THEODOR ESTERMANN, 1902–1991

OBITUARY

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Theodor Estermann, son of Leo and Rachel Estermann (née Brenner), was born in Neubrandenburg on 5 February 1902 and died on 29 November 1991.

His father, who came from Lithuania (and was therefore a Russian subject) worked in advertising and business management; his mother, a Latvian, was a skilled corsetiere. They had two children, of whom Theodor was the younger; his brother Immanuel (who was to become an outstanding physicist) was born in 1900.

Theodor's early schooling was first at the Talmud-Torah School in Hamburg, where (from the age of six) he learnt to read Hebrew as well as German, but a few years later he went to a state primary school in Berlin, which suited him much better.

Leo Estermann was an ardent Zionist (in fact, Theodor was named after the founder of modern Zionism, Theodor Herzl). In 1914 the family moved to Palestine and became Turkish subjects (transformed to Palestinian at the end of the First World War), and Theodor went to the Hebrew Grammar School in Jerusalem. Here the teachers translated their lesson notes from their own mother-tongues (often German) into Hebrew, and Theodor used to translate his notes back into German when he did his homework.

Before the end of the First World War, the family moved back to Hamburg. Here Theodor and his brother Immanuel went to different schools (determined by places available in the respective age groups), of which only Immanuel's had laboratories, so that only Immanuel was able to study science. It is a matter of speculation whether this lies at the root of their respective choices of careers in mathematics and physics. (Theodor did study physics at the University of Hamburg; the University regulations required three courses, one of which had to be a 'practical' science, and Theodor chose the two available pure mathematics courses together with physics).

When Theodor left school, his father arranged for him to be apprenticed to a farmer, because Zionism was a 'back to the land' movement. Fortunately (both for Theodor and for mathematics), the farmer not only recognised after a short time that he was unsuited to this work, but was sufficiently perceptive to tell his father that he ought to send Theodor to the University.

Theodor Estermann studied very briefly under Hilbert and Landau at the University of Göttingen, but was homesick and transferred to the University of Hamburg after a few months. Here he studied under Rademacher, and the degree of Doctor of Science was conferred on him in 1925.

After graduating, Estermann returned to Palestine, where his father had finally settled. He worked briefly as an 'usher' at Dr Biram's school in Haifa; this was a non-teaching post consisting mainly of supervising pupils' activities.

In 1926 he came to England (which he had first visited on holiday in 1924) to study at UCL. His aunt, Sarah Brenner, who ran her own corset-making business in London, housed him and looked after him. But funds provided for him by his father were inadequate, and he had very little money. In 1928, at the end of his very successful studies at UCL, he was awarded the prestigious London University degree of Doctor of Science.

Estermann then returned to Germany (it is believed to Hamburg) where he obtained temporary work, only to find himself unemployed when the professor on whose behalf he was giving lectures died about a year later.

When Estermann came to England for the degree ceremony in May 1929, the UCL professor under whom he had studied was surprised to find him unemployed and offered him an assistant lectureship in the Department of Mathematics at UCL. He was promoted to Lecturer in 1931 and to Reader in 1940. The title of Professor in the University of London was conferred on Estermann in 1965.

Professor Estermann retired in 1969 and the title of Emeritus Professor was conferred on him. His retirement was a very active one. He maintained his interest in mathematics, and was annually re-appointed as Honorary Research Fellow at UCL until 1987 (when he was unwilling to accept re-appointment because of failing eyesight); in addition, he took on various part-time school-teaching appointments. It was during this period that he discovered the remarkably simple new proof of the irrationality of $\sqrt{2}$ (which we reproduce below, at the end of the account of his work). Like all the best ideas, it is obvious once pointed out; but it took about two thousand years after Pythagoras for someone to point out this particular idea.

Estermann married Tamara Pringsheim, a granddaughter of the mathematician Alfred Pringsheim, in 1936. They had six children (five girls and one boy), and there are presently eight grandchildren. Estermann's nationality (previously successively Russian, Turkish and Palestinian) became British by naturalisation in 1948.

In view of Estermann's quiet and modest presence, one was surprised on better acquaintance to find that he was in fact a man of wide interests and many accomplishments. He spoke fluent grammatical German and Hebrew (and, of course, English). In addition, he knew some French and Latin, and studied Swedish in preparation for the Stockholm ICM. He had an extensive knowledge of literature, and knew large parts of Shakespeare's plays by heart. He enjoyed classical music, and as a young man was a keen gymnast. Klaus Roth recalls from his research student days that his supervisor, Dr Estermann, could still demonstrate faultless handstands with the same facility with which he made his many erudite (but apt) quotations from Shakespeare and Göthe.

Estermann's complete objectivity was legendary. A typical mathematical example of this objectivity in action was provided by the way that generations of specialist function-theorists made use of the technical concept of a 'piecewise-smooth path' (der eine machts dem andern nach); it was only Estermann who noticed that it was fatuous to differentiate between 'smooth' and 'piecewise-smooth', because the two concepts could easily be shown to be logically equivalent from their respective definitions. It was one of Roth's unfulfilled ambitions of his research student days to defeat his supervisor's objectivity by catching him out with a trick problem. Among Roth's many resulting disappointments was the following: Roth challenged Estermann to count the number of occurrences of the letter 'f' in a certain (specially constructed) sentence. Estermann quickly gave the correct total, but asked why he had been given such a trivial task. When Roth explained that nearly everyone gave an answer that was one less than the true total because for some mysterious reason only very few people counted the 'f' in the word 'of', Estermann remonstrated 'You should have told me to read the sentence. I did not read the sentence but merely counted occurrences of f.'

Estermann was a very kind man, and his heroic attempts against great odds to put Roth at ease during his PhD oral despite the latter's extreme examination nerves is a further source of amusing recollections.

Professor Estermann will be greatly missed. The fact that he had long since retired in no way lessened the profound sense of loss felt by all his colleagues on the news of his death.

