PAUL DIENES

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Paul Dienes was born at Tokaj in Hungary in 1882. His father, Barna Dienes, was a jurist, and owned local vineyards ; his mother was of Greek origin. He was attracted to Mathematics at an early age, and studied in Budapest and for a short while in Paris. After obtaining his Ph.D. at the former university, he was appointed Privatdocens, and remained in Budapest until 1919. During the Government of Béla Kun he took an active part in educational work in relation to the University. When this Government fell in 1919, Dienes had to leave Hungary in haste, with his life in danger. He has given me entertaining and thrilling accounts of his escape, and of his activities in the first period after his exile; for example, he escaped from Hungary in a cargo boat on the Danube which was supposed to be carrying beer to Vienna, and he occupied one cask instead of the beer.

He duly arrived in Vienna, and after doing a little film work, made his way to Paris. On his previous visit to Paris he had made close contacts with Borel and Hadamard. Dienes now researched at the Sorbonne, and was awarded the D. ès. Sc. He then wrote to Hadamard asking whether he knew of any British or American University which would like the services of a representative of the Paris School of Mathematicians. It was a curious coincidence that W. H. Young wrote to Hadamard a fortnight later, asking if Hadamard knew of anyone who would be willing to teach a little Mathematics (on Parisian lines) to his students in the University of Wales. In consequence, Dienes lectured at Aberystwyth from October 1921 until the resignation of Young in 1923, when he took up a lectureship at Swansea.

Until 1917 Dienes had researched mainly in the theory of functions of a complex variable, but after leaving Paris he became interested in relativity theory, and wrote several papers on tensors. During this period his only research student was E. T. Davies, and it was natural in these circumstances that Dienes put him on to Differential Geometry.

Towards the end of Dienes' stay in Swansea, G. H. Hardy and A. R. Richardson suggested that he should write a book on the Taylor series. From 1926 to 1930 Dienes was eagerly engaged in this task. " The Taylor Series " (Oxford, 1931) was a very fresh and interesting presentation of, firstly elementary properties of sequences, series and functions ; secondly, a proof of Jordan's theorem, followed by the usual well-known complex variable theory ; thirdly, more advanced theory, most of which is not to be found in any other book ; and lastly, a good introduction to infinite

* *Editorial note.* The author is in no way responsible for the delay in publishing this notice.

matrices and their application to divergent sequences and series. In 1929 Dienes was appointed to a Readership at Birkbeck College in the University of London.

Here his first research students worked on Differential Geometry, but soon his interest switched to Infinite Matrices, and I found myself sharing this interest. He had H. S. Allen, P. Vermes, and R. Henstock as research students in this subject, with a few others. As there was no book on the subject at this time, Dienes persuaded me (in 1944) to start writing one. The outlook of Dienes on infinite matrices was not the same as that of the older school interested in summability. His idea was to explore the properties of infinite matrices, these being the centre of interest ; whereas the older school were interested chiefly in the summability of divergent sequences and series. Moreover, the study of infinite matrices from Dienes's point of view is not just a particular case of linear operator theory on the lines of Banach (as some appeared to think), since multiplication of infinite matrices need not be associative, and some of their most interesting properties occur when products of them are not associative.

Dienes always had a serious interest in philosophical questions, and as far back as 1914 was immersed in symbolic logic, or axiomatics. In 1930 he returned to this for a short while, and then again from about 1942 until 1948. A. Robinson and R. L. Goodstein were among his research students in this subject.

In 1945 Dienes was appointed to the newly created Chair of Mathematics at Birkbeck College. During the brief period in which he held this post he modernised and greatly developed the department. He retired in 1948 and was surprised that his mind then turned to poetry ; he published through the Fortune Press a cycle of poems with the title " The Maiden and the Unicorn ". Michael Tippett, the composer, who was a close personal friend of Dienes, commented on him and his poetry : " The first thing to remember about him was that he had a philosophical and mathematical discipline as the core of his mind, but music had an almost Schopenhauerian significance for him .... I think, therefore, that his poetry represented an attempt at an amalgam of these various sensibilities, so that the sound pattern of the verse seems to overbalance the pattern of the sense. It is possible that I, as someone with a musical discipline, and abiding interest, though no aptitude, for philosophy, can savour Dienes's poetry better than most ".

