was always ready to discuss their latest successes and failures. Usually it was a conversation between him and one other; but the students all sat around waiting their turn to put a question. The conversation was almost entirely mathematical, nevertheless a regular habitué of the coffee house soon was aware of the Davenport philosophy regarding mathematics and life in general. 'Mathematicians are extremely lucky, they are paid for doing what they would by nature have to do anyway. One should not have a non-teaching fellowship too long, there comes a time when one must make a contribution to society. Great mathematics is achieved by solving difficult problems not by fabricating elaborate theories in search of a problem.' But usually the conversation became an effort to determine the answer to the mathematical problem posed."

Davenport would say that, when he was young he used to enjoy going to lectures, but he had long since reached the stage when he could only bear to go to a lecture if he was giving it himself, and was rapidly approaching the stage when he could hardly bear to go to a lecture even if he was giving it himself. In fact he never reached this last stage; he was an outstanding lecturer and clearly enjoyed exercising his skill. He was by nature rather conservative, taking the attitude that all change was for the worse, but his tact, in putting forward proposals to mitigate the evils of any change that seemed inevitable, usually resulted in the change being for the better rather than for the worse.

Davenport was essentially a problem solver, being impatient of abstract theories that merely systemised known results. Indeed the main bulk of his work was centred round a few key problems that he regarded as of outstanding importance: the distribution among the residue classes of the values of a polynomial with integral coefficients; the values taken, for integral values of the variables, by an algebraic form, with rational or real coefficients; Minkowski's conjecture on the product of non-homogeneous linear forms; the approximation and simultaneous approximation of real numbers by rational numbers. Although he often studied his problems from a geometrical point of view, sometimes drawing many diagrams, or even making paper models, he almost invariably recast his proofs in a severe analytical form, so that they could be easily checked and be seen to be correct. He often produced many draft versions of a single publication being dissatisfied until he had produced the strongest results that his methods would yield, and until the proofs had been explained in the best way he could find. His results seemed to come from hard and systematic study of his problems rather than by some sudden flash of inspiration; no doubt he had his flashes but they seemed to come only as a result of a considerable study.

In three subsequent sections, we give a brief survey of Davenport's main work. It remains here to mention his books and some of the distinctions he received. His first book *The Higher Arithmetic*, Hutchinson's University Library, London, 1952, made a very genuine effort to bring some of the most beautiful results of the Theory of Numbers within the reach of the man in the street. As he says in the introduction, he was well aware that it would not be read without effort by those who are not, in some sense at least, mathematicians. But it has undoubtedly been a source of joy and inspiration to many mathematicians both amateur and professional.

His second book, *Analytic methods for Diophantine equations and Diophantine inequalities*, Ann Arbor Publishers, Ann Arbor, Michigan, 1962, gives a most valuable introduction to the Hardy–Littlewood method, including Vinogradov's improvement. It discusses in considerable, but not complete, detail Davenport's modifications **[116, 134]** that enabled him to show that arbitrary cubic forms with integral coefficients and 16 or more variables represent zero non-trivially; the details are given for such forms with 17 or more variables. It discusses in detail the proof of Davenport and Heilbronn **[49]** that an indefinite diagonal quadratic form in five variables with real coefficients, not all in rational ratios, assumes arbitrarily small values. Further results of Birch, Davenport and Lewis are sketched. The whole serves as an excellent introduction to a rapidly developing branch of number theory.

His book *Multiplicative Number Theory* (Markham, Chicago, 1967) contains an extremely readable account of the analytic approach to the theory of the distribution of primes in arithmetic progressions, taking this as far as Bombieri's important theorem (E. Bombieri, 1965) and the Davenport–Halberstam proof of the basic sieve results of Roth and Bombieri.

Davenport received the Berwick Prize of the London Mathematical Society in 1954. He was elected an ordinary member of the Royal Society of Sciences of Uppsala in 1964. He received the Sylvester Medal of the Royal Society in 1967, and an honorary degree of D.Sc. from the University of Nottingham in 1968. After his death, *Mathematika* changed its title page to acknowledge its foundation by Davenport. The London Mathematical Society published the first part of the 23rd volume of the third series of its *Proceedings* as the Davenport Memorial Issue. *Acta Arithmetica* has also dedicated its volume 18 to Davenport's memory.

In the next section I give a brief account of Davenport's contribution to the Geometry of Numbers and to Diophantine approximation. I am most grateful to Dr. B. J. Birch for providing the section on the analytic theory of Diophantine equations and to Professor H. Halberstam and Professor D. A. Burgess for writing the section on multiplicative number theory.

C. A. Rogers

Work on the Geometry of Numbers and on Diophantine Approximation

When Davenport went to Manchester for the beginning of the session 1937–38, Mordell showed him a letter from Siegel proving an interesting result on the product of n non-homogeneous linear forms. Much later Davenport wrote " with the impertinence of youth I could not resist simplifying Siegel's proof and with great generosity Siegel insisted that I should publish my simplified version instead of his publishing anything ". This Davenport did in **[22]** explaining the circumstances. Let $L_1, \ldots, L_n$ be real linear forms

$$L_1 = a_{i1} u_1 + \ldots + a_{in} u_n, \qquad i = 1, \ldots, n, \tag{1}$$

with determinant

$$\Delta = |\det (a_{ij})|,$$

and let $c_1, \ldots, c_n$ be real constants. Siegel's result is that there is a constant γ_n, independent of $L_1, \ldots, L_n, c_1, \ldots, c_n$, such that the inequality

$$\left| \prod_{i=1}^{n} (L_i + c_i) \right| \leqslant \gamma_n \Delta, \tag{2}$$

always has a solution in integers $u_1, \ldots, u_n$. Minkowski had proved this result with $\gamma_2 = \frac{1}{4}$ when $n = 2$ and had conjectured that, in general, it holds with $\gamma_n = 2^{-n}$. Siegel's value for γ_n was too large for his result to be a good approximation to Minkowski's conjecture, but the ideas in this short note have been used repeatedly in later work. Problems connected with this conjecture of Minkowski on non-homogeneous linear forms recur in Davenport's subsequent work.