Estermann's mathematics

Early work

Estermann's early research topics had been suggested by Rademacher, who himself had only recently completed his studies with Carathéodory in real analysis. Although a visit by Brun to Hamburg had already engaged Rademacher's interest in number theory (and Estermann's inclination was probably towards number theory, having studied briefly with Landau in Göttingen), Rademacher apparently did not feel confident yet in offering problems in number theory. Thus Estermann's first paper, taken from his dissertation in Hamburg, is on a problem in measure theory closely associated with the work of Carathéodory.

His next work is on convex bodies. Blaschke **[4]** had posed the following problem. Let K denote an n-dimensional convex body with volume $V(K)$, and define the difference body $\mathrm{D}K$ by $\mathrm{D}K = \{x-y\colon x \in K,\ y \in K\}$. Then show that

$$2^n V(K) \leqslant V(\mathrm{D}K) \leqslant \binom{2n}{n} V(K),$$

and describe the extremal cases. The case $n = 2$ had been dealt with by Rademacher **[46]**. In [1927] and [1928d], Estermann settles the case $n = 3$ and in particular shows that in the second inequality, equality occurs only for a tetrahedron. This had been done independently by Süss **[54]** at the same time. Estermann's proof is the more natural, and is the one chosen by Bonnesen and Fenchel in their standard work **[5]**. The generalisation to n dimensions was not obtained until 1957, by Rogers and Shephard **[48, 49]**.

The papers [1928b] and [1928c] are two remarkable pieces of work. The first is concerned with the following question. Let F be a polynomial of degree k with the property that for every integer h, $F(h)$ is an integer, and suppose that $F(0) = 1$. Then put $g(p_1^{h_1} \dots p_m^{h_m}) = F(h_1) \dots F(h_m)$ and $f(s) = \sum_{n=1}^{\infty} g(n) n^{-s}$. The question, of great interest in number theory, is when can f be continued to the whole complex plane? This had been stimulated by a paper of Ramanujan **[47]** which contains several examples in which f can be written in terms of the Riemann zeta function and so can be continued with relative ease. In [1928b] Estermann completely solves the question, and in a very elegant way. Let $b_m = \sum_{h=0}^{m} F(h) \binom{k+1}{m-h} (-1)^{m-h}$. It follows that $b_m = 0$ for all large m. Let r be the largest m for which b_m is non-zero. Then b_r is a non-zero integer and $b_0 = 1$. Define α_m and α by $\sum_{m=0}^{r} b_m x^m = \prod_{m=1}^{r} (1-\alpha_m x)$ and $\alpha = \max(|\alpha_1|, \dots, |\alpha_r|)$. Then $|\alpha_1 \dots \alpha_r| \geqslant 1$, and so $\alpha \geqslant 1$. Estermann proves that when $\alpha = 1$, the function f can be continued to a meromorphic function in the plane, and that when $\alpha > 1$, it has the imaginary axis as a natural boundary, and it can be continued throughout $\sigma > 0$ to a function whose only singularities there are poles.

The function

$$\prod_{m=2}^{\infty} (1/(1-m^{-s}))$$

is introduced in MacMahon [42] and is studied in Oppenheim [43]. In [1928c] Estermann establishes a general theorem, a special case of which is that the function in question cannot be continued beyond the imaginary axis.

These problems had aroused great interest, and the solutions, for which no general techniques existed, represented considerable achievements. A remarkable feature is that the theory of the Riemann zeta function, which one might have expected to be essential, is hardly used at all.

Kloosterman's sums and applications

In a sequence of weighty papers, Kloosterman [33, 34, 35] had introduced a refinement of the Hardy–Littlewood method in order to obtain asymptotic formulae for the number of representations of large numbers by quaternary quadratic forms. In this work, an important rôle is played by the Kloosterman sum

$$K(q,u,v) = \sum_{\substack{x=1 \\ (x,q)=1}}^{q} e((ux+vx^*)/q), \tag{1}$$

where $e(.) = \exp(2\pi i.)$ and x^* denotes that residue class modulo q for which $xx^* \equiv 1 \pmod q$.

The use of power series in the Hardy–Littlewood method had meant that, realistically, only additive problems of a positive definite kind could be considered. In [1929c] Estermann makes a significant simplification and demonstrates in this case the connection with modular forms. He also shows that

$$K(q,u,v) \ll q^{3/4+\varepsilon}(u,q)^{1/4}, \tag{2}$$

by making use of an exact formula for the fourth moment of $K(q,u,v)$ when q is prime. By the way, this foreshadows Mordell's celebrated use of higher moments [40] to deal with complete exponential sums with polynomial arguments.

For positive definite quaternary quadratic forms, more precise error estimates are now available by the use of modular forms, as in Eichler [14], and the Hardy–Littlewood–Kloosterman method has been largely superseded in this case.

After Vinogradov's innovative introduction of finite sums to the Hardy–Littlewood method in the 1930s, it became possible to treat routinely indefinite problems. Here the Hardy–Littlewood–Kloosterman method still gives the best estimates currently in the literature.

In [1961], following Weil's proof [66] of the Riemann Hypothesis for rational function fields over finite fields, Estermann establishes the, essentially best possible, estimate

$$K(q,u,v) \ll q^{1/2+\varepsilon}(u,q)^{1/2}. \tag{3}$$

He then goes on to apply (3) in what is now considered to be the definitive formulation of Kloosterman's refinement of the Hardy–Littlewood method, that is, to the representation of a given integer by an indefinite diagonal quaternary quadratic form. Estermann characteristically gets to the root of the method, and displays its essence in a series of lemmas. One wishes to estimate an integral of the form

$$I = \int_0^1 f(\alpha)\,d\alpha,$$

where $f: \mathbb{R} \to \mathbb{C}$ is periodic with period 1, by dividing the interval $[0,1]$ into subintervals $\mathfrak{M}(q,a)$ associated with the Farey fractions a/q of order Q, say, and then to take advantage of some cancellation in the summation over a in

$$I = \sum_{q \leqslant Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} I(q,a),$$

where we have written

$$I(q,a) = \int_{\mathfrak{M}(q,a)} f(\alpha)\, d\alpha.$$

The problem is that as a/q ranges over the reduced fractions with denominators $q \leqslant Q$, the distance from a/q to its neighbouring Farey fraction varies appreciably. Let $a_{\pm}/q_{\pm}$ denote the neighbouring fraction to the right, left respectively. Then normally one takes

$$\mathfrak{M}(q,a) = \left[\frac{a_{-}+a}{q_{-}+q}, \frac{a_{+}+a}{q_{+}+q}\right].$$