Dienes was essentially of the Paris school of mathematicians, and he did much to promote good relations between British and French mathematicians. Immediately after the liberation of France in 1945, he arranged seminars at Birkbeck College, to be addressed by Mandelbrojt, Laurent Schwarz, Dieudonné, H. Cartan and others. The early work of Dienes, i.e., on functions of a complex variable, was perhaps his most important work, and was largely concerned with the behaviour of Taylor series on the circle of convergence. Writing throughout $f(z) = \sum_{m=0}^{\infty} a_m z^m$ with $s_n(z)$ for its n-th partial sum, and with radius of convergence 1, the following are among his chief results.

(i) *If* $a_n \to 0$ *and* $f(z)$ *is of bounded variation on an arc of* $|z| = 1$ *containing* $e^{i\phi}$, *then* $\sum_{n=0}^{\infty} a_n e^{in\phi}$ *converges.* [13], [16], [33]*.

(ii) *If* $f(z)$ *is meromorphic on* $|z| = 1$, *and if* $e^{i\phi}$ *is a pole of the highest order* r, *then* $\frac{s_n(e^{i\phi})}{n^r} \to \frac{B_r}{\Gamma(r+1)}$ *as* $n \to \infty$, *where* B_r *is the coefficient of* $(1-e^{-i\phi} z)^{-r}$ *in the Laurent expansion of* $f(z)$ *about* $z = e^{i\phi}$. [4], [33].

(iii) *If* $a_n \to 0$, $0 < r < 1$, *and in the neighbourhood of* $e^{i\phi}$

$$f(z) = \frac{P\left(\log 1/(1-e^{-i\phi} z)\right)}{(1-e^{-i\phi} z)^r} + f_1(z),$$

where $P(z) = A_q z^q + A_{q-1} z^{q-1} + \ldots + A_0$, *and the order* r' [33, §116, p. 487] *of* $f_1(z)$ *at* $e^{i\phi}$ *is less than* r, *then*

$$\lim_{n\to\infty} \frac{s_n(e^{i\phi})}{n^r \log^q n} = \frac{A_q}{\Gamma(r+1)}. \quad [14], [33].$$

(iv) *If* $n^{-r} a_n \to 0$, $r > 0$, *then the Cesàro means of order* r *of* $s_n(z)$ *converge on* $|z| = 1$ *at every point of negative order.* [13], [33].

(v) *If* $n^{-r} a_n \to 0$, $r \geqslant 1$, *and along every path inside* $|z| = 1$ *leading to* $e^{i\phi}$ *we have* $\lim_{z \to e^{i\phi}} f(z) = A$, *then the Cesàro means of order* r *of* $s_n(z)$ *converge to* A. [13], [16], [33].

(vi) *If* $n^{-r} a_n \to 0$, $r \geqslant 1$, $0 < \rho < r+1$ *and*

$$f(z) = \frac{P\left(\log 1/(1-e^{-i\phi} z)\right)}{(1-e^{-i\phi} z)^\rho} + f_1(z),$$

where $P(z) = A_q z^q + \ldots + A_0$, *and the order of* $f_1(z)$ *at* $e^{i\phi}$ *is* $\rho' < \rho$, *then*

$$\lim_{n\to\infty} \frac{s_n^{r-\rho}}{n^\rho \log^q n} = \frac{\Gamma(r-\rho+1)}{\Gamma(r+1)} A_q. \quad [16], [33].$$

(vii) *If, for every* $\epsilon > 0$, *the integral function* $E(z) = \sum_{k=0}^{\infty} g(k) z^k$, *where* $g(k) \geqslant 0$ *for every* k, *tends uniformly to zero in the region* $\epsilon \leqslant \arg z \leqslant 2\pi - \epsilon$ *when* $|z| \to \infty$, *then we have Mittag-Leffler's representation*

$$f(z) = \lim_{\omega\to\infty} \sum_{k=0}^{\infty} s_k(z) g(k+1) \omega^{k+1} / E(\omega)$$

valid in the principal star-domain of $f(z)$. [16], [33].

* Numbers in square brackets refer to the list of publications at the end of this notice.

In conjunction with the theorem of Le Roy and Lindelöf, I found this result basically useful in establishing sufficient conditions for a matrix of Mittag-Leffler type to be efficient for all Taylor series throughout the principal star-domain ; see my " Infinite Matrices and Sequence Spaces ", p. 184, (8.2, III).

(viii) *If* $f(z)$ *is of bounded variation on an arc of* $|z|=1$, *the Borel exponential means of* $s_n(z)$ *converge at the interior points of this arc.* [**13**], [**33**].

The same result extends to points on the polygon of summability and, more generally, for Mittag-Leffler matrices, to vertices of the principal star-domain of $f(z)$.