Davenport took up the problem of finding the arithmetic minimum of a product of three real linear forms, studying the problem by geometrical methods and drawing diagrams on triangulated graph paper. When he came to write up the work **[24, 25]** he eliminated all reference to the geometry he had used as a guide and presented a severely analytic proof that, in the above notation, there are always integers u_1, u_2, u_3, not all zero, satisfying

$$|L_1 L_2 L_3| \leqslant \tfrac{1}{7}\Delta.$$

Much later he presented a beautifully simple proof **[39]** of this result, and by a very complicated elaboration of his simplified method, showed **[42]** that there will be values of u_1, u_2, u_3, not all zero, satisfying

$$|L_1 L_2 L_3| < \frac{1}{9 \cdot 1} \Delta,$$

except when $L_1 L_2 L_3$ is equivalent to a multiple of the norm form of a cyclic cubic field of discriminant 49 or of discriminant 81. This provides an analogous " isolation " situation similar to the well-known Markoff results for quadratic forms. In a joint paper with Rogers **[79]** such isolation results and results asserting the existence of infinitely many solutions were discussed in a general setting. This work was taken further by Rogers (1953) and J. W. S. Cassels and H. P. F. Swinnerton-Dyer (1955). Very recently Swinnerton-Dyer, by very subtle use of an electronic computer, has found a chain of 18 special forms so that the inequality $|L_1 L_2 L_3| \leqslant (1/17)\Delta$ can be satisfied unless $L_1 L_2 L_3$ is equivalent to one of the 18 forms. Davenport and

Swinnerton-Dyer have studied the product of four linear forms and the result will be published, if the necessary calculations work out as expected.

Shortly after completing his "$\frac{1}{7}$" result, Davenport tackled the corresponding result for the product of three linear forms, one real and two conjugate complex, or equivalently the form $L_1(L_2{}^2+L_3{}^2)$, showing that there will be integral values of u_1, u_2, u_3 not all zero, satisfying

$$|L_1(L_2{}^2+L_3{}^2)| \leqslant \frac{2}{\sqrt{23}}\Delta.$$

Although the proof is presented in an analytic form, the geometry from which it was obtained is less well hidden, and indeed this is one of the very few of Davenport's papers that actually contains a diagram. No really simple proof of the result is available; the best that has been obtained so far is based on Mordell's reduction (Mordell, 1942) to the problem of a binary cubic form and Davenport's simplified discussion **[41]** of the binary cubic problem. The discussion of the cases of equality and of a limited type of "isolation" is given in **[79]**. An application to the problem of simultaneous Diophantine approximation was obtained jointly with Mahler in **[47]**.

Davenport **[28]** gave a very simple proof of Minkowski's theorem on the successive minima of a convex body, Minkowski's original proof being long and obscure. Davenport's proof has been criticised on the grounds that he merely remarked that functions ϕ_i, $i = 1, 2, \ldots, r-1$ could be defined to satisfy certain conditions, without explaining how this was to be done. These criticisms seem quite unjustified as it clearly suffices to take $\phi_i(x_r{}^*, \ldots, x_n{}^*)$, $i = 1, 2, \ldots, r-1$ so that $(\phi_1, \ldots, \phi_{r-1}, x_r{}^*, \ldots, x_n{}^*)$ is the centre of gravity of the section of $\lambda_{r-1} K$ by the linear space

$$x_r = x_r{}^*, \ldots, x_n = x_n{}^*,$$

or the Steiner point of this section, or, if K is taken to be closed rather than open, the nearest point of this section to the point $(0, \ldots, 0, x_r{}^*, \ldots, x_n{}^*)$.

In **[29]** Davenport gave a simple proof of Minkowski's conjecture (see above) for the product of three non-homogeneous linear forms; Remak's earlier proof was extremely difficult. Davenport explained his method at his seminar and in a few days F. J. Dyson had found his long and deep proof of the four dimensional case of the conjecture. The conjecture, which is trivial when the coefficients of the forms are rational, and which has been proved by Birch and Swinnerton-Dyer (1956) when the coefficient matrix of the forms is "nearly" diagonal, remains, in its general form, one of the most baffling problems in the Geometry of Numbers.

Davenport followed up his proof **[41]** of Mordell's results, giving the best possible inequalities for the arithmetic minimum of a binary cubic form

$$ax^3+bx^2y+cxy^2+dy^3,$$

by providing a reduction theory for such forms **[45, 46]** and by finding asymptotically the average number of such forms with integral coefficients and a numerically large discriminant

$$D = 18abcd + b^2 c^2 - 4ac^3 - 4b^3 d - 27a^2 d^2$$

[89, 90]. These results were applied later to a problem of Volkmann **[126]** and in collaboration with Heilbronn **[190, 191]** to determine the density of discriminants of cubic fields.

Davenport studied a series of special products of two or three linear forms, finding their non-homogeneous minima **[51, 52, 61, 62, 66]** and in some cases obtaining isolation results. He also showed that, in the notation already used, for any real constants c_1, c_2, c_3, there are integers u_1, u_2, u_3 satisfying

$$|(L_1 + c_1)^2 + (L_2 + c_2)^2 - (L_3 + c_3)^2| \leqslant \left(\frac{27}{100}\right)^{\frac{1}{3}} \Delta^{\frac{2}{3}},$$

and that this result is isolated **[63]**. These studies of non-homogeneous minima led through an intermediate result **[68]** to his paper **[70**, see also **77]**, which shows that given L_1, L_2, there are always real numbers c_1, c_2 such that

$$|(L_1 + c_1)(L_2 + c_2)| \geqslant \frac{1}{128} \Delta,$$

for all pairs of integers u_1, u_2, and which shows that Euclid's Algorithm can hold in the real quadratic field $k(\sqrt{m})$ only if $m \leqslant (128)^2$. This enabled him in the first place, with Chatland **[74]**, to show that Euclid's algorithm holds in no real quadratic fields beyond the last known example (believed at the time to be $\sqrt{97}$ but now, after the work of Barnes and Swinnerton-Dyer (1952), known to be $\sqrt{73}$), and secondly to show that Euclid's algorithm holds in only a finite number of cubic fields with negative discrimant **[76]** and in only a finite number of complex quartic fields with complex conjugate fields **[82]**. J. W. S. Cassels (1952) gives a unified account of these last results.

Davenport constantly returned to problems connected with simultaneous Diophantine approximation. He was the first to show that there are continuum many pairs θ, ϕ of irrational numbers that are badly approximable in that, for some c, there are only a finite number of integral solutions of the inequalities

$$\left|\theta - \frac{u}{q}\right| < \frac{c}{q^{\frac{3}{2}}}, \quad \left|\phi - \frac{v}{q}\right| < \frac{c}{q^{\frac{3}{2}}}, \quad q > 0.$$

The proof **[104]** was exceedingly intricate, θ and ϕ being chosen so that the linear form

$$\theta u + \phi v + w$$

has one infinite sequence of approximate representations as a multiple of a linear form with coefficients from one totally real cubic field and a second infinite sequence of such representations with coefficients from a second totally real cubic field. Later (see, for example [**144**]) he obtained much more general results using far simpler methods. Two of his latest papers [**185**, **189**] written with W. M. Schmidt discuss in great depth the circumstances in which Dirichlet's theorem on Diophantine approximation can or cannot be improved.