Given a, q, Q, the values of $a_{\pm}$, $q_{\pm}$ are readily deduced from the observations $a_{\pm}\, q - q_{\pm}\, a = \pm 1$ and $Q - q < q_{\pm} \leqslant Q$. Thus

$$\mathfrak{M}(q,a) = \left[\frac{b_{-}}{r_{-}}, \frac{b_{+}}{r_{+}}\right],$$

where $-r_{\pm} \equiv \pm 1 \pmod q$, $Q < r_{\pm} \leqslant Q+q$, $b_{\pm} = (\pm 1 + r_{\pm}\, a)/q$. Let

$$\mathfrak{N}(q,a) = \left[\frac{b_{-}}{r_{-}} - \frac{a}{q}, \frac{b_{+}}{r_{+}} - \frac{a}{q}\right].$$

Then in view of the above remarks, one sees that the characteristic function $g(\beta, q, a)$ of $\mathfrak{N}(q,a)$ can be put in the form

$$\sum_{h=1}^{q} c_h(\beta)\, e(ha^*/q),$$

where a^* denotes the residue modulo q for which $aa^* \equiv 1 \pmod q$ and $c_h(\beta)$ is given by

$$c_h(\beta) = \sum_{Q < r \leqslant \min(Q+q,\, 1/(q\beta))} e(-hr/q) \quad \text{when } \beta \geqslant 0,$$

$$c_h(\beta) = \sum_{Q < r \leqslant \min(Q+q,\, -1/(q\beta))} e(hr/q) \quad \text{when } \beta < 0.$$

Thus one obtains

$$\sum_{\substack{a=1 \\ (a,q)=1}}^{q} I(q,a) = \int_{-1/qQ}^{1/qQ} \sum_{h=1}^{q} c_h(\beta) \sum_{\substack{a=1 \\ (a,q)=1}}^{q} f\left(\frac{a}{q}+\beta\right) e(ha^*/q)\, d\beta$$

and

$$\sum_{h=1}^{q} |c_h(\beta)| \ll \log 2q.$$

In applications, one may be able to use the arithmetico-geometrico properties of the f under consideration to take advantage of some cancellation when one averages over

a. In the study of diagonal quaternary quadratic forms, the presence of the a^* leads naturally to the Kloosterman sum $K(q, u, v)$ defined above. In recent years, the Kloosterman refinement has been used by Heath-Brown [25], Hooley [28] and Ashton [1].

Questions as to the order of magnitude for the error term in asymptotic formulae for the number of representations of a number as the sum or difference of a product were, and still are, of great importance because of connections with the Riemann zeta function. Estermann makes a discovery of immense significance, namely that the error terms can be made to depend on Kloosterman sums.

In [1930a] Estermann investigates the question of the asymptotic formula for the number $D(n)$ of solutions of

$$xy + zt = n, \tag{4}$$

when n is large. This had been considered by Ingham [30], who had obtained the relatively weak asymptotic formula $(6/\pi^2 + o(1))\,(\sum_{d|n} d)\log^2 n$. By relating the error to the Kloosterman sum and applying (2), Estermann shows that

$$D(n) = n\sum_{r=0}^{2} (\log n)^r \sum_{s=0}^{2} c_{rs} \sum_{d|n} d^{-1}(\log d)^s + E(n),$$

with $E(n) = O(n^{7/8}(\log n)^{23/4}\sum_{d|n} d^{-3/4})$. By the way, Estermann acknowledges a simplifying suggestion by Hecke to use the Hurwitz zeta function rather than Dirichlet L-functions in the analysis.

In [1931d] Estermann examines the conjugate problem of the number $D(n; a)$ of solutions of the equation

$$xy - zt = a \tag{5}$$

in natural numbers x, y, z, t with $zt \leqslant n$. The problem is only superficially similar to that solved above. The indefinite nature of the problem prevented at that time an application of the Hardy–Littlewood–Kloosterman technique. Instead, Estermann develops a highly ingenious elementary method, in which the error term is expressed in terms of a function with a simple Fourier expansion. This leads to exponential sums which can be related to Kloosterman sums directly. Thus for n large, (2) gives the asymptotic formula

$$D(n; a) = c_2(a)\,n\log^2 n + c_1(a)\,n\log n + c_0(a)\,n + E(n; a), \tag{6}$$

where the error term satisfies $E(n; a) = O(n^{11/12+\varepsilon})$. Earlier, Ingham [30] had obtained the weaker asymptotic formula $(c_2(a) + o(1))\,n\log^2 n$.

The error term in the asymptotic formula for the fourth moment of the zeta function on the $\frac{1}{2}$-line can be made to depend on an estimate for $E(n; a)$ which is uniform in both n and a, and this has been exploited by Atkinson [3] and Heath-Brown [24].

For a long time, the only improvements in this area came about through better estimates for Kloosterman sums. For example, Halberstam [15] indicates that (3) would give $E(n) = O(n^{3/4}\log^3 n)$.

Quite recently, however, the theory associated with the equations (4) and (5), as well as the fourth moment of the zeta function on the $\frac{1}{2}$-line, has been revolutionised by the Kuznetsov [36, 37] trace formulae, which transform sums of Kloosterman sums into bilinear forms of the Fourier coefficients of cusp forms over the full modular group. Thus a number of authors have obtained improvements in the error terms. Most recently, Motohashi [41] has shown that

$$E(n;a) = O((n^2+na)^{1/3+\varepsilon}+(n^2+na)^{1/4+\varepsilon}a^{9/40}+a^{7/10}n^{\varepsilon})$$

uniformly for $a \leqslant n^{10/7}$ and

$$E(n) = O(n^{0.7+\varepsilon}).$$

See also Ivic [31], notes for Chapter 4, and Chapter 5, for an account of this material, especially as it relates to the Riemann zeta function.

