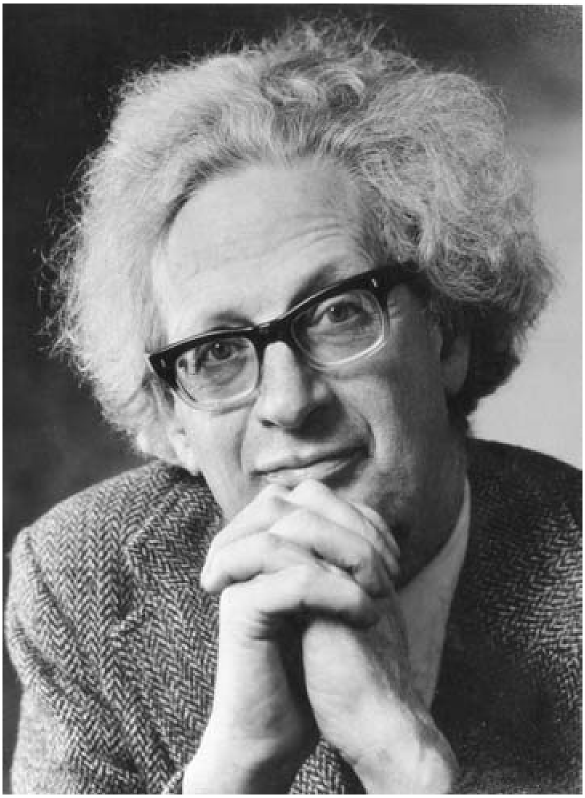
Bull. London Math. Soc. 38 (2006) 329–350
doi:10.1112/S0024609306018443

OBITUARY

ALBRECHT FRÖHLICH 1916–2001



After this it was noised abroad that Mr. Valiant-for-Truth was taken with a Summons... when he understood he called his Friends and told them of it... many accompanied him to the River side. So he passed over, and all the Trumpets sounded for him on the other side.

John Bunyan, *Pilgrim's Progress.*

1. *Life*

Ali Fröhlich was one of the truly great mathematicians of his generation. In addition to making fundamental contributions, he is one of the very few of whom it can be said, "he founded a new subject": namely that area now known as Galois module theory. In addition to establishing a theoretical basis for the subject, he also formulated some very deep and wonderful conjectures that would thrust him onto the world's stage. He also built up a school of researchers that played a central part in establishing British algebraic number theory as a world force. The outstanding nature of his contributions to mathematics was recognised in 1992 when the London Mathematical Society awarded him the De Morgan Medal: their highest and most distinguished prize, which is awarded triennially.

Albrecht Fröhlich was born in Munich on 22 May 1916. Ali, as he was to become known, was the youngest of the three children of Julius and Frieda Fröhlich. His brother Herbert, who was the eldest of the children, became a very distinguished physicist. His sister, Betti, became a Zionist Pioneer in Palestine; she would play a crucial role in the family's escape from Nazi Germany. Ali's father, Julius, was a cattle merchant who came from Rexingen, a village in the Black Forest where many of the residents, like the Fröhlichs, were Jewish. Following the privations of the First World War, Ali grew up in the happier and more hopeful period of the Weimar Republic. However, in his mid-teens he started to become increasingly involved in the political struggle against the rise of fascism.

Initially the menace of the rise of Hitler was not fully apparent: Herbert, his brother, teased the family that the Hotel Leopold in Munich had a sign in the window saying 'No dogs or Jews'; all the family burst into laughter when they realised that Herbert was joking. Within three months the hotel had exactly such a sign in place.

Matters deteriorated very rapidly indeed. His father lost his business and was beaten up by some business competitors and Nazi Brown Shirts. His brother Herbert was sacked from his post at the University of Freiburg. Ali himself was saved from the hands of Nazi Brown Shirts only by the presence of mind of a very helpful police officer, who understanding the situation, promptly took Ali into custody, pretending to arrest him as an enemy of the Third Reich; he was then released the next day, once it was considered safe. Ali tried to make the best of things and carry on. Then one day he went to a political bookshop which was near a Nazi headquarters. The owner was most helpful and told Ali to take as much political literature as he liked. Ali responded with alacrity, and filled first his pockets and then stuffed as many papers as he could down his trousers. There was then an almighty row when he got home and unloaded his booty, and his mother threw all the papers on the fire. Unfortunately Ali had been spotted from the Nazi offices. The next morning the wife of a policeman, who lived on the top flat of their apartment block, knocked on their door, pointed to Ali and said, "Are you still here?" The family understood that this was really a final warning, and so Ali packed his bags and went to the French Consulate, who were most helpful and gave him the necessary documentation to enter France.

It was agreed that his parents would follow. They tried to encourage Julius' brother to accompany them; alas he elected to stay behind, and later perished in Auschwitz. Ali led the way and fled to France, shortly to be followed by his parents, where they awaited passage to the British Mandate of Palestine. Betti, who was to prove to be the family's salvation, had been working as a Pioneer in a Kibbutz and so was able to obtain a visa for them. Many were far less fortunate, with many boats of escaping Jews being turned back when they tried to land in Palestine for lack of the required documentation. Ali gained temporary employment in a chocolate factory in Strasbourg. His time there was not without incident: he attended a political demonstration in Strasbourg, which was broken up by the French police. He was pursued by a policeman, but just managed to escape. He would later say, "The difference between me and the policeman was that I was running for my life. Had I been caught, I would have been deported to Germany to face certain death."

A year after fleeing to France, the family left for Palestine. Ali and his parents established a permanent home in Palestine approximately a year after their arrival there. Ali's education had, of course, come to an abrupt end: he now had to support his mother and father, working sometimes as a plumber and at other times as an electrician in a Palestine railway workshop. The latter provided Ali with a wonderful opportunity to learn English as spoken by the lower ranks of the British Army, and in the early days he would occasionally quite innocently address senior ranks with inappropriate language. While living in Palestine Ali became increasingly politically active. During this period, he had his first clash with British officialdom due to his political activities. As a result of these he was interned in Acre for six months during a period of politically tense relations between the Jewish community and the British authorities. Curiously, on one occasion, when Ali returned to Israel in the 1970s, he met his former jailer. As was typical of Ali, there was no rancour!

Whereas Ali and his parents had followed Betti and fled from Germany to go to Palestine, his brother Herbert had been dismissed from his post in Freiburg University but in 1934 had been able to obtain a post in Russia, at the University of Leningrad. Unfortunately, within a year, Stalin had expelled all non-communist foreigners from the universities and so once again Herbert found himself without a post. Happily, he was then offered a job by Professor Tyndall at the University of Bristol; this was subsequently to prove a crucial development also for Ali's future.

By the conclusion of the war, Herbert had become a Reader at the University of Bristol. In addition to supporting his parents, Herbert also arranged for Ali to study for a BSc at the University of Bristol, offering to provide him with all the necessary financial support. Initially Ali was refused a passage to the UK, as the huge number troops returning home were given priority and Ali's qualifications for university study looked distinctly thin. When Bristol University heard of his difficulties they sent a stinging rebuke to the Colonial Office, saying that "it has never been the policy of Bristol University to explain to the Colonial Office its reasons for admitting students." In consequence, Ali was now allowed to travel, but arrived too late for the start of the 1945 Michaelmas term. His arrival at Bristol occasioned his second encounter with British officialdom — now in the form of a university administration. However, this was a much more positive experience, and he was quite delighted by their flexibility and their ability to make exceptions: Ali had no formal qualifications in mathematics, and so the natural thing to do, having arrived late, would be for him to engage in some private study and wait until the start of the new academic year in 1946; then, hopefully after passing the examinations at the conclusion of the intermediate year, he could begin the honours course. Ali, however, was in a hurry to get on. The Dean was most understanding and helpful and, exceptionally, allowed Ali to progress straight into the honours course. His judgement was completely justified when Ali graduated with a First Class Honours Degree in 1948 after only eight terms' study — a truly remarkable achievement for someone whose education had stopped at the age of sixteen (when he had had to flee Germany) and who had been working as a manual labourer for the previous thirteen years.