Working in collaboration with Schmidt, he took up the problem studied by E. Wirsing (1960) of the approximation of irrational and algebraic numbers by algebraic numbers or by algebraic integers [**173**, **177**, **183**, **188**]. They obtained many striking results. For example, they proved that, if $n \geqslant 3$ and ξ is real but is not an algebraic number of degree at most $\frac{1}{2}(n-1)$, then there are infinitely many real algebraic integers α of degree at most n which satisfy

$$0 < |\xi-\alpha| \ll H(\alpha)^{-[\frac{1}{2}(n+1)]},$$

$H(\alpha)$ denoting the numerically largest coefficient in the defining equation for α.

C.A.R.

Work on the analytic theory of Diophantine equations

Heilbronn, who had met Davenport in Göttingen in 1933, came to Cambridge in 1935. In [**11**], they proved the best-possible result that every large positive integer is the sum of 17 fourth powers (the same theorem was proved independently at the same time by Estermann). Hardy and Littlewood had proved the result, but with 19 fourth powers instead of 17; Davenport and Heilbronn obtained their improvement using Vinogradov's version of the Hardy–Littlewood method. Their main extra device was Vinogradov's trick of taking the variables of different sizes—this trick was to become one of Davenport's trademarks, see for instance [**30**]. The style is terse and technical, with much quotation: a necessary lemma was proved in [**14**].

The collaboration with Heilbronn was continued. In [**18**], they proved by moderately conventional methods that almost all positive integers are of the form $x^2+y^3+z^3$. The methods of [**19**] are more exciting; here, they showed that almost all integers are the sum of a prime and a k-th power (for any fixed k). They used the Hardy–Littlewood technique to estimate the mean-square error $\sum_{n \leqslant p} (r(n)-\rho(n))^2$ where $r(n)$ is the number of solutions of $n = p+x^k$, and $\rho(n)$ is a plausible approximation to $r(n)$; the theorem follows when

$$\sum(r(n)-\rho(n))^2 = o(\sum\rho(n)^2).$$

The most interesting part of the proof is their estimate for the sum over primes, $\sum_{p<P} e(\alpha p)$; this depends on a formula for the number of primes in an arithmetic progression, with a good error term depending on Siegel's result (then very recent)

about the (presumably non-existent) real zero of an L-function. It is easy to modify the method to prove that almost all numbers have the form $p+x^k$ or $2p+x^k$ with p a prime congruent to 1 modulo 4, so almost all numbers have the form $x^2+y^2+z^k$.

In [30], Davenport announced a new method for constructing distinct sums of k-th powers, which enabled him to prove new Waring-type results for sums of k-th powers for $k = 3, 4, 5, 6$. The announcement was followed by several papers in which his method was described, developed, and applied to improve almost all the known results on Waring's problem for small exponents. Briefly, the method was described in [31] and improved in [34]; meanwhile, Erdős had produced a different, slightly weaker, method, and this was combined with Davenport's in [36]. As for applications, it was proved in [33] that, for large enough N, at least $N^{13/15-\varepsilon}$ integers less than N are sums of 3 cubes, and so almost all positive integers are sums of 4 positive cubes; in [35], he showed that every large enough positive integer not congruent to 15 or 16 modulo 16 is a sum of 14 fourth powers; and in [37] he showed that every large enough positive integer is the sum of 23 positive 5-th powers, and is the sum of 36 6-th powers. Some years later, in [87], Davenport used extreme ingenuity to prove that, for large enough N, at least $N^{47/54-\varepsilon}$ integers less than N are sums of 3 cubes; so far as I know, this is still the best known.

Between 1939 and 1956, Davenport almost deserted Waring's problem in favour of the Geometry of Numbers and Diophantine Approximation. Just two papers on the subject date from this period: [87] which we have just described, and [49], a very significant joint paper with Heilbronn. They attacked a conjecture of Oppenheim, that an indefinite quadratic form with real coefficients in 5 or more variables always takes small values, and proved it for diagonal forms. To be precise, they proved that if $\lambda_1, \ldots, \lambda_5$ are real numbers not all of the same sign, then there are integers $x_1, \ldots, x_5$ not all zero so that $|\Sigma\lambda_i x_i^2| < 1$. This is by no means a difficult paper, indeed it is one of the least complicated of all the many applications of the circle method, but it made it clear for almost the first time that Diophantine inequalities may be treated by the method as well as Diophantine equations. Papers on similar lines were soon published by Watson, Bambah, Roth and others; particularly interesting is the problem of reducing the number of variables from 5 to 4, but so far this has resisted all attacks.

Oppenheim's conjecture for non-diagonal forms is enormously harder than the diagonal problem; there are obvious difficulties in applying the essentially additive circle method to a problem which is not additive. In [112], after considerable effort, Davenport was able to solve the problem, at any rate when the form was sufficiently indefinite and involved enough variables; this was a breakthrough. To describe the ideas involved we cannot do better than quote from Davenport's presidential address to the London Mathematical Society [123]: "Let Φ be an indefinite quadratic form with arbitrary real coefficients; the problem is to prove that the inequality

$$|\Phi(x_1, \ldots, x_n)| < 1$$

is always soluble in integers (not all 0) if n is sufficiently large. The essential difficulty lies in estimating an exponential sum such as

$$\sum_{x_1=P}^{2P} \cdots \sum_{x_n=P}^{2P} e(\alpha\Phi(x_1, \ldots, x_n))$$

when P is large and α is a real number that is neither very small nor very large. It would suffice if one could prove that such a form is of lower order of magnitude than P^{n-2}. However, this cannot be true unconditionally, since it is possible (for example) that Φ may have integral coefficients and α may be 1, in which case every term in the sum is 1. It is natural to begin by investigating the consequences of the supposition that the desired estimate does not hold for a particular α; it is possible to prove that the form Φ must then represent a form in a small number of variables, say 5, whose coefficients are nearly integers. The arguments which lead to this conclusion are mainly concerned with systems of linear inequalities, and depend partly for their effectiveness on considerations taken from the geometry of numbers. If the new form, say $\Psi(y_1, \ldots, y_5)$ is indefinite, we can appeal to an elementary but elegant theorem discovered by Cassels in 1955. This states (in particular) that an indefinite quadratic form in 5 variables with integral coefficients represents zero with integral values of the variables which satisfy a simple estimate in terms of the size of the coefficients of the form. This result is applicable to the almost integral form $\Psi(y_1, \ldots, y_5)$, and leads to the conclusion that $\alpha\Phi$, and hence Φ itself, assumes arbitrarily small values. Thus, if the estimate needed for the success of the Hardy–Littlewood method fails to hold for *any* α, the desired final conclusion nevertheless follows. There remains, however, the difficulty of ensuring that the almost integral form $\Psi(y_1, \ldots, y_5)$ shall be indefinite, and at first I could do this only by imposing a condition on the signature of the original form Φ.' In his first attempt **[112]** Davenport was able to deal with forms of signature (r, s) when $r \geqslant 37$, $s \geqslant 37$; in a second paper **[114]** he mobilised the theory of successive minima from the geometry of numbers and reduced the 37 to 16. Improving the same method further, he proved in a joint paper with Ridout **[120]** that $r+d \geqslant 21$, $\min(r, s) \geqslant 6$ is enough.