The above work is complemented by two papers [1929a, 1932a] dealing with sums of three or more products by an application of the Hardy–Littlewood method. Also, in [1932b], the method used to deal with (6) is adapted to obtain a similar asymptotic formula for

$$\sum_{h=1}^{n} r(h)\,r(h+k),$$

where $r(m)$ is the number of representations of m as the sum of two squares.

Sums of squares, quadratic forms

The paper [1959] represents a technical achievement of the highest order. Hardy [18], Mordell [39] and Ramanujan (Hardy [19, Chapter 9]) had developed an analytic method for treating the representation of a number as the sum of s squares when s is odd. For $s \geqslant 5$ the method was comparatively straightforward, but for $s = 3$ it was an open question as to whether the method succeeded. The question reduces to the task of justifying directly an interchange in the order of summation of certain double sums. That the interchange is in fact valid was an indirect consequence of known results concerning the representation of numbers as sums of three squares. The problem of finding a direct justification remained unsolved for forty years despite the best efforts of Hardy, Mordell and others. In these circumstances, it is hardly surprising that Estermann's solution is one of great power and ingenuity.

The appendix to [1931e] contains a neat elementary argument for bounding the number of solutions in non-negative x and y to $ax^2+by^2 = m$. There are two basic situations. When a, b, m are positive, the bound obtained is $2d(m)$ where d denotes the divisor function, and when a, m are positive, b is negative, $-ab$ is not a square and x is restricted by $ax^2 \leqslant n$, the bound is $2(1+\log n)\,d(m)$. The bounds are close to best possible, for example when $a = b = 1$ and all the prime factors of m lie in the residue class 1 modulo 4. What is noteworthy and caused interest at the time is that the argument makes no use of the theory of quadratic fields. Also, the final conclusions are surprisingly tidy.

Waring's problem

In the mid 1930s, Vinogradov (see, in particular, [62]) had introduced some important refinements to the theory of the Hardy–Littlewood method as applied to Waring's problem, and Heilbronn had lectured on this at the British Mathematical Colloquium in June 1935. Vinogradov was concerned only to obtain an upper bound for $G(k)$, the smallest s such that every sufficiently large natural number is the sum of at most s kth powers, when k is large. It was apparent that his ideas had some relevance for smaller powers of k, and Estermann saw that at once. In [1936a] and [1937c] he shows that

$$G(4) \leqslant 17, \qquad G(5) \leqslant 29 \tag{7}$$

respectively, the previous best being $G(4) \leqslant 19$ and $G(5) \leqslant 35$ due to Hardy and Littlewood [22] and James [32] respectively. Davenport and Heilbronn [13] also pursued in this direction, and their proof of (7) appears immediately after Estermann's.

A little later, Davenport made important progress in the theory of Waring's problem when k is small. In particular, he showed that $G(4) = 16$ [10] and $G(5) \leqslant 23$ [11]. More precisely, if $G^{\#}(4)$ is the smallest s such that whenever $1 \leqslant r \leqslant s$, every sufficiently large natural number in the residue class r modulo 16 is the sum of at most s biquadrates, then he showed that $G^{\#}(4) \leqslant 14$. The last few years have seen some further advances. In Vaughan [59] and Vaughan and Wooley [60] it is shown that $G^{\#}(4) \leqslant 12$ and $G(5) \leqslant 17$ respectively.

Hua [29] had established the Hardy–Littlewood asymptotic formula

$$r_s(n) \sim \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} n^{s/k-1} \mathfrak{S}(n),$$

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} (S(q,a)\,q^{-1})^s e(-an/q),$$

$$S(q,a) = \sum_{x=1}^{q} e(ax^k/q),$$

for the number of representations $r_s(n)$ of a large natural number n as the sum of s kth powers of natural numbers when $s \geqslant 2^k+1$. This was by an ingenious refinement of the differencing argument for dealing with exponential sums over polynomials introduced by Weyl [67] in his seminal paper on uniform distribution. Earlier, Hardy and Littlewood [20] had required $s \geqslant (k-2)\,2^{k-1}+5$, and just prior to Hua's paper, Vinogradov [61, 63, 64] had reduced this to asymptotically $6k^2 \log k$ for *large* k. Estermann [1948b] gives a very simple elegant exposition of Hua's theorem, and it was from this paper that one of us [Vaughan] learnt the Hardy–Littlewood method. More recently, the required lower bound has been reduced somewhat, by Vaughan [57, 58] to 2^k for all $k \geqslant 3$, by Heath-Brown [26] to $\frac{7}{8}2^k+1$ for all $k \geqslant 6$, by Boklan [2] to $\frac{7}{8}2^k$ for all $k \geqslant 6$ and by Wooley [68] to approximately $2k^2 \log k$ for all $k \geqslant 10$.

Sieve theory

The paper [1932d] is Estermann's only essay on the sieve method. For many years sieve theory was considered to be a 'difficult' area. The papers were hard to understand, technically complicated and gave only partial results. As a result of a visit to Hamburg by Brun, Rademacher [45] had become interested in Brun's sieve method and had shown that every sufficiently large even number is the sum of two products of seven or fewer primes. It seems certain that Estermann was also present in Hamburg during this visit. In [1932d] Estermann simplifies the notation and refines the argument of Rademacher's paper, and is thereby able to replace the seven by six.

With the advent of new sieve ideas introduced by Selberg (see his collected papers [51, 52]), a good deal of progress has been made in this area, culminating in the celebrated theorem of Chen [6] (see also Halberstam and Richert [16] and Ross [50]) that every sufficiently large even number is the sum of a prime and a number having at most two prime factors.

