LEOPOLD FEJÉR

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1. Why did Hungary produce so many mathematicians of our time? Many people have asked this question which, I think, nobody can fully answer. There were, however, two factors whose influence on Hungarian mathematics is manifest and undeniable, and one of these was Leopold Fejér, his work, his personality.

The other factor was the combination of a competitive examination in mathematics with a periodical. The examination, named after the physicist Roland Eötvös, was given once a year (since 1894) and it was open to young people who, having passed the terminal examination of the secondary schools in that year, were about to enter the university. The periodical *Középiskolai Mathematikai Lapok* was a well-conducted mathematical journal for the pupils of secondary schools; it contained mainly problems proposed by the editors, the best solutions sent in by the readers, and the names of all those who sent in correct solutions. Doing the problems of the periodical was regarded as a sort of standard preparation for the examination. Examination and periodical cooperated in stirring up the ambition and the curiosity of many future mathematicians—and Fejér was one of them.

Leopold (Lipót) Fejér was born on 9 February, 1880, in Pécs, an old, middle-sized, provincial town in Hungary. He was educated in his native town and passed there the terminal examination of the secondary school. He had an early interest in mathematics. He sent solutions to the school mathematics journal we have mentioned above; he took the Eötvös examination and won second prize. This happened in 1897. In the same year he began his university studies in Budapest and, in 1902, he received his Ph.D. degree. In between, however, he spent one academic year, 1899-1900, at the University of Berlin, and this year decisively influenced his scientific career : he was led, probably by a remark of H. A. Schwarz, to the discovery on Fourier series which he presented in a *Comptes Rendus* Note of 1900 and in his Ph.D. dissertation, and about which we shall have more to say.

Fejér rose quickly from the academic ranks. He was elected a corresponding member of the Hungarian Academy of Sciences in 1908 (his election to full membership took place much later, in 1930). He was appointed full professor at the University of Budapest in 1911, at the age of 31—there was no higher academic position in Hungary.

From this date on he lived in Budapest, through all the political upheavals and calamities that befell Hungary. (His election to full membership in the Academy was delayed by racial prejudice. He was

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ousted from his professorship by the fascist regime in 1944 and narrowly escaped being killed.) He never sought a position outside his native country, to which he was deeply attached by all the links of affection and habit. He travelled, however, extensively outside Hungary, especially in Germany, where he had many friends, of which I wish to mention C. Carathéodory, E. Landau and I. Schur. He came to the United States as Visiting Lecturer of the American Mathematical Society in 1933.

Fejér died on 15 October, 1959, after a prolonged illness which dimmed his manifold talents one by one; the gift of which sparks stayed with him almost to the last, I am told, was his characteristic sense of humour.

2. In a *Comptes Rendus* Note presented to the Paris Academy on 10 December, 1900, Fejér stated and proved the proposition that is now usually quoted as "Fejér's theorem": *Let $f(x)$ denote a bounded (Riemann) integrable function, periodic with period 2π, and $s_n(x)$ the n-th partial sum of the Fourier series of $f(x)$. Set*

$$S_n(x) = \frac{s_0(x)+s_1(x)+\ldots+s_{n-1}(x)}{n} = \int_0^{2\pi} \frac{1}{2n\pi} \frac{1-\cos n(t-x)}{1-\cos(t-x)} f(t)\, dt. \tag{1}$$

Then
$$\lim_{n\to\infty} S_n(x) = [f(x-0)+f(x+0)]/2, \tag{2}$$

provided that the two limits indicated on the right-hand side exist.

Restated in more "colloquial" language (less precisely, less sharply, in present-day terminology): *The Fourier series is summable (C, 1) to the value of the function at each point of continuity.*

This theorem of Fejér, of which the date 1900 deserves to be remembered, started a mass of research; it gave new impetus to the theories of Fourier series, of divergent series, of approximations; its applications, refinements and analogues pursued by many authors form a vast literature. It certainly became the centre of Fejér's scientific work as the following survey attempts to show. As I shall have to quote it repeatedly, I shall use the abbreviation F.T. (Fejér's Theorem) for it.