As a result of this experience and the help given to him, in later years he would always feel it was his duty to try to help others coming from an unusual background or facing similar kinds of problems. He would also always feel a very deep sense of gratitude to his brother for his great generosity in bringing him to the UK and financing his studies. In fact, in addition to supporting Ali in Bristol, Herbert also supported his parents on what must have been a relatively meagre salary.

Whilst he was an undergraduate Ali joined the International Student Society, and he signed up for their Youth Hostel trip to the South West. He first came to the attention of the then President, Evelyn Ruth Brooks, because he was the last student to pay for the trip. He and Ruth were further drawn together on various adventures during the trip — and in particular fate singled them out as the only two in the whole student group who were not sick during a very rough boat ride on a lake. The relationship flourished, and they were married in 1950. Ruth would become a popular and very highly regarded general practitioner in the Wimbledon area, where they lived from 1960–1992.

In the course of his undergraduate studies, Ali realised that he wanted to pursue a research career in number theory and he asked Heilbronn to take him on as a PhD student. Initially, Heilbronn was reluctant, telling Ali that he was too old; however, in time he relented and accepted Ali as a prospective research student, subject to the usual condition of achieving a satisfactory performance in the final examinations. Attending a post-finals party with Heilbronn, Ali, like so many students, became depressed at the possibility of his not doing as well in the exams as he had hoped. Heilbronn expressed total confidence in Ali's ability and insisted that they talk about research plans for the next year that very evening. Ali wrote his thesis, entitled 'On topics of representations of groups and in class-field theory', in only two years, and was awarded his PhD in 1951. When one reviews his research over the period of nearly fifty years, it is most striking to see the profound impact of his thesis work: the interplay between representation theory and algebraic number theory was to lie right at the heart of a considerable amount of his work.

After completing his doctorate, Ali Fröhlich took up two appointments, each of just a few years' duration: the first at the University of Leicester (1950–52) and the second at the College of North Staffordshire (1952–55). He continued to develop the work on genus fields that he had begun in his thesis. Later, he would feel that the value of some aspects of this work had not been fully appreciated. He was particularly upset by an account of his research in *Mathematics Reviews*, which completely misunderstood the significance of one of his papers. Again, sage advice from Heilbronn saved the day: he told Ali to forget all about the mistaken review, and he put the matter into perspective, saying that one had to accept that such things happened. Later, when Ali was a George Miller Professor in Illinois, he would give a course on this and related work, drawing the threads together and providing a lovely presentation of the theory of genus fields and presentations of associated Galois groups. This was published in the book [**89**].

In 1955, Ali noticed that a Readership at King's College, London, had been advertised. Initially he was hesitant about applying; he had had quite a struggle to master the English language — to the level required in British universities — and he was also a little unsure as to whether he had quite the right calibre of publications list. However, his wife Ruth encouraged him to apply, and he was duly called for interview. He found the interview to be a gruelling experience: Mordell gave him a hard time, but Davenport, Hall and his future new Head of Department, Semple, were more supportive. This was a period which saw a great diversification in his research interests. There was a feeling that great insight into the structure of groups could be obtained by studying the near ring generated by the endomorphisms of a given group, and Ali's work in this area prospered greatly. For this and other work he was appointed to a chair at King's in 1962. In 1969, he became Head of Department (at King's), a position he was to retain until his retirement in 1981. He proved to be a very fine Head of Department indeed, creating a lively department, with the pure mathematics group having a distinctive focus on algebra, geometry and number theory. He had a wonderful way of establishing a good relationship with incoming Vice-Chancellors. He would endeavour to have a major argument with them when they first tried to exact their will on him. In this way he would gain their respect, and a good relationship based on mutual respect would thereby be established. Being a Head of Department was considerably less stressful a task than it is today: he was able to find solace during meetings of the Academic Board by thinking about mathematics and, allegedly, writing up his ideas later that same evening. The success of his department was due in substantial measure to the energetic research programme that he maintained: he had numerous research students, and would put on special lecture courses for them; he also maintained a very active visitors' programme involving a number of German colleagues and visitors from the United States (especially from Urbana, Illinois), together with a regular stream of young mathematicians from the University of Bordeaux.

In 1965, together with J. W. S. Cassels, he organised a conference on behalf of the London Mathematical Society at the University of Sussex, with the aim of making the class-field theory more widely accessible to the number theory community. With hindsight, this can be seen as a watershed in the development of algebraic number theory in the twentieth century: centre stage was given to the new cohomological approach, and the requisite new ideas and methods were made available to the participants by numerous mini-lecture courses. The resulting book, now known affectionately as the 'Brighton proceedings', has been essential reading for aspiring algebraic number theorists for nearly forty years.

In the 1970s many of the strands of his earlier work were drawn together in a most spectacular manner, when he established a completely unexpected relationship between the Galois module structure of rings of algebraic integers and the so-called 'Artin constant' in the functional equation of certain Artin L-functions. This was a 'crazy idea' of Jean-Pierre Serre, building, in part, on previous work of Jacques Martinet and J. V. Armitage, and which he further described as "not a serious conjecture but rather wishful thinking. Trop beau pour être vrai." Ali's proof of this unlikely result was a sensation; invitations and honours followed in rapid succession. He was elected FRS in 1976, and he drew special satisfaction from the fact that he and his brother Herbert were one of very few pairs of siblings who were both Fellows of the Royal Society. That same year, he was awarded the London Mathematical Society's Senior Berwick Prize; the timing of this award was particularly gratifying, as the LMS awarded their other Senior prize that year (the Senior Whitehead Prize) to C. T. C. Wall, with whom Ali had collaborated very effectively over many years.

It is famously often said that mathematicians achieve their best work before the age of forty. G. H. Hardy asserts in his book *A mathematician's apology* that "mathematics is a young man's game . . . I do not know of a major mathematical advance initiated by a man past 50." Ali Fröhlich's work on the Galois module structure of rings of integers provides an excellent counter-example, and is an inspiration to the older mathematicians: he was 56 when he made his major breakthrough.

In the years that followed, his major breakthrough blossomed considerably. He would sometimes refer to this as the 'period of grace': it seemed that all his conjectures were being proved to be correct. The general picture became much clearer, and new directions relating to his earlier work on quadratic forms and hermitian forms opened up. His main conjecture asserted that the Galois module structure of rings of algebraic integers of tame extensions was determined by symplectic Artin root numbers; this was proved by the author in 1981. However, Ali noticed that the converse did not apply, and that there were certain families of Galois groups (such as quaternion groups of order 2^n with $n \geqslant 4$), where the rings of integers are always stably free over the group ring — regardless of the signs of the symplectic Artin root numbers. This led him to formulate his second main conjecture, which asserted that the quadratic Galois structure, afforded by rings of integers endowed with their trace form, should determine not only the tame symplectic Artin root numbers but even the local Langlands symplectic root numbers. This was proved by the author together with Philippe Cassou-Noguès in 1983.

His recognition by the British mathematical community was doubly welcome; indeed, until the mid-1970s, it had seemed that in some respects his work was best appreciated abroad — in France and Germany, and in the United States in particular.