Additive problems involving primes

The paper [1937a] is a curious pre-echo of the practically contemporaneous celebrated work of Vinogradov [**65**] on sums of three primes. Estermann shows that every sufficiently large natural number is the sum of two primes and a number having exactly two prime factors. The exponential sum

$$\sum_{p_1}\sum_{p_2} e(p_1 p_2 \alpha)$$

used by Estermann is what is now sometimes called a good 'Type II' bilinear form. That is, one can view it as a value of the bilinear form uMv^*, where the matrix $M = (a_{rs})$ has general entry $e(rs\alpha)$ when r and s lie in intervals I and J respectively, and general entry 0 otherwise, and the vectors u and v are given the values of the characteristic functions of the primes in I and J respectively. Then a non-trivial estimate for the exponential sum can be obtained by observing that the bilinear form is bounded by $\|u\|\,\|v\|\,\lambda^{1/2}$, where λ is the largest eigenvalue of MM^*, and a good estimate for λ can be obtained by observing that the largest eigenvalue of a Hermitian matrix (b_{rs}) is bounded by the maximum over r of $\sum_s |b_{rs}|$. Of course, in each case the necessary estimates were obtained by directly applying Cauchy's inequality to appropriate sums.

The paper [1938] shows that Estermann was really on the ball at this time. Vinogradov's celebrated work [**65**] is rapidly mastered, and the method applied to show that the exceptional set $E(x)$, the number of even numbers not exceeding x which are *not* the sum of two primes, satisfies for any fixed positive A

$$\operatorname{card}(E(x)) = O(x(\log x)^{-A}).$$

The same theorem was obtained independently by Chudakov [**8**] and van der Corput [**9**]. It was not improved upon until Vaughan [**56**]. A little later, Montgomery and Vaughan [**38**] showed that there was a fixed positive number δ such that

$$\operatorname{card}(E(x)) = O(x^{1-\delta}).$$

This may be compared with the original work of Hardy and Littlewood [**21**] where, on the assumption of the Generalised Riemann Hypothesis, the above is obtained for any $\delta < \frac{1}{2}$.

Siegel zeros and the distribution of primes

One of the most important and tantalising questions in number theory concerns whether the Dirichlet L-function

$$L(s,\chi),$$

defined for $\Re s > 0$ and χ a non-principal Dirichlet character by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

has, when χ is real, any real zeros β on $(0, 1)$, and how close any such β can be to 1. Prior to important work by Siegel [**53**], the best available general bounds had been relatively weak. Page [**44**], in work which is the basis of his PhD thesis written under Estermann's supervision, had shown that there is a suitable positive constant c such that of all the primitive characters χ to moduli $q \leqslant z$, there is at most one for which $L(s,\chi)$ has a real zero β satisfying

$$\beta > 1 - \frac{c}{\log z},$$

and then χ is real and non-principal, and $L(s,\chi)$ has only one such zero. Thus the main point of interest is the nature of any possible 'exceptional' zero.

Siegel shows that for each positive number ε there is a positive number $c(\varepsilon)$ such that for each natural number q and non-principal real character χ modulo q, any real zero β of $L(s,\chi)$ must satisfy

$$\beta < 1 - c(\varepsilon)\, q^{-\varepsilon}.$$

Siegel's proof is decidedly difficult as well as requiring a good deal of knowledge of quadratic number fields. A number of attempts had been made at simplification (see, for example, Heilbronn [**27**]), but Estermann was the first to get right to the root of the matter. In [1948a] he shows that Siegel's theorem can be obtained by a straightforward argument based on classical estimates for L-functions and a simple function-theoretic method of Landau. Estermann's argument is that now usually given in expositions, such as Davenport's monograph [**12**], and, of course, his own book on prime number theory [1952] which is a concise and excellent introduction to the subject.

Other works

As well as his book on prime number theory, which formed the basis of a postgraduate course given at University College over a number of years, Estermann also wrote a text on complex numbers and functions [1962d]. This makes an excellent companion to Titchmarsh's book [**55**] on the theory of functions. Whereas Titchmarsh provides an overview of the ideas underlying the subject (but presented in a very informal fashion), Estermann's book constitutes an ideal source of concise and accurate proofs (but without emphasis on motivation). Estermann's treatment of the Jordan curve theorem and of analytic continuation (particularly his definition of non-isolated singularities) are of especial interest.

Estermann was most adept at finding simple elementary proofs. The early paper [1928a] (see also [1929b]) establishes elementarily an asymptotic formula for $\sum_{m=1}^{n} d(mv+c)$, where d is the divisor function, which had been stated without proof by Ramanujan [**47**], and in [1931c] Estermann obtains in an elementary way an asymptotic formula for

$$\sum_{\substack{n \leqslant x \\ (n,l)=1}} 1/\phi(n),$$

where ϕ is Euler's function, which had been obtained earlier by Titchmarsh by complex analytic methods. The latter sum was of interest as it arises in sieve theory.

The paper [1933a] gives an interesting proof of Kronecker's theorem and is one of those selected for their famous *Introduction* by Hardy and Wright [**23**, §23.8].

Perhaps the most celebrated short proof is that [1945] giving the sign of the Gauss sum

$$\sum_{x=1}^{q} e^{2\pi i x^2/q}.$$

This is still the shortest way, with the minimum of knowledge, of establishing the sign of the Gauss sum, and has been reproduced in various places, for example in Chowla [7, Chapter 2].

The paper [1953] is concerned with establishing

$$\lim_{u \to \infty} (u^{-1}g(u, a)) = a,$$

where $g(u,a)$ denotes the number of lattice points in the set $\{(x,y): 0 < x \leqslant u, \alpha x - a < y \leqslant \alpha x\}$. The proof acknowledges an idea remembered from a lecture of Hecke 30 years earlier but not traceable in the literature, and again reveals the influence of Hecke in Estermann's formative years.

In retirement, Estermann discovered [1975] an elegant proof of Pythagoras's theorem which is actually simpler than the original and is sufficiently short to be included verbatim.