In the series of joint papers **[115]**, **[118]**, **[119]**, written with Birch, a different technique is applied for the case $r+s \geqslant 21$, $\min(r, s) \leqslant 4$; using a method of Brauer, one can show that a non-diagonal quadratic form in many variables can almost be diagonalised, in the sense that one can find integral vectors $\mathbf{u}_1, \ldots, \mathbf{u}_5$ so that $\Phi(\Sigma t_i \mathbf{u}_i) = \Sigma\lambda_i t_i^2 + \Psi(\mathrm{t})$, where Ψ has small coefficients. The crux of the matter accordingly is **[119]**, where one proves a refined version of **[49]**, which tells us that for any $\delta > 0$ there is a constant C_δ so that $|\Sigma\lambda_i t_i^2| < 1$ always has integral solutions with $0 < \Sigma|\lambda_i t_i^2| < C_\delta|\lambda_1 \ldots \lambda_5|^{1+\delta}$. The proof of this estimate was a wonderful example of Davenport's power and industry (and, at the time, a wonderful example for a young student). The crucial period was 13–19 August, 1957, during which (in four separate, complete, drafts of the paper) he improved the estimate from $0 < \Sigma|\lambda_i t_i^2| < C_\delta \max|\lambda_i|^{40+\delta}$ to $0 < \Sigma|\lambda_i t_i^2| < C_\delta|\lambda_1 \ldots \lambda_5|^{2+\delta}$; at that stage, he did not think the result would be improved further, but he was able to reduce the exponent further during the Christmas vacation, obtaining the published result $0 < \Sigma|\lambda_i t_i^2| < C_\delta|\lambda_1 \dots \lambda_5|^{1+\delta}$. Subsequently, the proof has been made more transparent by Dr. Pitman (it is mentioned in her thesis). Ridout (1958) finally used the Birch–Davenport method to deal with the case $r+s \geqslant 21$, $\min(r, s) = 5$, so establishing Oppenheim's conjecture for indefinite forms in 21 or more variables.

Davenport often proposed the analogous theorem for definite quadratics: if $Q(x)$ is a definite quadratic form in (say) 21 variables, that is not proportional to a rational form, then there is an N_0 such that $|Q(x)-N| < 1$ is soluble for all $N > N_0$. So far, no-one has succeeded in proving this—the difficulty is that the result ceases to be true for rational forms.

In **[116]** Davenport proved that a rational cubic form in 32 variables always has a rational zero. It was a paper of which he was (justly) proud; in his own words, "it kept him deeply engaged for many months". The circle method is applied, and as usual the difficulty is with the minor arcs. The methods developed in **[112]** and **[114]** of estimating exponential sums are applied to the sum $\Sigma \exp(2\pi i\alpha f(x))$, where $f(x)$ is a cubic form; the "main lemma" states that *either* the sum is small *or* α is well approximable *or* $f(x_1, \dots, x_n)$ represents a form of shape $ay_0^3+g(x_1, \dots, x_m)$ for a suitable $m < n$. The crux of the paper was to deduce this useful geometric condition on f from the awkward selection of inequalities that the method throws up; some geometric condition is essential, as otherwise the lemma is simply untrue. The logic of the paper is complicated; one starts with a form $f_0(x_1, \dots, x_{32})$ and shows that either the circle method works, leading to an asymptotic formula for the number of zeros of f_0, or f_0 represents a form of the shape $a_1y_1^3+f_1(x_1, \dots, x_{24})$; either the circle method applies to $a_1 y_1^3+f_1$ or it represents a form of shape

$$a_1 y_1^3+a_2 y_2^3+f_2(x_1, \dots, x_{18});$$

and so on. After seven steps, one has a diagonal form in 8 variables, which is easily dealt with. A feature of this paper, a foretaste of difficulties to come, is that the p-adic problem involved in proving the singular series positive is no longer trivial.

In **[130]**, a very difficult paper, Davenport improved the 32 variable result to one involving 29 variables. A year later, he found the proper geometric condition that should occur in the "main lemma", and was able to prove quite simply that a rational cubic in 17 variables always has a non-trivial zero. This simple proof was written up in his book; in the paper **[134]**, he saves a little extra, and proves the result for cubics in 16 variables, but the proof is no longer simple. There the problem rests; no better result is known for cubics, no simple results are to be expected for forms of even degree, and though results have been proved for "general" forms of odd degree, and for forms of odd degree in stupendous numbers of variables, no attempt to prove a decent unconditional result for quintics has yet succeeded.

From now on, most of Davenport's papers were written jointly. This applies to all the papers on forms in many variables. The more important of these later papers may be taken in three groups. In **[135]** and **[148]**, he works out in collaboration with

D. J. Lewis the application of the ideas of **[116]** and **[134]** to, first, cubic exponential sums and, second, inhomogeneous cubic equations—their paper **[148]** was complemented by some exceedingly intricate work by G. L. Watson (1969). Then, in **[124]**, **[128]** and **[131]**, collaborating in turn with Chowla, Birch and Lewis, various favourable cases are treated, where it is particularly easy to use the circle method to prove that certain Diophantine equations are soluble; for instance, in **[128]** it is shown that if $F(x_1, ..., x_n)$ is the sum of n d-th powers of linear forms over an algebraic extension field, then $\sum \exp(2\pi\alpha F(x))$ may be estimated as well as if F were a sum of rational d-th powers; and in **[131]** it is shown that if K is an algebraic number field, then every large enough integer is the sum of two K/Q norms and a d-th power. This set of papers is probably most useful for the refinement of the technique.