His relationship with German mathematics and German culture are particularly noteworthy. It was typical of his warm and generous nature that, even though his family had suffered so terribly under Hitler, he retained a great fondness for German life and German culture. Initially after the war, for obvious reasons, he ceased to speak German, and when he first returned to Germany, he was initially reluctant to do so again. However, during a train journey with his wife Ruth, he became so incensed that no one would offer her a seat, that he exploded into a stream of extremely effective and well-chosen German, which immediately had the desired effect. Thereafter, he had absolutely no inhibitions in using his mother tongue. He regularly attended conferences in Oberwolfach, where he was very fond of both the institute and the locality. His love of things German manifested itself in many ways: from his contacts, exchanges and support for German mathematics, to his appreciation for German coffee houses and especially German 'Kuchen'. His favourite coffee house was in Schiltach (not far from Oberwolfach), where his capacity for eating 'Kuchen' and his mop of white hair are still fondly remembered by one of the employees. German mathematics reciprocated with a great appreciation of Ali. He received numerous invitations to visit German departments including Essen, Cologne, Augsburg, Karlsruhe, and Heidelberg. He received the Alexander von Humboldt Prize in 1992, he was the DFG Richard Merton Professor at Heidelberg in the summer of 1980, he was also a Visiting Gauss Professor in Göttingen (1983), and in 1982 he received the splendid accolade of being elected a corresponding member of the Heidelberg Academy of Sciences.

It was at the Oberwolfach meeting on Algebraic Number Theory in 1968 that he first met Jacques Martinet from Bordeaux. Martinet had made important progress on the Galois structure of rings of integers. He had shown that the rings of integers of tame extensions of the rationals with certain dihedral Galois groups, were all free-Galois modules; that is, they had a so-called 'normal integral basis'; by contrast, Martinet had shown that this was not necessarily the case when the Galois group was the quaternion group with eight elements. Ali was profoundly interested by his work. Not only would this ultimately lead Ali to his spectacular breakthrough, but this would also prove the beginning of a long and exceedingly fruitful collaboration with the University of Bordeaux. In the years that followed, there were numerous exchange visits by Ali and Martinet, and also by their many students. In time, Ali would declare that Bordeaux was, for him, a second home. Despite his great enthusiasm for visits to Bordeaux, he never quite made the effort to master the French language; he would buy books to teach himself, but when he tried it out, he often found that Hebrew came out instead. This led him to decide that mastering two languages was quite enough. There was, however, one notable exception to this decision: when he was awarded an honorary doctorate by the University of Bordeaux, he delivered his vote of thanks in impeccable French.

Ali also greatly enjoyed visiting colleagues in the United States, and especially at the universities of Arizona and Cornell. However, his closest work and collaboration was with Leon McCulloh, Irv Reiner and Steve Ullom from the University of Illinois at Urbana. Ali was appointed a George Miller Visiting Professor at Urbana for the academic year 1981–82. Irv Reiner used Ali's visit as a focal point for a year of tremendous sustained activity. The list of those who visited Urbana in the course of Ali's stay reads like the *Who's Who* of algebraic number theory. The visits of Ted Chinburg and Phil Kutzko were particularly fruitful, and led to important collaborations with Ali's group. Ali gave two courses during his stay there: the first was on a new approach to local class field theory, which he had discussed and developed jointly with H. W. Leopoldt during Leopoldt's visit to King's; the second concerned genus fields and descriptions of Galois groups by generators and relations; this was subsequently written up and published in [**89**]. The year in Urbana was a particularly fruitful and intense period of research activity for Ali — he could devote himself wholeheartedly to research, with no administrative duties whatsoever. A typical day would begin discussing mathematics at a local coffee shop called 'The Daily Grind'. There would then be further discussions on the blackboard when he got into the department. Researchers would then gather for lunch in Bevier Hall, where we all compared ideas. In the afternoon there were often seminars, and in the evening we would often relax, go shopping or play cards.

Ali's infectious love and enthusiasm for mathematics made him keen to discuss mathematics just about anywhere: on a walk, in a café, on a train, or in a taxi. He always seemed to have time for people and mathematics. By today's standards, he had a relatively relaxed pace of life; he prioritised his use of time, and mathematics was the absolute top priority. Discussing mathematics with him was an uplifting experience: not only was he invariably encouraging, but in addition, he had the great gift of making you feel confident that you could make a valuable contribution. He had an outstandingly fertile mathematical imagination, and generated far more ideas than he could hope to develop himself. At times it seemed that he had the mathematical equivalent of the gardeners' notion of 'green fingers', so that he could make progress on the most unlikely and unpromising of areas. He was exceptionally generous with his mathematical ideas, and this was undoubtedly a central reason why he was able to build up such a strong research school, and why he attracted so many visitors. Ali also possessed considerable social skills: he was a very warm and charming individual with tremendous wit and a great sense of humour. All these various personal qualities, together with his outstanding mathematical creativity, made him a quite exceptional research leader.

Ali retired from his chair at King's College, London, in 1981 and became an Emeritus Professor of the College. There was immediately a certain amount of competition by other leading institutions to benefit from his research expertise. He was elected to a Senior Research Fellowship at Imperial College, London, and to a Fellowship at Robinson College, Cambridge. He greatly enjoyed the new friendships at both institutions. He relished the stimulating mathematical environments at Imperial College and Cambridge; he and Ruth settled readily into the welcoming atmosphere of Robinson College, where he entered into the corporate life of the College with enthusiasm. He used these new positions to maintain his research, and he continued to produce mathematical papers up to the age of 82. With offices in Wimbledon, Imperial and Cambridge, he said that he found it quite difficult to keep on top of the paperwork: it always seemed to him that at any given time there would be a vital document in one of the other two offices.

His declining health, and in particular his severely impaired vision, increasingly cut him off from the world of mathematics for which he cared so much. Even so, he made every effort to keep abreast of developments. He was always keen to discuss trends which he felt would prosper and ideas which he felt had been neglected in the past but whose time he felt had now come. Only two weeks before his death he managed to attend a seminar on Fröhlich fibre bundles, delivered by the author at the new Cambridge Mathematics School.

Ali was a distinguished scholar and an internationally distinguished mathematician. Throughout his life, he was a family man. As a proud father, he endeavoured to support his children in their various activities. He would take his son, Shaun, for sailing lessons, encouraging a passion that grew and is with him still, and he derived great pleasure from playing piano duets with his daughter Sorrel. Ali loved attending conferences and visiting other departments of mathematics, but he was always happiest when he could arrange for Ruth to accompany him.

The celebration of his 80th birthday at Robinson College was a most impressive event, attended by numerous famous mathematicians. However, in the minds of many, the most memorable moment was the musical presentation made by his grandchildren at the reception that followed at Ruth and Ali's home in Barton Road. He was quintessentially a family man, and his devoted family cared for him and loved him right to the end.

2. *Work*

2.1. *Class groups of number fields.* [**1–8**, **15, 18**, **19, 29**, **30**]

Ali's earliest work concerns the development and interplay between two themes: firstly, the study of class groups as modules over Galois groups, and the development of the required representation-theoretic techniques; secondly, the study of Galois extensions of a given number field of nilpotency class two (that is to say, Galois groups with the property that the commutator subgroup is contained in the centre of the group). These two themes are closely related for the following reason: given an abelian extension K of the rationals with known ramification, then the knowledge of the maximal class-two extension of the rationals with prescribed ramification can be used to derive information about the maximal sub-field M of the Hilbert class field of K which has the property that $\mathrm{Gal}(M/Q)$ has class two. This fundamental observation can be used to obtain a considerable amount of information about the class group of K.

In later life, Ali became conscious that many of these early papers had become hard to read: the notation was not easy and, in particular, the style of class-field theory that he used had been replaced by idelic and cohomological techniques. He therefore decided to re-write much of this work — resulting in the beautiful little book [**89**].