Let S be the set of those natural numbers n for which $n\sqrt{2}$ is an integer. If S were not empty, it would have a least element k, say. Consider the number $(\sqrt{2}-1)k$. Then

$$(\sqrt{2}-1)k\sqrt{2} = 2k - k\sqrt{2},$$

and, since $k \in S$, both $(\sqrt{2}-1)k$ and $2k - k\sqrt{2}$ are natural numbers. So, by definition $(\sqrt{2}-1)k \in S$. But $(\sqrt{2}-1)k < k$, contradicting the assumption that k is the least element of S. Hence S is empty, which means that $\sqrt{2}$ is irrational.

Estermann's influence was far wider than a perusal of his published works would suggest. For example, Kestelman in his classic text [**17**] on measure theory, including what was for a long time one of the few accessible accounts in English of the Lebesgue integral, states that the notes of Estermann's postgraduate lectures giving an introduction to Lebesgue integration have been incorporated and form its essential nucleus. Moreover, although Estermann had only four doctoral students, Page, Halberstam, Roth and Vaughan, his influence has continued through them and their students, including Anderson, Chen, Choi, Filaseta, Ghosh, Hall, Nair, Ross, Woodall and Wooley. Also, the University College Mathematics Department Library houses a large collection of MSc theses by students who were taken under his wing.

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References

1. R. Ashton, PhD Thesis, London University, 1993.
2. K. D. Boklan, 'The asymptotic formula in Waring's problem', *Mathematika*, to appear.
3. F. V. Atkinson, 'The mean value of the zeta-function on the critical line', *Proc. London Math. Soc.* (2) 47 (1941) 174–200.
4. W. Blaschke, Aufgabe 573, *Arch. Math. Phys.* (3) 28 (1920) 74.
5. T. Bonnesen and W. Fenchel, *Theorie der Konvexen Körper* (Springer, Berlin, 1934).
6. Jing-Rung Chen, 'On the representation of a larger even integer as the sum of a prime and the product of at most two primes', *Sci. Sinica* 16 (1973) 157–176.
7. S. Chowla, *The Riemann hypothesis and Hilbert's tenth problem* (Gordon and Breach, New York, 1965).
8. N. G. Chudakov, 'On the Goldbach problem', *Dokl. Akad. Nauk SSSR* 17 (1937) 335–338.
9. J. G. van der Corput, 'Sur l'hypothèse de Goldbach', *Proc. Akad. Wet. Amsterdam* 41 (1938) 76–80.
10. H. Davenport, 'On Waring's problem for fourth powers', *Acta Math.* 40 (1939) 731–747.
11. H. Davenport, 'On Waring's problem for fifth and sixth powers', *Amer. J. Math.* 64 (1942) 199–207.
12. H. Davenport, *Multiplicative number theory* (1st edn, Markham, Chicago, 1966); 2nd edn, revised by H. L. Montgomery, Graduate Texts in Math. 74 (Springer, Berlin, 1980).
13. H. Davenport and H. Heilbronn, 'On Waring's problem for fourth powers', *Proc. London Math. Soc.* (2) 41 (1936) 143–150.

14. M. Eichler, 'Quaternäre quadratische Formen und die Riemannsche Vermutung für die Kongruenzzeta function', *Arch. Math.* 5 (1954) 355–366.
15. H. Halberstam, 'An asymptotic formula in the theory of numbers', *Trans. Amer. Math. Soc.* 84 (1957) 338–351.
16. H. Halberstam and H.-E. Richert, *Sieve methods* (Academic Press, London, 1974).
17. H. Kestelman, *Modern theories of integration* (1st edn, Oxford University Press, 1937; 2nd edn, Dover Publications, New York, 1960).
18. G. H. Hardy, 'On the representation of a number as the sum of any number of squares, and in particular of five', *Trans. Amer. Math. Soc.* 21 (1920) 255–284.
19. G. H. Hardy, *Ramanujan* (Cambridge University Press, 1940).
20. G. H. Hardy and J. E. Littlewood, 'Some problems of "Partitio Numerorum": IV The singular series in Waring's problem', *Math. Z.* 12 (1922) 161–188.
21. G. H. Hardy and J. E. Littlewood, 'Some problems of "Partitio Numerorum": V A further contribution to the study of Goldbach's problem', *Proc. London Math. Soc.* (2) 22 (1923) 46–56.
22. G. H. Hardy and J. E. Littlewood, 'Some problems of "Partitio Numerorum": VI Further researches in Waring's problem', *Math. Z.* 23 (1925) 1–37.
23. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers* (5th edn, Oxford University Press, 1989).
24. D. R. Heath-Brown, 'The fourth power moment of the Riemann zeta-function', *Proc. London Math. Soc.* (3) 38 (1979) 385–422.
25. D. R. Heath-Brown, 'Cubic forms in ten variables', *Proc. London Math. Soc.* 47 (1983) 225–257.
26. D. R. Heath-Brown, 'Weyl's Inequality, Hua's Inequality, and Waring's problem', *J. London Math. Soc.* (2) 38 (1988) 216–230.
27. H. Heilbronn, 'On Dirichlet series which satisfy a certain functional equation', *Quart. J. Math.* 9 (1938) 194–195.
28. C. Hooley, 'On nonary cubic forms', *J. Reine Angew. Math.* 386 (1988) 32–98.
29. L.-K. Hua, 'On Waring's problem', *Quart. J. Math.* 9 (1938) 199–202.
30. A. E. Ingham, 'Some asymptotic formulae in the theory of numbers', *J. London Math. Soc.* 2 (1927) 202–208.
31. A. Ivic, *On mean values of the Riemann zeta function*, Lectures at the Tata Institute of Fundamental Research (Springer, Berlin, 1991).
32. R. D. James, 'On Waring's problem for odd powers', *Proc. London Math. Soc.* (2) 37 (1934) 257–291.
33. H. D. Kloosterman, 'Over het uitdrukken van geheele positieve getallen in den vorm $ax^2+by^2+cz^2+dt^2$', *Verslag Amsterdam* 34 (1925) 1011–1015.
34. H. D. Kloosterman, 'On the representation of numbers in the form $ax^2+by^2+cz^2+dt^2$', *Acta Math.* 49 (1925) 407–464.
35. H. D. Kloosterman, 'On the representation of numbers in the form $ax^2+by^2+cz^2+dt^2$', *Proc. London Math. Soc.* (2) 25 (1925) 143–173.
36. N. V. Kuznetsov, 'Petersson's hypothesis for parabolic forms of weight zero and Linnik's hypothesis. Sums of Kloosterman sums', *Mat. Sb.* 111 (1980) 334–383.
37. N. V. Kuznetsov, 'Convolution of the Fourier coefficients of the Eisenstein–Maass series', *Zap. Nauchn. Sem. LOMI* 129 (1983) 43–84.
38. H. L. Montgomery and R. C. Vaughan, 'The exceptional set in Goldbach's problem', *Acta Arith.* 27 (1975) 353–370.
39. L. J. Mordell, 'On the representation of a number as a sum of an odd number of squares', *Trans. Cambridge Philos. Soc.* 22 (1919) 361–372.
40. L. J. Mordell, 'On a sum analogous to a Gauss's sum', *Quart. J. Math.* 3 (1932) 161–167.
41. Y. Motohashi, 'The binary additive divisor problem', *Ann. Sci. École Norm. Sup.* (4) 27 (1994) 529–572.
42. Major P. A. MacMahon, 'Dirichlet series and the theory of partitions', *Proc. London Math. Soc.* (2) 22 (1924) 404–411.
43. A. Oppenheim, 'On an arithmetic function', *J. London Math. Soc.* 1 (1926) 205–211.
44. A. Page, 'On the number of primes in an arithmetic progression', *Proc. London Math. Soc.* (2) 39 (1935) 116–141.
45. H. Rademacher, 'Beiträge zur Viggo Brunschen Methode in der Zahlentheorie', *Hamburger Abh.* 3 (1924) 12–30.
46. H. Rademacher, 'Über den Vektorenbereich eines konvexen eberen Bereichs', *Jahresber. Deutsch. Math.-Verein.* 34 (1925) 64–79.
47. S. Ramanujan, 'Some formulae in the analytic theory of numbers', *Messenger Math.* 45 (1916) 81–84.
48. C. A. Rogers and G. C. Shephard, 'The difference body of a convex body', *Arch. Math.* 8 (1957) 220–233.
49. C. A. Rogers and G. C. Shephard, 'Convex bodies associated with a given convex body', *J. London Math. Soc.* 33 (1958) 270–281.
50. P. M. Ross, 'On Chen's theorem that each large even number has the form p_1+p_2 or $p_1+p_2p_3$', *J. London Math. Soc.* (2) 10 (1975) 500–506.