Finally, there is a set of rather long papers (**[138]**, **[167]**, **[184]**, **[186]**) written jointly with Lewis in which diagonal equations are treated. The first paper of the set proves a particularly pleasant result; if $c_1, ..., c_s$ are integers not all of the same sign, then the equation $c_1 x_1{}^k + ... + c_s x_s{}^k = 0$ is soluble in integers $x_1, ..., x_s$ not all zero so long as $s \geqslant k^2+1$, except possibly when $7 \leqslant k \leqslant 17$; the theorem is the best possible when $k+1$ is prime. As usual, the proof falls into two parts: first one shows that the equation is p-adically soluble for every p, and then one uses analytic techniques to deduce, from the "local" p-adic solubility, the "global" solubility in rational integers. The new feature of the proof is that, while the analytic part of the argument is fairly standard, though skilful, the p-adic part (almost trivial in most applications of the circle method) is in this case decidedly difficult. In later papers of this set, it becomes even clearer that the analytic techniques are well understood though complicated to apply, but that the solution of p-adic equations is not. The later papers are all concerned with the solution of simultaneous diagonal equations; **[167]** deals with two cubics in 18 variables, and then in **[184]** they prove a more general but less precise result for R simultaneous diagonal forms of degree k in n variables: if k is odd and $n > 9R^2 k \log 3Rk$ then the forms certainly have a common zero. As they say in the introduction, they have not exercised great economy in the analytical work, as the answer is dominated by the p-adic problem. In **[186]**, more precise p-adic results are proved for two simultaneous diagonal forms; however, a long-standing project of Davenport and Lewis to prove a good general result about the rational solubility of a pair of general cubic equations has been frustrated by the lack of p-adic information.

B.J.B.

Multiplicative number theory

1. *Work on character sums.* Davenport submitted his first paper **[1]** in October 1930, on a subject suggested to him by Littlewood: to compute the number R_n of sets of n consecutive quadratic residues (or non-residues) mod p among $1, 2, ..., p-1$, p being an odd prime; the result to be expected was that R_n would be about $p/2^n$. In 1906 Jacobsthal had evaluated R_2 exactly, as well as R_3 when $p \equiv -1 \bmod 4$; and

he had been able to show also that if $p \equiv 1 \bmod 4$, $R_3 = p/2^3 + O(p^{\frac{1}{2}})$. After that, there was little progress for a long time, but in 1930 appeared a partial result about R_4, due to H. Hopf; this came to Littlewood's notice, and so to Davenport's, and the latter was able to prove that

$$R_n = \frac{p}{2^n} + O(p^{\frac{3}{4}}) \quad \text{when} \quad n = 4 \text{ and } 5. \tag{1}$$

Davenport was fortunate in his first investigation—unspectacular as the problem appeared at first sight, it raised questions of the greatest importance. The estimation of R_n quickly reduces to the estimation of sums of the type

$$S_r(a_1, \ldots, a_r) = \sum_{n=0}^{p-1} \left(\frac{(n+a_1)\ldots(n+a_r)}{p} \right), \tag{2}$$

where the integers $a_1, \ldots, a_r$ are incongruent mod p and (x/p) is the Legendre symbol of x modulo p; and this turns out to be, in general, extremely difficult if $r \geqslant 3$. Davenport was able to show, in an elementary but highly ingenious way, that

$$S_r(a_1, \ldots, a_r) \ll p^{\frac{3}{4}} \quad \text{if} \quad r = 3 \text{ and } 4,$$

and from this he derived (1). Although Davenport did not make this observation at the time, his result was significant in another way. The number N_r of solutions x, y of the congruence

$$y^2 \equiv (x+a_1)\ldots(x+a_r) \bmod p$$

is given by

$$N_r = \sum_{x=0}^{p-1} \left(1 + \left(\frac{(x+a_1)\ldots(x+a_r)}{p} \right)\right) = p + S_r(a_1, \ldots, a_r),$$

so that, by the results of **[1]**,

$$N_r = p + O(p^{\frac{3}{4}}) \quad \text{when} \quad r = 3 \text{ and } 4;$$

and one may obviously raise similar questions in regard to the number $N_f(k)$ of solutions of the congruence

$$y^k \equiv f(x) \bmod p, \qquad \deg f = r, \tag{3}$$

and even of more general polynomial congruences in two (or more) variables. Although Davenport did not realise this until the following year, when he went to stay with Hasse in Marburg, Artin had already conjectured (implicitly) that

$$N_f(k) = p + O(p^{\frac{1}{2}}), \tag{4}$$

where the constant implied by the O-notation depended only on k and the degree of f; and this conjecture implied that the sums Davenport had considered in **[1]** were $\ll p^{\frac{1}{2}}$.

Davenport was not to settle Artin's conjecture. It occurred to no-one at the time that (3) might best be viewed as a question of algebraic geometry over $GF(p)$, and, indeed, Weil's celebrated work was still almost ten years away. But, stimulated by Mordell and Hasse, both of whom he had managed to interest in these questions, Davenport developed his elementary methods over the next few years in a series of technically brilliant papers. In [3] he extended the ideas of [1] to higher power residues mod p, in [6] he gave a method for dealing with the sums (2) for $r > 4$, and in [27], his first *Acta Mathematica* memoir, he improved all his previous results and presented them in a very general setting; in so doing, he pushed his elementary methods about as far as they could go. [27] contained also an elementary account (as compared with Hasse's) of L-functions associated with an algebraic function field generated by an equation of the form $y^k = f(x)$, and establishes for them a functional equation, thus confirming a conjecture of Hasse. (This was to be the first of many occasions when Davenport gave technically elementary versions of proofs of other people. Often these lead to actual improvements of the results.) In [5] Davenport, following up some pioneering work of Mordell, studied the related exponential sums

$$\sum_{x=0}^{p-1} \exp\left(2\pi i f(x)/p\right) \quad \text{and} \quad \sum_{x=1}^{p-1} \exp\left(2\pi i(ax^n + bx^{-n})\right)$$

—the latter a generalisation of Kloosterman's sum—and applied his results to partial (incomplete) character sums. This work, too, was, in due course superseded by the deep results of Weil. In later years Davenport would sometimes raise with his students, or in a seminar, the question of making these results of Weil accessible to a wide circle of mathematicians, and to number-theoreticians in particular. From the point of view of the latter group, J. V. Armitage, a former student of Davenport, has recently developed some promising approaches through the geometry of numbers.