His thesis is written as two distinct parts, and both parts were to influence the direction of his mathematical research for many years. The first part treats the representations of a finite group G into the group of automorphisms of a finite abelian group A, with the high point being a simple classification theory for such representations when the orders of G and A are coprime. This was later published in [**1**]. In the second part he goes on to apply the preceding work to the case where A is the class group of a ring of integers of a number field with Galois group G acting on the class group in the natural way. It would appear that this was the first time that Galois action on class groups was looked at in such a systematic way. Such an approach bore great fruit later in the hands of many others, including Golod, Shafarevich, and Iwasawa. One of the main results of the second part (later published in [**2**]) is a description of the class group of a number field N which is abelian over a sub-field K in terms of the class groups of cyclic sub-extensions of K in N.

In the papers [**3**] and [**6**], he develops ideas from his thesis to give, *inter alia*, characterisations of various maximal extensions, M, of the rationals with nilpotency class 2. The study of central extensions of number fields had really been initiated by A. Scholz, in the latter part of the 1930s — although there had previously been contributions from Rédei and Reichardt which had used methods of central extensions. Class field theory gives a powerful tool for describing the abelian extensions of a number field; Ali's idea, here, is to push class-field theoretic techniques to obtain information on central extensions of abelian extensions of the rationals. He describes the Galois groups of such maximal extensions M in terms of generators and relations which are derived in a pleasingly simple and elegant manner from ramification data allowed in M. It is interesting to note that, since Ali's work, there has been a considerable amount of further work in this area: the work of Demuskin, Serre and Shafarevich has been particularly influential. For a nice and readable account, see H. Koch's article in [**74**], and see also the concluding section of [**89**] entitled 'Some remarks on history and literature'.

In [**4**], [**7**] and [**8**], as explained previously, he was able to use the above work on class-two extensions to obtain strong divisibility information on the class numbers of number fields in terms of ramification data. His strongest results are obtained for an l-power cyclic extension of the rationals, and can be seen as building on earlier work of Rédei and others on the divisibility properties of class numbers of quadratic fields. A particularly striking result that he obtains is that the narrow class number of such an ℓ-power cyclic extension L is coprime to ℓ if and only if there is only one prime number which ramifies in L; this result is extended to more general base fields in [**15**].

In [**18**] and [**19**] he carried out a detailed analysis of the genus theory of various families of number fields including: absolutely abelian number fields, non-normal cubic number fields, number fields of the form $\mathbb{Q}(\sqrt[n]{m})$, and certain number fields whose Galois groups are symmetric groups.

2.2. *Field theory and constructive existence proofs* [**10**, **11**]

When Ali was at Keele he collaborated with Shepherdson, in Bristol, on questions posed by B. L. van der Waerden in his *Modern algebra* ⟨**1**, §42⟩, concerning 'field theoretical operations in a finite number of steps'; that is, the question as to whether the proofs of the existence of a splitting field for a polynomial over a field can be replaced by constructive existence proofs. In [**10**] and [**11**], the former being a résumé of the latter, they take up both the positive results, due to Kronecker, and a pioneering negative result, due to van der Waerden, and express them in terms of the notions of algorithm and computability and of the theory of recursive functions, in which they draw on and explain results of Church, Kleene and Turing (and others) in order to discuss the effective construction of field extensions.

Van der Waerden defined an 'explicitly given field' as one whose elements are uniquely represented by distinguishable symbols with which the basic operations can be performed in a finite number of steps. He had shown that there is no general algorithm applicable to all explicitly given fields K, for splitting polynomials in $K[x]$. In [**10**], Fröhlich and Shepherdson strengthened that result by exhibiting a particular given field K_0, for which there is no splitting algorithm. Their proof uses a result, due to Kleene, to the effect that if $\lambda(n)$ denotes a recursive function, defined for all positive integers n, such that $n \neq m$ implies that $\lambda(n) \neq \lambda(m)$, then the integers m satisfying $(\exists n)(\lambda(n) = m)$ form a non-recursive class. The argument, which is, in a sense, typical of those in these two papers, runs as follows. Denote by p_n the nth prime number, and write $\vartheta_n = \sqrt{p_{\lambda(n)}}$. Then the field $K_0 = \mathbb{Q}(\vartheta_1, \vartheta_2, \ldots)$ has an explicit representation. Now $x^2 - p_m$ is reducible in K_0 if and only if $(\exists n)(\lambda(n) = m)$, but there is no recursive algorithm for deciding whether that holds for an arbitrarily given m, and so there can be no recursive algorithm for deciding whether an element of $K_0[x]$ is reducible or not, and hence no splitting algorithm.

They also give explicit constructions of fields $K, \bar{K}$ such that $\bar{K}$ is a simple, non-separable extension of K and such that K has a splitting algorithm but $\bar{K}$ has not. The proof of that is tedious (to quote the authors), and is completed (as with other claims in [**10**]) in [**11**]. They also show that that example leads to an ingenious paradox, for the field $\bar{K}$ that has no splitting algorithm turns out, as an abstract field, to be isomorphic to a field K' obtained by adjoining $\aleph_0$ independent transcendentals — but surely that is an explicit field with a splitting algorithm. The resolution of the paradox shows that the properties of those fields depend not only on the field as an abstract field, but also on the particular explicit representation.

The paper [**10**] concludes with a discussion of ideas of Krull on splitting algorithms in a 'narrow' and in an 'extended' sense, and whether or not bounds can be given for the number of steps required in the process. They argue that the distinction is irrelevant for the purpose of 'high-speed computing machines', or perhaps arises from an intuitionist objection to the law of the excluded middle. The reviewer in *Mathematical Reviews* ⟨**2**⟩ argues that Krull wasn't concerned with such examples.

The longer paper [**11**] not only gives detailed proofs completing the work in [**10**], but also affords a fascinating and attractive introduction to the general ideas and the notion of explicit rings and extension rings, splitting algorithms and the concepts of explicit isomorphism and homomorphism. That leads to a classification of the types of extension fields that can be effectively constructed. The papers continue to make attractive reading.

2.3. *Algebraic period* (*non-abelian homology and near-rings*) [**12–14, 23**, **24, 26**–**28**]

Between the years 1958 and 1962 Ali published eight papers developing the theory of distributively generated (d.g.) near-rings, and seeking to use this theory to lay the foundations for a non-abelian homological algebra. Dickson, in 1905, and Zassenhaus, in 1935, had studied in some depth, near-fields (fields lacking one distributive law) which had arisen in a geometrical setting. Shortly before, in 1954, H. Neumann had hoped to use near-rings to solve a problem in the theory of varieties of groups, but without success. More generally, there were the beginnings of a general theory of near-rings (rings lacking commutativity of addition and one distributive law). For example, there was the work of Blackett, in 1950, and then later the work of Betsch, a pupil of Wielandt, who had been interested in near-rings from the 1930s, but who had chosen to concentrate rather on subnormal subgroups. Ali's papers were the first systematic study, and the only one dealing with d.g. near-rings (near-rings generated as additive groups by a multiplicative semigroup of distributive elements).

While near-rings in general act on groups as mappings, distributively generated near-rings act on groups as the mappings generated additively by the endomorphisms of the group. Ali concentrated on this aspect, and this theme is developed in the first three papers: in [**12**] and [**13**], he set the foundations of the theory of d.g. near-rings and their representations on groups; [**14**] applies the theory to simple non-abelian groups. In [**23**] and [**24**] he studied d.g. near-rings in their categorical setting. This work was the basis for the three papers developing a non-abelian homological algebra [**26–28**].