In order to render the following survey less long and less heavy, I shall state most results only in colloquial language, restricting myself to prominent particular cases, and referring for details to the Bibliography.†

† Numbers in heavy print refer to the short Bibliography at the end of this Notice. There is a complete bibliography of Fejér's papers to 1950 in *Matematikai Lapok*, 1 (1950), 267–272, to which five more papers, printed after 1950, are added in Paul Turán's Notice on Fejér.

I use here the numbering of that complete bibliography. When a paper is written in Hungarian, but there is an essentially coinciding paper in another language (which happens to be German in all the cases here considered) I quote the latter. For instance, [**9**] is not Fejér's dissertation, but a shortened German translation of his dissertation.

I was permitted to see the manuscript of Professor Turán's Notice on Fejér, for which I wish to thank him. I wish to thank also Professor G. Szegö for advice on various points.

3. I wish to include within the abbreviation F.T. some (easily proved) extensions of the theorem quoted in Section 2, which are all mentioned in Fejér's dissertation, see [**9**], especially the following: If the function $f(x)$ is everywhere continuous, $S_n(x)$ converges *uniformly* to $f(x)$.

Here are a few (easily derivable) *corollaries* of F.T.: If the Fourier series converges at a point of continuity of the function, its sum is the value of the function at that point [**9**].—Weierstrass' approximation theorem: A function continuous in a closed interval is a uniform limit of polynomials [**9**].—Poisson's integral yields a valid solution for Dirichlet's problem for the circle [**9**]. This fact was proved in a classical paper by H. A. Schwarz, who thought that a proof of the fact by the use of Fourier series, as had been attempted before, was hardly possible. This opinion of Schwarz was the challenge that led Fejér to the discovery of his theorem.

Another also easily derivable and very beautiful corollary was much later noticed by Fejér [**49**]: The power series whose sum conformally maps the interior of its circle of convergence onto the interior of a Jordan curve, is uniformly convergent also on the *boundary* of the circle of convergence.

4. We pass now to *analogues* of F.T. To emphasise the decisive point, I introduce a definition. Let us call a functional operation *positive*, if applied to a positive function (*i.e.* one taking values $\geqslant 0$) it yields a positive function, or a non-negative number. The functional operation that applied to $f(x)$ yields $S_n(x)$, see (1), is positive, and the great simplicity of F.T.'s proof is due to this circumstance. Fejér was keenly conscious of this connection; each of the following analogues of F.T. is based by him on an appropriate positive functional operation.

While the Fourier series aims at representing a function defined on the periphery of a circle, the analogous Laplace series, whose terms are surface harmonics, aims at representing a function defined on the surface of a sphere. *The Laplace series is summable* $(C, 2)$ *to the value of the function at each point of continuity* [**28**].—The usual trigonometric interpolation formula (at $2n+1$ periodically equidistant points) is analogous in a certain way to the partial sum $s_n(x)$ of the Fourier series. Fejér constructed an interpolation formula that is analogous in the same way to $S_n(x)$ and is *uniformly convergent* for $n \to \infty$ to the interpolated function when it is continuous; the same construction was independently found by Dunham Jackson [**53**].—Analogous to the passage from $s_n(x)$ to $S_n(x)$, or from the usual trigonometric interpolation to the Fejér-Jackson interpolation, is the passage from the Newton-Lagrange interpolation (by polynomials of degree $\leqslant n-1$ for n given points) to interpolation by "step-parabolas" (degree $\leqslant 2n-1$ for n given points); this latter *converges uniformly* to the interpolated function when it is continuous, at least for certain systems of abscissas, in particular for the system $\cos(2k-1)\pi/(2n)$, $k = 1, 2, \ldots, n$, of the "Chebyshev abscissas" in the interval $-1 \leqslant x \leqslant 1$ [**53**].—Also the "mechanical quadrature" with the Chebyshev abscissas turns out to be a positive functional operation from which fact convergence follows [**80**].

5. Other results of Fejér, without being applications of, or analogues to, F.T. are still *closely connected* with it. Let us look at a few examples.

To see the conclusion (2) of F.T. in the right light, we must know that the partial sum $s_n(x)$ may have no limit where $S_n(x)$ has one. That the Fourier series of a continuous function may diverge, was known before, but Fejér constructed examples that satisfied his taste for clear and concrete detail. He explicitly stated the law of the coefficients of a trigonometric series which diverges at a given point, or in an everywhere dense set of points, although it is the Fourier series of a continuous function. His essentially simple method of construction enabled him to produce other types of singular behaviour in trigonometric and power series the existence of which was not known before [**32, 54**].