In his work on tensors, Dienes gave, in several papers [**21**], [**22**], [**24**], [**25**], [**26**], [**27**], [**28**], [**29**], [**30**], [**34**], [**36**], [**37**], [**38**], [**39**], [**40**], [**41**], a substantial contribution to the theory of tensor submanifolds, resolution of tensors, connexion and differentiation, etc. Some of these ideas were further developed later in a joint paper with E. T. Davies [**43**].

In his paper [**35**] on infinite matrices, Dienes expressed in a clear way for the first time several simple fundamental properties of infinite matrices and gave some new theorems on reciprocals of these. He also modified Helly's criteria for the bound of a matrix so as to make it apply to a wider class of matrices. Even this, however, was subsequently found to be not altogether satisfactory, and the criteria were then modified again as follows (see my book referred to above, pp. 26-27). If $\mathscr{F}$ is a class of (infinite) matrices which contains the scalar matrices, and is closed under finite sum and finite product, we suppose that the bound $|A|$ of matrices in $\mathscr{F}$ satisfies the following conditions, where A and B are matrices of $\mathscr{F}$, c is any scalar, and I is the unit matrix:

(i) $|cA|=|c|\,|A|$, $|I|=1$; (ii) $|A+B|\leqslant|A|+|B|$;

(iii) $|AB|\leqslant|A|\,|B|$; (iv) $|a_{ij}|\leqslant|A|$.

It will be observed that (iv) is now a condition for *every* bound, instead of merely for *regular* bounds.

Dienes also lectured at Birkbeck College on sequence spaces, introducing several new ones, with their properties; new results due to him are given in my book, Chapter 10. He further wrote a joint paper with me [**44**]. Dienes wrote papers on symbolic logic at such widely separated dates as 1917, 1930, 1935, 1938, 1949 [**19**], [**20**], [**32**], [**42**], [**45**], [**46**], [**48**]. Finally, three papers outside the main fields mentioned above are on transfinite sets of real numbers [**23**], on the exponential function in linear algebras [**31**], and on the Riemann-Stieltjes integral [**47**]; also there was a " popular " textbook [**17**].

Dienes was married twice, and had two sons by his first wife, Valerie, whom he trained in Mathematics, and with whom he wrote six joint papers [**7**], [**12**], [**14**], [**15**].

He also trained his son, Z. P. Dienes, now lecturer in Mathematics at Leicester ; his other son, G. P. A. Dienes, became expert in at least twelve languages, and holds an official position at Budapest. This first marriage was dissolved when Dienes left Budapest in 1919. Shortly after coming to Swansea, he brought back his second wife, Sári, after a visit to the Continent. She was a painter, studying in a (then) ultra-modern school in Paris, and often returned to Paris for a while to carry out further work.

Dienes had also a great love of music of certain types, either the very ancient or the very modern, but nothing in between, i.e., he was definitely anti-romantic; Michael Tippett's remark concerning Dienes's musical tastes quoted above appears to me to be very appropriate.

Dienes had a most charming personality, and was much loved by both his colleagues and his students; when he died from a heart attack in March 1952, only a month after the death of my wife, I felt, as an extra blow, that I had lost a good friend.

List of Publications

[*C.R.* denotes *Comptes Rendus* (Paris)]

1. " La série de Taylor sur le cercle de convergence ", *C.R.*, 140 (1905), 489–491.
2. " A Taylor sor az összetartási körön ", *Bulletin de l'Académie Hongroise des Sciences*, 23 (1905), 505–511.
3. " Adalékok az analytikai függvények elméletéhez ", *Mathematikai és Fizikai Lapok*, 14 (1905).
4. " Sur les singularités des fonctions analytiques ", *C.R.*, 147 (1908), 1388–1390.
5. " Sur les singularités des fonctions analytiques en dehors du cercle de convergence ", *C.R.*, 148 (1909), 694–698.
6. " Essai sur les singularités des fonctions analytiques ", *Journal de Math.* (*de Liouville*) (6), 5 (1909).
7. (Conjointly with V. Dienes) " Sur les singularités algébrologarithmiques ", *C.R.*, 149 (1909), 972–974.
8. " Analytikai függvények negativrendü szinguláris helyeinek vizsgálata ", *Math. és Termész. értesitö* ", 27 (1909), 58–63.
9. " Analytikai függvények viselkedése az összetartási körön ", *Math. és Fizikai Lapok*, 18 (1909).
10. " Analytikai függvények végtelenségi helyeinek vizsgálata ", *Math. és Fizikai Lapok*, 18 (1909).
11. " Sur un problème d'Abel ", *C.R.*, 151 (1910), 294–296.
12. (conjointly with V. Dienes) " Altalános tételek az algebrai és logarithmikus szingularitásokrol ", *Math. és Termész. értesitö* ", 28 (1910), 26–31.
13. " Sur la sommabilité de la série de Taylor,", *C.R.*, 153 (1911), 802–805.
14. (conjointly with V. Dienes) " Recherches nouvelles sur les singularités des fonctions analytiques ", *Acad. Sci. de l'Ecole Normale* (3), 28(1911), 389–457.
15. (conjointly with V. Dienes) " Analytikai függvények algebrai és logarithmikus szingularitásairol ", *Math. és Fizikai Lapok*, 20 (1911) and 21 (1912) (3 papers).
16. *Leçons sur les singularités des fonctions analytiques* (Collection Borel (Paris), 1913).
17. *Valóság és Matematika—Betekintés a mennyiségtan fogalomrendszerébe* " (Budapest, 1914, Haladés, Villágosság nyomda, pp. 68, Galilei füzetek).
18. " Kisérlet a funkcionál-számitás rendszeres megalapozására, Parts I and II ", *Math. és Termesz. értesitö* ", 34 (1916), 154–194, 656–692.
19. " Leibniz logikai és matematikai eszméi ", in the volume " Leibniz " of the Hungarian Society of Philosophy (1917).