These papers showed great insight, and provided the springboard for the flourishing of a major strand of the theory of near-rings, one that is most closely related to the theory of groups. Ali had a major influence in the early development of d.g. near-rings, and indeed was a contributor at the inaugural Oberwolfach meeting on near-rings, held in 1968. His research student R. R. Laxton was for many years a leading light in the circle of near-ring theorists. A close friend of Ali's, J. R. Clay, who was one of the most influential people working in this area, often paid tribute to the inspiration given him by Ali. Although he never returned to near-rings, his contributions to, and influence on, the subject were immense.

2.4. *Discriminants* [**20–22**, **31**, **33, 38**, **39, 67**, **68**]

Ali's interest in discriminants began with the papers [**20**] and [**22**], where he used idelic methods to define a fine discriminant, which contained more information than the usual discriminant ideal. He used this to obtain a very elegant criterion for the existence of relative bases of rings of integers, which has a neater feel than the earlier criterion due to Artin. Thus, for an extension L/K of number fields, the ring of integers O_L is free over O_K if and only if the fine discriminant is principal. He also showed that the class of the fine discriminant in the class group of fractional O_K ideals is always a square — a result which is closely related to Hecke's theorem that the class of the different of L/K is always a square in the class group of O_L. In [**21**] he developed the techniques of these papers to study the O_K module structure of arbitrary ideals of L. In [**33**] Ali, in joint work with J.-P. Serre and J. Tate, gave an example of a different of an extension of function fields whose different has class which is not a square. This body of work seems to have marked the start of Ali's interest in parity questions. He would go on to be interested in whether conductors of real-valued characters were squares; this in turn led to questions about the signs of Artin root numbers — an issue that lay right at the heart of his work on Galois modules.

He realised that, for Galois extensions, his discriminant techniques could be refined over the different irreducible characters of the Galois group; this then gave rise to his theory of resolvents and module conductors, which was developed in the papers [**33, 38**, **39, 67**, **68**]. His notion of a resolvent (a generalisation of the classical Lagrange resolvent) was quite crucial because it turned out to be exactly the right tool for calculating the classes of arithmetic modules in class groups of group rings (see Section 2.6 below). Similarly, they turned out also to be Pfaffians of trace forms in his Hermitian class group, and so played an essential role here too.

2.5. *Quadratic forms* [**41, 45**, **46**, **49**, **51**, **53**, **73**, **84**, **86**, **95**]

Ali's early work on quadratic and Hermitian theory concerned Grothendieck groups and Witt groups for quadratic and Hermitian modules over rings with a group action.

The papers [**45**] and [**46**] were both written jointly with his former student A. M. McEvett. [**45**] is a foundational study of quadratic and Hermitian theory for non-commutative rings. It deals with both their Morita theory and also their associated Grothendieck and Witt groups. These ideas are then applied to representations of groups and a detailed study of their associated Grothendieck and Witt groups for group rings in [**46**]. Ali then proceeded to the more arithmetic aspects of this approach in [**51**]: here he studies the Grothendieck groups of quadratic forms over rings of integers and number fields. In particular, he obtains complete descriptions of the kernel and image of the extension map from the ring of integers to its field of fractions. These really are most striking and beautiful results.

In [**53**] he considers various Grothendieck rings of orthogonal and symplectic representations. The use here of Clifford algebras with a group action was to prove a very useful tool in his later joint work with Terry Wall on equivariant Brauer groups.

Although his involvement with this topic was relatively brief, he left an enduring legacy of extremely useful techniques, which are particularly well adapted to use in calculations. In addition to its intrinsic value, this early quadratic work paved the way for two major further developments: firstly, his work on Hermitian class groups, and secondly his work on what are now frequently called 'Fröhlich-twists of quadratic forms'. The latter work is in [**95**]. This built on Serre's seminal work relating Hasse–Witt invariants of trace forms to Stiefel–Whitney invariants. Ali showed how such formulae could be obtained for a wider class of forms obtained by twisting Galois invariant forms over the base by the trace form of an extension. This was a splendid piece of work, which gave Ali great pleasure and which even now continues to bear fruit. The author gave a seminar in Cambridge on arithmetic-geometric developments of Ali's work, at which Ali was present, only two weeks before he died.

2.6. *Class groups of group rings* [**37**, **40**, **50**, **55**, **57**, **59, 63**, **64**, **72**, **104**]

In his thesis, Ali had understood the importance of classifying the ways in which a finite group can act on an arithmetic modules — such as a class group or a group of units. In [**37**], partly encouraged by Olga Taussky-Todd, he began to develop methods for dealing with non-projective modules over an order. These methods were somewhat ahead of their time (at least from the arithmetic point of view). They lay comparatively dormant until the 1980s, when Ali initiated a major study of rings of integers of wildly ramified extensions, so that the ring of integers is not projective over the group ring; the results of this paper then led to what became known as his *factorisability theory*, a topic that was further developed by two of his research students, Adrian Nelson and David Burns.

Returning to the case of projective modules, the papers [**50**] and [**55**] signalled the start of a long period of study of class groups of group rings. Such class groups classify the projective modules over group rings, in the same way that the class group of a number field classifies the isomorphism classes of the ideals of a ring of integers. In these two papers he studied such class groups for finite abelian groups, with a strong emphasis on abelian p-groups. One of the main results was to give formulae for the asymptotic growth of such class groups. The papers [**57**] and [**63**] represent some of his first attempts to deal with non-abelian groups. The latter paper was written jointly with Irv Reiner and Steve Ullom, with whom he frequently collaborated on class groups of group rings. The account of Ali's work on class groups and group rings in ⟨**3**⟩ did much to make his work on this topic more accessible to a wider mathematical audience. One of his greatest contributions in this area was his introduction of idelic methods in [**64**]. This, together with Eichler's strong approximation theorem, took our understanding of such class groups to a new level and, in particular, paved the way for the calculation of class groups of a number of important families of class groups — such as the calculations for 2-power quaternion and dihedral class groups in [**60**] (with his two previous students M. Keating and S. M. J. Wilson). This new approach assumed its final form in the appendix of the seminal paper [**72**], where he recast his idelic method into a dual form, so that classes were now represented by idele-valued character functions. This approach became known informally as the *Fröhlich–Hom description*; it was more powerful than earlier descriptions of class groups of group rings (for example, as found in the work of Jacobinski) because of its excellent functorial properties. It provided an extremely powerful method, both for the calculation of class groups and also for working with the classes of specific modules. This presentation of the class group in terms of character functions gave substance to the idea, which he had been aware of for some time, that arithmetic classes were represented by character functions, such as Artin root numbers or certain L-function values.

2.7. *Galois module theory, root numbers, parity problems* [**32**, **33**, **54**, **56**, **58**, **60**, **62, 65**, **66, 69**, **70**, **72, 75–78, 80**, **81**, **83, 98**, **99, 102**, **103, 106–109**]

Each generation produces a number of mathematicians who make striking contributions and breakthroughs. Ali was one of the very few of his generation to open up a new subject area: namely arithmetic Galois module theory of rings of integers. Approximately one quarter of his publications deal directly with this topic, and his work on class groups, discriminants and Hermitian theory all played a vital supporting role.

The subject of the Galois module structure of algebraic integers has its roots in Satz 132 of Hilbert's *Zahlbericht*, which establishes the existence of a normal basis for the ring of integers of an absolutely abelian number field that has the property that the prime divisors of the degree of the extension are all unramified in the extension. Further progress was then made when Emmy Noether essentially showed that, for a given Galois extension N/K of p-adic fields, the local ring of integers O_N has a normal integral basis if and only if the extension N/K is at most tamely ramified. The subject then really started to take off with Leopoldt's complete description in 1959 of the Galois structure of rings of integers of absolutely abelian number fields.