51. A. SELBERG, *Collected papers*, Vol. I (Springer, Berlin, 1989).
52. A. SELBERG, *Collected papers*, Vol. II (Springer, Berlin, 1991).
53. C. L. SIEGEL, 'Über die Classenzahl quadratischer Zahlkörper', *Acta Arith.* 1 (1935) 83–86.
54. W. SÜSS, 'Über den Vektorenbereich eines Eikörpers', *Jahresber. Deutsch. Math.-Verein.* 37 (1928) 87–90.
55. E. C. TITCHMARSH, *The theory of functions* (2nd edn, Clarendon Press, Oxford, 1950).
56. R. C. VAUGHAN, 'On Goldbach's problem', *Acta Arith.* 22 (1972) 21–48.
57. R. C. VAUGHAN, 'On Waring's problem for cubes', *J. Reine Angew. Math.* 365 (1986) 122–170.
58. R. C. VAUGHAN, 'On Waring's problem for smaller exponents, II', *Mathematika* 33 (1986) 6–22.
59. R. C. VAUGHAN, 'A new iterative method in Waring's problem', *Acta Math.* 162 (1989) 1–71.
60. R. C. VAUGHAN and T. D. WOOLEY, 'Further improvements in Waring's problem, I', *Acta Math.*, to appear.
61. I. M. VINOGRADOV, 'An asymptotic formula for the number of representations in Waring's theorem', *Mat. Sb.* 42 (1935) 531–534.
62. I. M. VINOGRADOV, 'On Waring's problem', *Ann. of Math.* 36 (1935) 395–405.
63. I. M. VINOGRADOV, 'On the asymptotic formula in Waring's problem', *Mat. Sb.* (2) 1 (1936) 169–174.
64. I. M. VINOGRADOV, 'On some new inequalities in the theory of numbers', *C. R. Hebd. Séanc. Acad. Sci. Paris* 202 (1936) 1361–1362.
65. I. M. VINOGRADOV, 'Some theorems concerning the theory of primes', *Rec. Math.* 2 (44) (2) (1937) 179–195.
66. A. WEIL, 'On the Riemann hypothesis in functions fields', *Proc. Nat. Acad. Sci. USA* 27 (1941) 345–347.
67. H. WEYL, 'Über die Gleichverteilung von Zahlen mod Eins', *Math. Ann.* 77 (1916) 315–352.
68. T. D. WOOLEY, 'On Vinogradov's Mean Value Theorem', *Mathematika* 39 (1993) 379–399.