Years later, Davenport returned to the estimation of exponential sums in [135], with Lewis, and in [150], with Bombieri. In these papers the sum

$$\sum_x \sum_y \exp\left(2\pi i f(x, y)/p\right)$$

is estimated, where f is a non-degenerate cubic polynomial. The estimate $O(p^{\frac{3}{2}})$ of [135] is strengthened to $O(p)$ in [150] using the multiple exponential sum estimates of Bombieri (1966), which, in turn, were based on Dwork's work on congruence ζ-functions. [150] also discusses the problem raised by Mordell, of finding a small m not assumed as a value of the polynomial $f(x)$ modulo p. They obtained the estimate $m = O(p^{\frac{1}{2}} \log p)$ provided $[f(x)-f(y)]/(x-y)$ has an absolutely irreducible factor, a condition that was shown to be necessary by MacCluer (1966–67).

Davenport's visit to Hasse was fruitful in several ways. Not only did he learn much, as is shown in the joint paper [8] with Hasse, where some very general results of Hasse are examined in a number theoretic context and then applied to some concrete arithmetical questions, but he turned Hasse's interest to residue problems, one result of which was Hasse's famous proof (Hasse, 1936) of (4) in the case (*cf.* (3)) $k = 2, r = 3$. Mordell tells the story that Davenport challenged Hasse to demonstrate the usefulness of abstract algebra, and this was Hasse's reply!

Although Davenport did not continue in the mainstream of these important developments, what he learnt in these early years about character and exponential sums, about polynomials and algebraic number theory in general, was to play a decisive part in much of his later work. A distinctive characteristic of his highly organised mind was that he learnt always to turn experience to good account. His remarkable handling of exponential sums in applications and adaptations of the Hardy–Littlewood method illustrates this admirably—for an account see the previous section written by Birch.

Davenport returned once more, in **[101]** (jointly with Erdős), to the distribution of quadratic and higher power residues. They studied the distribution of the "partial" sum

$$S(x, h) = \sum_{n=x+1}^{x+h} \left(\frac{n}{p}\right),$$

showing that the distribution of $h^{-\frac{1}{2}} S(x, h)$ is asymptotically normal; and they were able to improve slightly Vinogradov's famous estimates of the least quadratic (and, more generally, k-th power) non-residue mod p. A few years later, Davenport proposed to his then pupil D. A. Burgess the problem of improving on **[101]**; he had in mind the possibility of a further slight improvement on Vinogradov's results, but was delighted to see Burgess go far beyond anything that might have been hoped for (Burgess, 1957).

Two other papers, **[20]** and **[141]** (with Lewis), belong to this chain of results. They deal with character sums in general finite fields: the first is concerned with the distribution of primitive roots in such fields, by the method of Vinogradov, and the second deals with the estimation of "partial" character sums in the manner of Pólya–Vinogradov and Burgess. An interesting result in **[20]** is the proposition that if θ is an arbitrary generator of $GF[p^n]$, n fixed, and if $p > p_0(n)$, there exists a primitive root of the form $\theta + x$. A somewhat similar problem, posed by Conway and solved in **[175]**, consists of showing the existence in $GF[p^n]$ of a primitive root θ for which $\theta, \theta^p, \ldots, \theta^{p^{n-1}}$ is a normal basis. The results of **[141]** have been improved, for some finite fields, by Jordan (1967) and Burgess (1967).

An interesting paper to be mentioned in this context is **[78]**, dealing ingeniously with the signs of partial sums of $L(1, \chi)$; and also **[125]** which improved P. J. Cohen's famous contribution to Littlewood's problem of finding a good lower bound for

$$\int_0^1 \left| \sum_{j=1}^{N} \exp(2\pi i n_j x) \right| dx,$$

where $n_1, \ldots, n_N$ are any N (N large) distinct integers. Cohen used the Hahn–Banach theorem to obtain the lower bound

$$c\left(\frac{\log N}{\log\log N}\right)^{\frac{1}{8}}$$

with c a positive absolute constant; and Davenport improved the exponent $\frac{1}{8}$ to $\frac{1}{4}$. Characteristically, Davenport also gave a completely elementary account of Cohen's proof.

The last group of results to be mentioned under this heading concerns Davenport's work on the large sieve with Bombieri and Halberstam. Despite his deep interest in character sums, and his knowledge of analytic methods in number theory, Davenport had made no contribution to the study of prime numbers (**[96]** is a popular expository account) up to the time that he visited Bombieri in Milan in 1965. It may have been at Bombieri's suggestion, or perhaps it was because Davenport was then helping to edit Hardy's collected papers on prime number theory; but they decided to look at the problem of estimating

$$E = \liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n},$$

a problem that had been the theme of Hardy and Littlewood's unpublished memoir "Partitio Numerorum VII" of 1926, and was later the subject of a series of papers by Rankin. It is easy to show that $E \leqslant 1$; if the prime twin conjecture were true, $p_{n+1}-p_n$ would equal 2 infinitely often so that probably $E = 0$. Actually it proved difficult to show even that $E < 1$—this was first done by Erdős using Brun's method, and later several other authors (Rankin, Ricci) obtained explicit numerical estimates, all rather close to 1. Bombieri and Davenport, like Rankin, set out to put into effect the original programme of Hardy and Littlewood—a weighted form of the Hardy–Littlewood circle method using major arcs only—and they did so in a most elegant and efficient manner; the chief new idea they introduced was the use of Bombieri's theorem on primes in arithmetic progressions—actually they had planned to use an older result of Rényi which would have led them to $E < \frac{2}{3}$—but Bombieri discovered his theorem in the course of the joint work, and this led to the improved inequality $E \leqslant \frac{1}{2}$. The final step from this to $E \leqslant 0{\cdot}46650\ldots$ was accomplished by an appeal to the Selberg upper bound sieve, also in conjunction with Bombieri's theorem.

In a last section they discussed a related problem proposed by Erdős, namely that of estimating

$$E_r = \liminf_{n\to\infty} \frac{p_{n+r}-p_n}{\log p_n} \qquad (r = 2, 3, \ldots);$$

and they showed that $E_r < r-\frac{1}{2}$. This result has since been improved by Davenport's former student, M. N. Huxley (1969) who showed, for example, that $E_2 \leqslant 1{\cdot}451\ldots$.