Ali's first paper on the topic would seem to be [**32**], which was published in 1962, and which came hot on the heels of Leopoldt's big success. With hindsight, the paper can be seen to have contained a wealth of ideas that would come to fruition at various different times. One of the principal features of this paper is the work on Kummer orders, where the operating ring is a maximal order. This work would play an increasingly important role in Galois module theory when the importance of Hopf algebras became more fully appreciated. He also gave a criterion for the existence of a normal integral basis in terms of class group invariants. The techniques that he developed here can be seen to have laid the basis for his whole approach to class groups of group rings and the calculation of classes of arithmetic modules.

In [**56**] in joint work with J. Queyrut, he proved a conjecture of Serre that asserted that the Artin root numbers of orthogonal Galois representations are always +1, so that the associated Artin L-functions are always symmetric about the critical point $s = 1/2$. It was therefore very natural to ask whether symplectic Galois representations could yield Artin root numbers which were −1, and so give interesting zeros of Artin L-functions at $s = 1/2$. Work of J. V. Armitage showed that there are indeed such symplectic representations with negative root numbers. Following a suggestion of Serre, and also stimulated by work of Martinet and Armitage, in [**54**] Ali showed that the ring of integers of a tame Galois extension of the rationals with Galois group H_8 is free over the group ring $\mathbb{Z}[H_8]$ precisely when the Artin root number of the irreducible non-linear representation of the Galois group (which is of course symplectic) is +1.

The paper [**54**] was undoubtedly the high point of his mathematical life: it related the algebraic Galois structure of rings of integers to an analytic invariant in a totally new and sensational way. This thrust him and his subject to the fore on the world stage (see, in particular, his ICM talk [**70**]). He was awarded the LMS Senior Berwick prize in 1976 for his paper [**58**]. Following the appearance of [**56**] and [**58**], a general picture began to emerge. Various special families of Galois groups were considered: quaternion groups in [**60**], and generalised dihedral groups in [**69**].

These early examples served to demonstrate a crucial underlying principle: it was important first to study the congruence properties of group ring determinants, and then secondly to try and establish the corresponding property for Gauss sums. At this point, Ali sensed the importance of the use of Adams operations. He encouraged the author to use them on the class groups of group rings of p-groups, and this led to the development of the so-called *group logarithm*, which was to provide a fundamental tool in dealing with the general conjecture.

Simultaneously, Ali was also developing general methods: class groups were developed as described in Section 2.6 above; the corresponding tools for determining arithmetic classes in these class groups were dealt with in [**37**], [**39**], [**40**] and [**61**]. The culmination of this multi-pronged attack on the Galois structure of rings of integers came to full fruition in the famous *Fröhlich conjecture*, which, in its simplest form, asserts that the Galois structure of tame rings of integers is determined by the signs of the symplectic Artin root numbers. The precise formulation of quite how the symplectic root numbers determine the group ring class of the ring of integers used an important idea due to Cassou-Noguès. This conjecture was underpinned by the seminal paper [**72**], which drew together numerous different threads of his research, into what is surely the magnum opus of his research career. The conjecture was proved by the author in 1981.

Ali then began a second phase of Galois module theory: namely, Hermitian (or quadratic) Galois structure, where one considers rings of integers endowed with their trace form. Here again, he found himself in the position of having a number of key-ideas and techniques ready for immediate use from his earlier work on quadratic forms with a group action. He again employed a similar strategy to the one that he had used so successfully for standard Galois module theory: on the algebraic side, he developed a new and powerful theory of Hermitian class groups which classified locally free modules over a group ring which support a group invariant form (see especially [**84**]); the vital new ingredient which he used to classify such quadratic structures was his new version of the Pfaffian, which for trace forms is very closely related to his generalised Lagrange resolvents. On the arithmetic side, his work led to his formulation of the *second Fröhlich conjecture* (later proved by the author and Cassou-Noguès) which asserted that both global and local tame Artin root numbers could be determined by the Hermitian–Galois structure of rings of integers when endowed with their trace form. The fundamental work here is contained in [**73**] and [**77**]. The book [**90**] contains a definitive account of both the algebraic and arithmetic aspects of his Hermitian class group theory.

It is interesting to note that the above two phases of his work have given rise to interesting arithmetic-geometric developments, and, in particular, T. Chinburg, G. Pappas and the author have established versions of both Fröhlich conjectures in the context of higher-dimensional arithmetic varieties.

The third phase of his interest in Galois module theory concerned the Galois structure of algebraic integers where there is wild ramification. Whereas in the tame case, by a result of Emmy Noether, one knows that the ring of algebraic integers is a projective module over the group ring, it is not at all clear what limitations or constraints there are on the Galois structure of integers of wildly ramified extensions. He obtained a fundamental insight into this hard problem by returning to ideas in [**37**] and developing the notion of *factorisable modules*: essentially modules which enjoy certain relations on distinguished sub-modules (see [**98**], [**102**], [**108**] and [**109**]). He felt that the importance of this emerging wild theory was not fully appreciated by the mathematical community, and he would often say that this was the area that he would introduce a really bright young student to! He was acutely aware that many of his ideas and techniques would be applicable to Galois modules other than rings of integers: indeed, his thesis had dealt with the Galois module structure of ideal class groups.

His proposed programme of studying multiplicative Galois modules, such as units and class groups, was launched in [**103**] and [**106**]. There is particularly well-written account of Ali's wild theory and multiplicative theory in [**107**]. The interplays between additive and multiplicative Galois module theory, and Stark's conjecture had become apparent through the work of Ted Chinburg, who had visited Ali in Urbana in 1982, and then visited both the author and Ali in Cambridge in 1983. These aspects of the subject are currently developing at a great pace in the context of equivariant Iwasawa theory. Particularly notable is the contribution by J. Ritter and A. Weiss, with their *lifted root number conjecture*, which was in fact initiated by suggestions and questions from Ali himself. The recent work of Ali's former student D. Burns (with some of his work being joint with M. Flach and some with C. Greither) is really very striking indeed, as it relates Galois module structure issues to equivariant Bloch–Kato conjectures.

2.8. *Gauss sums and Langlands theory* [**79**, **82**, **88, 91–94, 96**, **97**, **100**]

Ali was fascinated by Gauss sums and their ubiquity throughout mathematics. His tame Galois module theory had depended on the crucial relationship between Galois Gauss sums and his generalised Lagrange resolvents. This was a truly remarkable relationship: whilst his resolvents had a relatively straightforward definition, in terms of determinants, by contrast, the Gauss sums (or, equivalently, the epsilon constants) of representations of local Galois or Weil groups have a less direct definition — through inductivity (on virtual representations of degree zero) and explicit formulae for one-dimensional representations.

The paper [**82**] (written jointly with the author) was inspired by the striking similarities between Gauss sums and resolvents, and it provided a new characterisation of tame local Galois Gauss sums in terms of the arithmetic of the local field in question.

Apart from [**82**], the main papers here concern the problem of understanding Gauss sums within the context of the Langlands programme — when such matters were almost entirely conjectural. The key point is that, under the disguise of the 'local constant', the Langlands programme identifies a Galois Gauss sum (of an n-dimensional irreducible representation of the Galois group of a local field F) with a number attached to an irreducible supercuspidal representation of the general linear group $\mathrm{GL}(n, F)$. A variant of the programme deals with irreducible representations of $\mathrm{GL}(1, D)$, where D is a central F-division algebra of dimension n^2.