Bibliography of Theodor Estermann

[1925] 'Über Carathéodorys und Minkowskis Verallgemeinerungen des Längenbegriffs', *Abh. Hamburg* 4 (1925) 73–116.
[1927] 'Zwei neue Beweise eines Satzes von Blaschke und Rademacher', *Jahresber. Deutsch. Math.-Verein.* 36 (1927) 197–200.
[1928a] 'On the divisor problem in a class of residues', *J. London Math. Soc.* 3 (1928) 247–250.
[1928b] 'On certain functions represented by Dirichlet series', *Proc. London Math. Soc.* (2) 27 (1928) 435–448.
[1928c] 'On a problem of analytic continuation', *Proc. London Math. Soc.* (2) 27 (1928) 471–482.
[1928d] 'Über den Vektorenbereich eines konvexen Körpers', *Math. Z.* 28 (1928) 471–475.
[1929a] 'On the representations of a number as the sum of three products', *Proc. London Math. Soc.* (2) 29 (1929) 453–478.
[1929b] 'Über die Summe der Teilerzahlen der Glieder einer arithmetischen Reihe', *Jahresber. Deutsch. Math.-Verein.* 38 (1929) 48–49.
[1929c] 'Vereinfachter Beweis eines Satzes von Kloosterman', *Abh. Hamburg* 7 (1929) 82–98.
[1929d] 'Bemerkung zu den Kontinuitatsbeweisen der Abbildungssatzes für Polygone', *Abh. Hamburg* 7 (1929) 11–13.
[1930a] 'On the representation of a number as the sum of two products I', *Proc. London Math. Soc.* (2) 31 (1930) 123–133.
[1930b] 'On the representation of a number as the sum of two products II', *J. London Math. Soc.* 5 (1930) 131–137.
[1931a] 'On the representations of a number as the sum of two numbers not divisible by *k*th powers', *J. London Math. Soc.* 6 (1931) 37–40.
[1931b] 'On the representation of number as the sum of a prime and a quadratfrei number', *J. London Math. Soc.* 6 (1931) 219–221.
[1931c] 'On an asymptotic formula due to Titchmarsh', *J. London Math. Soc.* 6 (1931) 250–251.
[1931d] 'Über die Darstellungen einer Zahlen als Differenz von zwei Produkten', *J. Math.* 164 (1931) 173–182. Berichtigung. *J. Math.* 166 (1931) 64.
[1931e] 'Einige Sätze über quadratfrei Zahlen', *Math. Ann.* 105 (1931) 653–662.
[1932a] 'On the representations of a number as the sum of three or more products', *Proc. London Math. Soc.* (2) 34 (1932) 190–195.
[1932b] 'An asymptotic formula in the theory of numbers', *Proc. London Math. Soc.* (2) 34 (1932) 280–292.
[1932c] 'On Ostrowski's gap theorem', *J. London Math. Soc.* 7 (1932) 19–20.
[1932d] 'Eine neue Darstellung und neu Anwendungen der Viggo Brunschen Methode', *J. Math.* 168 (1932) 106–116.
[1933a] 'A proof of Kronecker's theorem by induction', *J. London Math. Soc.* 8 (1933) 18–20.

[1933b] 'Über die totale Variation einer stetigen Funktion und den Cauchyschen Integralsatz', *Math. Z.* 37 (1933) 556–560.
[1934a] 'Bemerkung zu meiner Arbeit "Über die totale Variation einer stetigen Funktion und den Cauchyschen Integralsatz"', *Math. Z.* 38 (1934) 641.
[1934b] (with C. L. BARHAM) 'On the representation of a number as the sum of four or more N-numbers', *Proc. London Math. Soc.* (2) 38 (1934) 340–353.
[1936a] 'Proof that every large integer is a sum of seventeen biquadrates', *Proc. London Math. Soc.* (2) 41 (1936) 126–142.
[1936b] 'On the representations of a number as a sum of squares', *Acta Arith.* 2 (1936) 47–79; and *Prace Mat.-Fiz.* 45 (1937) 93–125.
[1936c] 'Some recent work in the additive theory of numbers', IT British Assoc. Adv. Sci. Report (1936) 332–333.
[1937a] 'A new result in the additive prime-number theory', *Quart. J. Math. Oxford* 8 (1937) 32–38.
[1937b] 'Proof that every large integer is the sum of two primes and a square', *Proc. London Math. Soc.* (2) 42 (1937) 501–516.
[1937c] 'On Waring's problem for fourth and higher powers', *Acta Arith.* 2 (1937) 197–211.
[1938] 'On Goldbach's problem: proof that almost all even positive integers are sums of two primes', *Proc. London Math. Soc.* (2) 44 (1938) 307–314.
[1942] 'A new proof of a theorem of Minkowski', *J. London Math. Soc.* 17 (1942) 158–161.
[1945] 'On the sign of the Gaussian sum', *J. London Math. Soc.* 20 (1945) 66–67.
[1946] 'Note on a theorem of Minkowski', *J. London Math. Soc.* 21 (1946) 179–182.
[1947] 'Elementary evaluation of $\zeta(k)$', *J. London Math. Soc.* 22 (1947) 10–13.
[1948a] 'On Dirichlet's L functions', *J. London Math. Soc.* 23 (1948) 275–279.
[1948b] 'On Waring's problem: A simple proof of a theorem of Hua', *Sci. Rep. Nat. Tsing-Hua Univ. Ser. A* 5 (1948) 226–239.
[1951] 'On sums of squares of square-free numbers', *Proc. London Math. Soc.* (2) 53 (1951) 125–137.
[1952] *Introduction to modern prime number theory*, Cambridge Tracts Math. and Math. Phys. 41 (Cambridge University Press, 1952).
[1953] 'On the number of primitive lattice points in a parallelogram', *Canad. J. Math.* 5 (1953) 456–459.
[1956] 'On the fundamental theorem of algebra', *J. London Math. Soc.* 31 (1956) 238–240.
[1959] 'On the representation of a number as a sum of three squares', *Proc. London Math. Soc.* (3) 9 (1959) 575–594.
[1961] 'On Kloosterman's sum', *Mathematika* 8 (1961) 83–86.
[1962a] 'Another proof that bounded modular functions are constants', *Illinois J. Math.* 6 (1962) 128–130.
[1962b] 'A new application of the Hardy–Littlewood–Kloosterman method', *Proc. London Math. Soc.* (3) 12 (1962) 425–444.
[1962c] 'Note on a paper of A. Rotkiewicz', *Acta Arith.* 8 (1962/3) 465–467.
[1962d] *Complex numbers and functions* (University of London, Athlone Press, 1962).
[1966] 'A theorem implying the irrationality of π^2', *J. London Math. Soc.* 41 (1966) 415–416.
[1971] *Notes on Landau's proof of Picard's 'great' theorem*, Stud. Pure Math., presented to Richard Rado (Academic Press, London, 1971) 101–106.
[1975] 'The irrationality of $\sqrt{2}$', *Math. Gaz.* (408) 59 (1975) 110.

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