In the following year Davenport gave a graduate course of lectures in Ann Arbor on prime number theory leading up to Bombieri's theorem (these lectures were later published as *Multiplicative Number Theory*, the first volume of a new lecture notes series published by Markham, Chicago); and in the course of these lectures Davenport and Halberstam (who also was visiting Michigan at the time) found a simple version **[168]** of the basic large sieve inequality in the following form:

If $x_1, \ldots, x_N$ are any R real numbers such that

$$||x_r - x_s|| \geqslant \delta > 0 \quad \text{for} \quad r \neq s$$

($||x||$ denotes the distance of x from the nearest integer), and the a_n are any complex numbers, then

$$\sum_{r=1}^{R} \left| \sum_{n=M+1}^{M+N} a_n \exp(2\pi i n x_r) \right|^2 \leqslant K(N, \delta) \sum_{n=M+1}^{M+N} |a_n|^2, \tag{5}$$

with $K(N, \delta) = \frac{11}{5} \max(N, \delta^{-1})$. They also derived a corresponding inequality for character sums, but the proof they gave (also in *Multiplicative Number Theory*) was unsound except when the a_n are zero for composite n; a correct account of an even better result was, however, given in **[179]**. In **[170]** Davenport and Halberstam sharpened a result of Barban's, an interesting second mean analogue of Bombieri's theorem; this was later improved by Gallagher (1967) (who found an even simpler proof of (5)) and extended by Montgomery (1970). Davenport corresponded with Bombieri about this work, and in **[172]**, **[181]** they sharpened the form of $K(N, \delta)$ in (5) in various ways; they also gave some significant applications. The analysis in **[181]** is especially difficult and delicate. Inequality (5) attracted considerable interest, by virtue of its form and the simplicity of the argument, and gave rise to many other investigations. Notable among these are the papers of Montgomery who discovered a far-reaching extension of (5) and was able to prove with its help a new "world record" for gaps between consecutive primes—he showed that, for any $\varepsilon > 0$ and $x \geqslant x_0(\varepsilon)$, there is a prime between x and $x + x^{(3/5)+\varepsilon}$; and the papers of Huxley who extended much of this work to algebraic number fields. Both Montgomery and Huxley were students of Davenport at the time of his death. One might mention here too the paper of Burgess and Elliott (1968), the latter also a former student, who proved, using the large sieve, that the least primitive root mod p is, on average, $\ll \log^2 p (\log \log p)^4$.

2. *Polynomials and Diophantine equations.* Davenport travelled extensively after his move to Cambridge, mostly to the U.S., and when he was in residence in Cambridge there were invariably mathematical visitors there eager to work with him. In this way he entered upon several fruitful collaborative ventures, notably with Lewis, Schinzel and Bombieri. Among the results were a number of attractive papers on polynomials.

In **[129]**, with Lewis and Schinzel, is a study of the Diophantine equation $f(x) = g(y)$, where f and g are polynomials with integer coefficients. In view of

Siegel's fundamental theorem on the solubility of Diophantine equations in two variables, the problem becomes one of deciding when $f(x)-g(y)$ is irreducible over the complex field, a question of independent interest; if $f(x)-g(y)$ is thus irreducible, then, according to Siegel, $f(x)-g(y) = 0$ has at most finitely many solutions if the genus of the equation is positive. Davenport and Lewis found a simple condition which ensures irreducibility of $f(x)-g(y)$ as well as, in general, positive genus; and as a special application they showed that if

$$f(x) = x^n+x^{n-1}+\ldots+x, \quad g(y) = y^m+y^{m-1}+\ldots+y \qquad (n > m > 1),$$

then $f(x) = g(y)$ has at most a finite number of solutions.

In **[146]** Davenport and Schinzel also studied irreducibility of polynomials of several variables. In particular, and in answer to some questions posed by the latter, they succeeded in characterizing polynomials $f(x, y, z)$ with complex coefficients, irreducible over the complex field but reducible as polynomials in x and y for infinitely many values of z.

[145], with Lewis and Schinzel, sets out from the well known result that if $f(x)$, a polynomial with integer coefficients, is a k-th power for every positive integer x, then $f(x) = (g(x))^k$ identically, for some polynomial g with integer coefficients, and proves that the same result is true provided only $f(x)$ is a k-th power for some x in every arithmetic progression. Some other general results of this kind are proved, from which a similar kind of condition is seen to determine when $f(x)$ is identically the sum of two polynomial squares. This investigation was taken further in **[158]**. The topic is one where there appear to be other interesting questions still unanswered—for example, is there a "two cubes" theorem of this type?

In **[140]** and **[159]** Davenport settled two specific problems about polynomials. In the former he answered an interesting question addressed to him by N. J. Fine, showing that there do not exist rational functions F_1, F_2, F_3, with real coefficients, of $x_1, \ldots, x_4$ such that $x_1^2+\ldots+x_4^2 = F_1^2+F_2^2+F_3^2$. Since then Cassels (1964) has taken this further, into a more general context. In the latter he settled a conjecture of Birch, Chowla, M. Hall Jr. and Schinzel that if f, g are polynomials with arbitrary real or complex coefficients, then

$$\deg(f^3-g^2) \geqslant \tfrac{1}{2}\deg f+1,$$

except when $f^3 = g^2$ identically.

Finally, in **[182]** Davenport and Baker dealt with a question raised by van Lint at the 1968 Oberwolfach meeting: Is 120 the only value of N for which

$$1, 3, 8, N$$

are such that the product of any two, increased by 1, is a perfect square? This problem reduces to showing that the simultaneous Diophantine equations

$$3x^2-2 = y^2, \qquad 8x^2-7 = z^2$$

have no solution sets other than (1, 1, 1) and (11, 19, 31); and this was the specific question settled in this paper. The method was based on the deep results of Baker.

3. *Dirichlet and other series.* Davenport did not at any time become deeply involved with the study of Riemann's ζ-function and allied functions; but he was always well up in the subject, and made several interesting and useful contributions. In **[2]**, an early work suggested to him by Littlewood, he showed that, for *fixed* $s = \sigma + it$, $L(s, \chi) \ll k^{\frac{1}{2}(1-\sigma)}$ if χ is a non-principal character mod k; this improved by a logarithmic factor on what was known at the time, and provided a k-analogue of the classical estimate $\zeta(\sigma+it) \ll t^{\frac{1}{2}(1-\sigma)}$ for fixed σ. In **[10]** he gave an alternative proof of an integral mean-value theorem of Ingham (1933) for $\zeta(s)$, and extended its range of validity; and in **[171]**, written while he was Gauss professor at Göttingen, he extended slightly the region near $s = 1$ known to be free of zeros of $L(s, \chi)$, the point being that though his results were weaker than Siegel's, all the constants appearing in his results are capable of explicit computation.

In **[12]** Davenport and Heilbronn showed that the function

$$\zeta(s, a) = \sum_{n=1}^{\infty} (n+a)^{-s} \qquad (0 < a < 1, \quad a \neq \tfrac{1}{2}),$$

in contrast to $\zeta(s)$, has infinitely many zeros in $Rs > 1$ when a is rational or transcendental. The more difficult case of a algebraic was settled many years later by Cassels (1961). They began a similar investigation in regard to the Epstein zeta function, and completed this in **[15]**. Their results have since been completed by others (see Stark (1967)).