In the case of $\mathrm{GL}(1, D)$, it is easy to define an explicit Gauss sum attached to an irreducible representation — just by imitating the classical formula for $\mathrm{GL}(1, F)$; and this Gauss sum is related to the local constant in an easy way. Unfortunately, at that time, there was no Langlands correspondence for $\mathrm{GL}(1, D)$. However, by comparing the two sorts of Gauss sums, in [**88**] he and his former student Colin Bushnell produced a candidate for the Langlands correspondence for $\mathrm{GL}(1, D)$ when n is not divisible by the residual characteristic of the valuation ring of F. Colin Bushnell informs me that the correspondence proposed in [**88**] is very nearly, but not exactly, right, and that the proof for the amended correspondence (which is a substantial result in its own right) is unlikely to appear for some time. The notion of Gauss sum for $\mathrm{GL}(n, F)$ is much more subtle, and gave rise to one of Ali's great ideas in [**94**]. The normaliser K of a principal order A in the matrix ring $M(n, F)$ is a maximal compact mod centre subgroup of $G = \mathrm{GL}(n, F)$. It has many structural features common to the case $n = 1$ (when $K = F^{\times}$). By means of an explicit formula, one can attach a Gauss sum to an irreducible representation ρ of K. This Gauss sum can be zero, and ρ is called *non-degenerate* if its Gauss sum is non-zero. Ali and Bushnell showed that, if an irreducible supercuspidal representation π of G contains a non-degenerate representation ρ of K, then there is a formula connecting the local constant of π and the Gauss sum of ρ, thereby generalising part of Tate's thesis in the case $n = 1$. (Note also that this works equally well for $\mathrm{GL}(m, D)$.)

This beautiful result has been developed in two ways. Bushnell and his collaborators have used it as the starting-point for their investigation of the structure of representations of G, culminating in the Bushnell–Kutzko classification theory. Twenty years on, this technique is still the only good way of computing local constants for $\mathrm{GL}(n)$!

Ali took matters on in a somewhat different way, concentrating on his favourite tamely ramified representations in the papers [**93**, **97**, **100**]. Here he developed a substantial theory of Gauss sums for tame representations of 'chain groups' K, which is in many ways parallel to Green's treatment of finite general linear groups. The connection with representations of G is a bit loose in general; there is, however, a good connection for cuspidal representations.

2.9. *Collaboration with C. T. C. Wall* [**48**, **52**, **61**, **110**]

Ali and C. T. C. Wall worked together on different projects over a period of more than thirty years. Their first paper [**48**] used the tool of graded categories to extend some of the then recent K-theoretic work of H. Bass to the equivariant setting. Thereafter they would meet regularly in London, with Terry Wall often being invited back to Ali's home in Wimbledon. They used these meetings to sketch out a considerable programme of work that could be developed, building on their separate interests. The second paper [**52**] was written-up by Ali, sketching out the number-theoretic applications that he envisaged. The paper deals with the equivariant Brauer group associated to a ring with a group action, and develops techniques for their calculation. Although they both made considerable progress with their common problems, they failed to agree on the best way to write this up. They had rather contrasting mathematical styles, with Ali favouring lengthy detailed verifications of cocycle-type identities, which Wall preferred to leave to the reader. The paper [**61**] on graded monoidal categories was essentially written by Wall. The core of the paper constructed the cohomological theory for graded categories, and was conceived as groundwork for a paper to follow, containing the details and applications. However, by the mid 1970s their interests started to drift apart, with Wall moving increasingly into singularity theory, while Ali's work on Galois module structure was really taking off. In consequence, their efforts at writing-up the sequel gradually petered out.

The final paper [**110**] establishes some fundamental exact sequences involving equivariant Brauer groups. It was the last paper that Ali ever published, and was written when Wall realised that the conference proceedings in question offered an ideal last chance to place on record the progress that they had made together. It was highly satisfying to both authors to have this opportunity to rescue their ideas from total oblivion.

Terry Wall records that their collaboration was based on more than merely a shared mathematical interest: despite friendly disagreements on style, they got on very well together. Ali always seemed to have time to talk (universities thirty years ago were less pressured than today!); their mathematical discussions were constructive on both sides, and their informal discussions also were most congenial.

2.10. *Books*

(a) *Algebraic number theory* ('The Brighton proceedings') [**42**]. This book was a watershed in algebraic number theory. As a consequence of the 1965 Brighton Conference, and this resulting proceedings volume, class-field theory became widely available and accessible as a tool. The book builds on the work of Chevalley, Artin–Tate and others, to present a systematic treatment of class-field theory in terms of ideles and cohomology, together with all the requisite foundational material. The courses were delivered by leading experts who had reputations for outstanding communication skills. The book also rendered a very great service to the mathematical community by publishing Tate's thesis — a piece of work which was to become extremely influential in the years that followed. Finally, it is amusing to note that Ian Cassels put a reference to Joseph Stalin in the author index at the back of the book. However, when the book was translated into Russian, to the delight of Ali and Ian Cassels they found that someone had very carefully censored out this reference!

(b) *Central extensions, Galois groups, and ideal class groups of number fields* [**89**]. Ali had been unhappy for some years that his early work (see Section 2.1 above) had not achieved the full recognition that he felt it deserved. He was fully aware that his early papers had become really quite difficult to read. He describes the situation very well in the preface to this book. In consequence, when Ali was a George Miller Professor for the academic year 1981/2 at the University of Illinois at Urbana, he seized this opportunity to present this body of work in a much more accessible and modern setting. It was a real success, and he was greatly pleased by the ease of presentation that these new methods and techniques afforded him. The book contains only 86 pages; it is fast-moving, and is a very elegant and comfortable read.

(c) *Algebraic number theory* (with the author) [**105**]. Whilst the author had his own agenda of wishing to write an algebraic number theory text that contained many examples and which would encourage students to do many examples and calculate, Ali, on the other hand, wanted to use this book as a vehicle to pass on many of the insights and instructive examples that he had learned over many years of presenting courses on this topic. The marriage between these two approaches was an exceedingly happy one: both authors greatly enjoyed writing the book; it was also something of a commercial success, with many graduate courses adopting it as the course book. For some years, this book was, I believe, the second-best seller in the Cambridge University Press Pure Mathematics list — a little behind Cassels' book on elliptic curves.

(d) *Algebraic number fields* ('The Durham proceedings') [**74**]. This book is the proceedings of the 1975 Durham Symposium, which Ali organised. To many participants, this meeting very much had the feel of a sequel to the 'Brighton proceedings'. As usual, Ali exhibited impeccable taste in the choice of material. He was particularly pleased by the success of H. Stark's account of his new conjectures. Ali was rather rushed in the delivery of his first talk, and thereby reduced the impact of his subsequent talks on his rapidly evolving body of work, based around his recent spectacular discoveries. By contrast, though, the write-up of his course of talks provides an excellent introduction to this material.

(e) *Formal groups* [**47**]. This book is based on a lecture course that Ali delivered on this topic at King's College London in 1966/7, and is based on notes provided by his former student Abe Lue. Here again, Ali showed his unerring eye for spotting a winner: this book is a very readable account, which came out just when this topic was taking off; in consequence, for many, this became the standard text on the subject. Ali was particularly pleased with his account of Kummer theory for formal groups, which later became important in work of Coates and Wiles, and also in the study of generalised explicit reciprocity laws.

(f) *Galois module structure of algebraic integers* [**87**]. Towards the end of the 1970s, Ali and his school had produced numerous results contained in very many different papers. It was therefore apparent that there was a need for a definitive account of the subject which would present the material in a unified and coherent manner. This book certainly met that need. However, with hindsight, it can be seen to have adopted an excessively encyclopaedic style, with the result that, whilst the book was excellent for workers in the area, it did little to make the subject more accessible to outsiders.