The functional operation that applied to $f(x)$ yields $S_n(x)$ is positive and, therefore, the value of $S_n(x)$, for all n and all x, lies between the minimum and the maximum of $f(x)$ [**9**]. Hence, knowledge of the approximating $S_n(x)$ for any (small) n yields some *definitive information* about the approximated $f(x)$. To find such informative relations between the approximating function and the approximated function became one of Fejér's lasting interests. Thus, he explored certain higher Cesàro-means of the Fourier series and found that they must correctly mirror the sign, the monotonic increase, or the convexity of the approximated function in certain simple cases whereas the partial sum $s_n(x)$ or the first Cesàro mean $S_n(x)$ need not do so [**82**].

I do not think that Fejér followed in detail the vast literature that was initiated by F.T. but he was intent on harmonising the principal refinements of his theorem with his own circle of ideas. He was rather exacting in this respect and it often took many years till he found a proof that satisfied his taste. I refer here to Lebesgue's condition for the convergence of $S_n(x)$ to $f(x)$ [**62**] and to Hardy and Littlewood's theorem on the "strong summability" of the Fourier series [**96**].

6. The foregoing survey is far from being exhaustive; I have just attempted to show the great unity of Fejér's mathematical work through all its variety, unity due to the loving care and unremitting industry with which he worked on all the details, implications and suggestions of his fundamental discovery, F.T.

Yet he worked in other domains too. He had wide-ranging interests and the gift to recognise significant problems capable of a simple solution. He often spotted such problems in the work of his mathematical friends. In connection with Landau's research on Picard's theorem, he very simply proved an elegant estimate for the first root of certain polynomials (with gaps in the sequence of coefficients, [**23**]).—A joint paper he wrote with Carathéodory is also connected with Picard's theorem (and with the Carathéodory-Toeplitz theory of functions with positive real part analytic in a circle, [**39**]).—Jointly with Frédéric Riesz, he gave a simple and suggestive inequality for analytic functions of which the often considered Hilbert inequality is a striking corollary [**59**].

We cannot list here Fejér's remarks on several other subjects.

7. Fejér's great influence on Hungarian mathematicians is due not only to his scientific work but also to his personality.

He had artistic tastes. He deeply loved music and was a good pianist. He liked a well-turned phrase. "As to earning a living," he said, "a professor's salary is a necessary, but not a sufficient, condition." Once he was very angry with a colleague who happened to be a topologist, and explaining the case at length he wound up by declaring: ". . . and what he is saying is a topological mapping of the truth." He had a quick eye for human foibles and miseries; in seemingly dull situations he noticed points that were unexpectedly funny or unexpectedly pathetic. He carefully cultivated his talent of *raconteur*; when he told, with his characteristic little gestures, of the little shortcomings of a certain great mathematician, he was irresistible. The hours spent in continental coffee houses with Fejér discussing mathematics and telling stories are a cherished recollection for many of us. Fejér presented his mathematical remarks with the same verve as his stories, and this may have helped him in winning the lasting interest of so many younger men in his problems.

8. Fejér talked of a paper he was about to write up. "When I write a paper," he said, "I have to rederive for myself the rules for differentiation and sometimes even the commutative law of multiplication." These words stuck in my memory and years later I came to think that they express an essential aspect of Fejér's mathematical talent: his love for the intuitively clear detail.

It was not given to him to solve very difficult problems or to build vast conceptual structures. Yet he could perceive the significance, the beauty, and the promise of a rather concrete, not too large problem, foresee the possibility of a solution, and work at it with intensity. And, when he had found the solution, he kept on working at it with loving care, till each detail became fully intuitive and the connection of the details in a well-ordered whole fully transparent.

It is due to such care spent on the elaboration of the solution that Fejér's papers are very clearly written and easy to read, and most of his proofs appear very clear and simple. Yet only the very naive may think that it is easy to write a paper that is easy to read, or that it is a simple thing to point out a significant problem that is capable of a simple solution.

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