Papers **[17]**, **[21]** and **[155]** deal with delicate convergence problems. In particular, the first two study conditions on θ under which the formal identity

$$\sum_{n=1}^{\infty} \frac{a_n}{n} \{n\theta\} = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{A_n}{n} \sin 2n\pi\theta, \qquad A_n = \sum_{d \mid n} a_d,$$

arising from the Fourier expansion of $\{n\theta\}$, is valid; here

$$\{t\} = \begin{cases} t-[t]-\tfrac{1}{2}, & t \neq [t], \\ 0, & t = [t]. \end{cases}$$

The special cases $a_n = \mu(n)$, $\lambda(n)$, $\Lambda(n)$ are shown in **[17]** to lead to valid relations for all rational θ, and for almost all θ; and in **[21]**, Vinogradov's work on trigonometric sums with prime arguments having appeared in the meantime, the relations are shown to be valid uniformly in θ when $a_n = \mu(n)$ (and $\lambda(n)$ is stated to give a similar result). The results of **[21]** are based on the truth of estimate

$$\sum_{n \leqslant x} \mu(n) \exp(2\pi i n\theta) \ll x \log^{-h} x$$

for any fixed h, uniformly in θ; and the proof of this was not simple even given

Vinogradov's results. **[155]** established a power series representation of

$$f(x) = \sum_{-\infty}^{\infty} \frac{a_n \exp(2\pi i n\gamma)}{1-x\exp(2\pi i n\alpha)}, \quad |x|<1, \quad \alpha, \gamma \text{ real},$$

and the asymptotic behaviour of $f(x)$, $x = r\exp(2\pi i k\alpha)$ ($k = [k]$ and α irrational), as $r \to 1$ from below, under rather weak conditions on the a_n.

Paper **[69]**, with Pólya, is a souvenir of Davenport's visit to Stanford. The problem arose from the study of the vibration of a membrane stretched across a rectangular frame; and it asked, specifically, for conditions on the sequences $[u_n]$, $[v_n]$ of positive numbers so that, if

$$\sum_0^{\infty} w_n x^n = \sum_0^{\infty} u_n x^n \sum_0^{\infty} v_n x^n,$$

the sequence $\{w_n\}$ is (i) monotonic or (ii) logarithmically convex (in the sense that $w_n^2 \leqslant w_{n-1} w_{n+1}$). Some simple conditions, comparing u_n and v_n with certain binomial coefficients, were obtained.

4. *Miscellaneous results.* Davenport was, in his youth, an avid student of Landau's *Handbuch*; from this interest sprang **[4]**, a study of the arithmetic function

$$\phi_a(n) = \sum_{\substack{m=1 \\ (m,n)=1}}^{n} m^a.$$

In [7] Davenport studied another arithmetic function, $\sigma(n) = \sum_{m|n} m$; in particular, he showed that the sequence of k-abundant numbers, i.e. numbers n for which $\sigma(n) \geqslant kn$, possesses an asymptotic distribution function, continuous in k. Davenport's method was based on a paper of I. Schoenberg. Behrend and Chowla proved the same result at much the same time, and Erdős has, since then, taken the study of abundant numbers much further.

If n is abundant, so is every (positive) multiple of n. If n is abundant, but no divisor of n is abundant, one refers to n as *primitive* abundant; so that the sequence of abundant numbers arises as the set of all distinct multiples of the primitive abundant numbers. This led Erdős to the concept of a "primitive" sequence, i.e. one having no one member dividing another. Erdős proved that the logarithmic density of any primitive sequence is 0; and in **[16]** Davenport and Erdős proved that, on the other hand, if a sequence has positive logarithmic density, it is imprimitive to the extent that it contains a subsequence in which each term divides the next. They based their argument on a Tauberian theorem of Hardy–Littlewood; but, later, in **[92]**, they gave an elementary proof. There is a systematic account of this interesting field in Chapter 5 of *Sequences* (by Halberstam and Roth, Clarendon, 1966); the particular problem investigated by Davenport and Erdős has been further studied by Erdős, Sárközi and Szemeredi (1966, 1968).

Still on the subject of general sequences, the " $\alpha+\beta$-hypothesis " on the addition of integer sequences was a notorious open question in the mid-thirties, and Davenport left his mark on the subject with his famous mod p analogue **[9]**: if the $a_i \bmod p$ ($i = 1, \ldots, m$) and $b_j \bmod p$ ($j = 1, \ldots, n$) are two sets of distinct residue classes mod p, then the number l of distinct residue classes representable as $a_i + b_j \bmod p$ satisfies

$$l \geqslant \min(m+n-1, p).$$

Many years later Davenport (see **[60]**) found this result in Cauchy's work; and this useful result has justly become known as the Cauchy–Davenport theorem. The result can be seen in its proper setting in *Sequences* or in H. B. Mann's *Addition theorems* (Interscience, 1965). The rather simple extension to composite moduli was accomplished by I. Chowla and recorded in Landau's *Fortschritte*.

A number between 0 and 1 is said to be *normal* (in the scale of 10), if in the decimal representation of the number, every finite combination of digits occurs with the proper frequency. Champernowne proved that 0·12345 ... is normal, and Besicovitch that 0·1491625 ... (*i.e.* $0{\cdot}1^1\, 2^2\, 3^2\, 4^2 \ldots$) too is normal. In **[93]** Davenport and Erdős showed that $0{\cdot}f(1)f(2)f(3)\ldots$ is normal, f being an integer valued polynomial; and some stronger results are obtained too. Weyl's inequality for exponential sum was used.

In **[152]**, **[154]**, **[192]** Davenport and Schinzel had occasion to look at an attractive combinatorial problem, namely, to find the greatest length $N_d(n)$ of a sequence consisting of elements from $1, 2, \ldots, n$, such that

(i) no consecutive terms are equal,

(ii) there is no subsequence of type $abab \ldots$ with $d+1$ terms and $a \neq b$.

(Thus $N_2(n) = n$, $N_3(n) = 2n-1$.) They obtained upper and lower estimates for $N_d(n)$ if $d > 3$.

Finally, in **[139]** Davenport and Lewis discuss the analogue for power series fields of characteristic 0 of Littlewood's problem: for any real θ, ϕ and any $\varepsilon > 0$, does there exist an integer $n > 0$ such that

$$n\,||n\theta||\;||n\phi|| < \varepsilon?$$

They gave an inductive construction of a pair $\theta(t)$, $\phi(t)$ for which the conclusion is false. Baker (1964) gave an explicit example of such a pair and Armitage (1970) has since extended Baker's construction to the case of characteristic > 3.

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