20. " Logika és Matematika ", *Athénaeum* (1917).
21. " Sur la connection du champ tensoriel ", *C.R.*, 174 (1922), 1167–1170.
22. " Sur le déplacement des tenseurs ", *C.R.*, 175 (1922), 209–211.
23. " Sur les suites transfinies de nombres réels ", *C.R.*, 176 (1923), 67–69.
24. " Sur la théorie électromagnétique relativiste ", *C.R.*, 176 (1923), 238–241.
25. " Sur la géometrie tensorielle ", *C.R.*, 176 (1923), 370–372.
26. " Sur l'intégration des equations du déplacement parallèle de M. Lévi-Cività ", *Rendiconti di Palermo*, 42 (1923), 144–152.
27. " Sur la structure mathématique du calcul tensoriel ", *Journal de Math.* (*de Liouville*) (9), 3 (1924), 79-106.
28. " Sur les differentielles secondes et les dérivations des tenseurs ", *Proc. Rome Academy*, (1924), 265–269.
29. " Déterminants tensoriels et la géometrie des tenseurs ", *C.R.*, 178 (1924), 682–685.
30. " On tensor geometry ", *Annali di Mat.* (IV), 3 (1926), 247–295.
31. " The exponential function in linear algebras ", *Quarterly Journal of Math.* (Oxford), 1 (1930), 300–309.
32. " A new treatment of the theory of inference ", *The Monist* (1930).
33. *The Taylor Series* (Oxford, 1931).
34. " On the fundamental formulae of the geometry of tensor submanifolds ", *Journal de Math.* (*de Liouville*) (9), 11 (1932), 255–282.
35. " Notes on linear equations in infinite matrices ", *Quarterly Journal of Math.* (Oxford), 3 (1932), 253–268.
36. " Sur le déplacement d'un n-uple et sur une interprétation nouvelle des coefficients de rotation de Ricci ", *Proc. Rome Acad.* (6), 17 (1933), 119–122.
37. " Sur la déformation des espaces à connexion linéaire générale ", *C.R.*, 197 (1933), 1084–1087.
38. " Sur la déformation des sous-espaces dans un espace à connexion linéaire générale ", *C.R.*, 197 (1933), 1167–1169.
39. " On the deformation of tensor manifolds ", *Proc. London Math. Soc.* (2), 37 (1933), 512–519.
40. " Sur un théorème de M. Fermi ", *Proc. Rome Acad.* (6), 18 (1934), 369–372.
41. " On curves in a space of general linear connection ", *Journal London Math. Soc.*, 9 (1934), 259–266.
42. " On symbols mostly mathematical ", *The Modern Quarterly* (1935).
43. (conjointly with E. T. Davies) " On the infinitesimal deformations of tensor submanifolds ", *Journal de Math.* (*de Liouville*) (9), 16 (1937), 111–150.
44. (conjointly with R. G. Cooke) " On the effective range of generalised limit processes ", *Proc. London Math. Soc.* (2), 45 (1938), 45–63.
45. *The logic of Algebra* (Hermann, Paris, 1938).
46. " The crisis in Mathematics ", *The Modern Quarterly* (1938).
47. " Sur l'intégrale de Riemann-Stieltjes ", *Revue Scientifique* (Paris), 85 (1947), 259–274.
48. " On ternary logic ", *Journal of Symbolic Logic*, 14 (1949), 85–94.

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