(g) *Classgroups and Hermitian modules* [**90**]. This book can be seen as being derived from three earlier themes in Ali's work: his work on 'fine' discriminants, his work on quadratic forms, and his work on arithmetic Galois modules. As is explained in Section 2.5 above, following his spectacular success with the Galois module structure of rings of integers, he very quickly realised that the strongest results were to be obtained by considering rings of algebraic integers when endowed with the Galois-invariant quadratic form given by the trace. He developed a new use of Pfaffians to form very fine equivariant discriminants; these new invariants provided a key new measure as to how non-trivial a quadratic structure is. Ali was very excited to discover that these new invariants were essentially the same thing as some of his generalised Lagrange resolvents. This book contains a wealth of original ideas and new techniques, and it also provided all the requisite infrastructure necessary for the second Fröhlich conjecture.

Acknowledgements. The author wishes gratefully to acknowledge help and input from Bryan Birch, Colin Bushnell, Philippe Cassou-Noguès, Ruth Fröhlich, Leon McCulloh, John Meldrum, Jürgen Ritter, Steve Ullom, Terry Wall, and Ali's daughter Sorrel Young. A very special mention goes to Vernon Armitage for all his help and support in this most challenging of tasks.

References

⟨**1**⟩ B. L. van der Waerden, *Modern algebra*, revised English edn, Vol. 1 (Frederic Ungar Publishing Co., New York, 1949) 134–137.
⟨**2**⟩ G. Kreisel, *Math. Rev.* 17 (1954) 119.
⟨**3**⟩ C. W. Curtis and I. Reiner, *Methods of representation theory with applications to finite groups and orders*, vols I and II (Wiley, New York, 1990 and 1994).

Publications of Albrecht Fröhlich

1. 'The representation of a finite group as a group of automorphisms on a finite Abelian group', *Quart. J. Math. Oxford* (2) 1 (1950) 270–283.
2. 'On the class group of relatively Abelian fields', *Quart. J. Math. Oxford* (2) 3 (1952) 98–106.
3. 'On fields of class two', *Proc. London Math. Soc.* (3) 4 (1954) 235–256.
4. 'On the absolute class group of Abelian fields', *J. London Math. Soc.* 29 (1954) 211–217.
5. 'A note on the class field tower', *Quart. J. Math. Oxford* (2) 5 (1954) 141–144.
6. 'The generalization of a theorem of L. Rédei's', *Quart. J. Math. Oxford* (2) 5 (1954) 130–140.
7. 'A remark on the class number of Abelian fields', *J. London Math. Soc.* 29 (1954) 498.
8. 'On the absolute class group of Abelian fields II'., *J. London Math. Soc.* 30 (1955) 72–80.
9. 'Non Abelian laws of prime decomposition', *Proc. Int. Math. Congress* (1954).
10. (with J. C. Shepherdson) 'On the factorisation of polynomials in a finite number of steps', *Math. Z.* 62 (1955) 331–334.
11. (with J. C. Shepherdson) 'Effective procedures in field theory', *Phil. Trans. Royal Soc. London, A* 248 (1956) 407–432.

12. 'Distributively generated near-rings (I. Ideal theory)', *Proc. London Math. Soc.* (3) 8 (1958) 76–94.
13. 'Distributively generated near-rings (II. Representation theory)', *Proc. London Math. Soc.* (3) 8 (1958) 95–108.
14. 'The near ring generated by the inner automorphisms of a finite simple group', *J. London Math. Soc.* 33 (1958) 95–107.
15. 'On a method for the determination of class factors in number fields', *Mathematika* 4 (1957) 113–131.
16. 'The restricted biquadratic residue symbol', *Proc. London Math. Soc.* (3) 9 (1959) 189–207.
17. 'The rational characterisation of certain sets of relatively Abelian extensions', *Philos. Trans. Royal Soc. London* A (1959) 385–425.
18. 'The genus field and the genus group in finite number fields', *Mathematika* 6 (1959) 40–45.
19. 'The genus field and the genus group in finite number fields II', *Mathematika* 6 (1959) 142–146.
20. 'Discriminants of algebraic number fields', *Math. Z.* 74 (1960) 1–28.
21. 'Ideals in an extension field as modules over the algebraic integers in a number field', *Math. Z.* 74 (1960) 29–38.
22. 'The discriminants of relative extensions and the existence of integral bases', *Mathematika* 7 (1960) 15–22.
23. 'On groups over a d.g. near-ring (I). Sum constructs and free R-groups', *Quart. J. Math. Oxford* (2) 11 (1960) 193–210.
24. 'On groups over a d.g. near-ring (II). Categories and functors', *Quart. J. Math. Oxford* (2) 11 (1960) 229–228.
25. 'A prime decomposition symbol for certain non-Abelian number fields', *Acta Sci. Math.* 21 (1960) 229–246.
26. 'Non-Abelian homological algebra, I. Derived functors and satellites', *Proc. London Math. Soc.* (3) (1961) 239–275.
27. 'Non-Abelian homological algebra, II. Varieties,' *Proc. London Math. Soc.* (3) 12 (1962) 1–28.
28. 'Non-Abelian homological algebra, III. The functors EXT and TOR', *Proc. London Math. Soc.* (3) 12 (1962) 739–768.
29. 'On the ℓ-class group of the field $P(m^{1/l})$', *J. London Math. Soc.* 37 (1962) 189–192.
30. 'A remark on the class field tower of the field $P(m^{1/l})$', *J. London Math. Soc.* 37 (1962) 193–194.
31. 'Discriminants and module invariants over a Dedekind domain', *Mathematika* 8 (1961) 170–172.
32. 'The module structure of Kummer extensions over Dedekind domains', *J. Reine Angew. Math.* 209 (1962) 39–53.
33. (with J. P. Serre and J. Tate) 'A different with an odd class', *J. Reine Angew. Math.* 209 (1962) 6–7.
34. 'On non-ramified extensions with prescribed Galois group', *Mathematika* 9 (1962) 133–134.
35. 'Baer invariants of algebras', *Trans. Amer. Math. Soc.* 109 (1963) 221–244.
36. 'Galois algebras,' *J. Reine Angew. Math.* 216 (1964) 1–11.
37. 'Invariants for modules over commutative separable orders', *Quart. J. Math. Oxford* (2) 16 (1965) 193–232.
38. 'Some topics in the theory of module conductors', *Oberwolfach Berichte* 2 (1966) 59–83.
39. 'Resolvents, discriminants, and trace invariants', *J. Algebra* 4 (1966) 543–662.
40. 'Radical modules over a Dedekind domain', *Nagoya Math J.* 27 (1966) 173–198.
41. 'Quadratic forms "à la" local theory', *Math. Proc. Camb. Philos. Soc.* 63 (1967) 579–586.
42. (with J. W. S. Cassels (joint editors)) *Algebraic number theory*, proceedings of the Brighton Conference of 1965 (Academic Press, 1967).
43. 'Local fields', [**42**] 1–41.
44. (with J. V. Armitage) 'Class numbers and unit signatures', *Mathematika* 14 (1967) 94–98.
45. (with A. M. McEvett) 'Forms over rings with involution', *J. Algebra* 12 (1969) 79–104.
46. (with A. M. McEvett) 'The representation of groups by automorphisms of forms', *J. Algebra* 12 (1969) 114–133.
47. *Formal groups*, Lecture Notes in Math. 74 (Springer, 1958) 1–140.
48. (with C. T. C. Wall) *Foundations of equivariant algebraic K-theory*, Proceedings of Hull Conference, Lecture Notes in Math. 108 (Springer, 1969) 12–27.
49. 'Hermitian and quadratic forms over rings with involution', *Quart. J. Math. Oxford* (2) 20 (1969) 297–317.
50. 'On the classgroup of integral group rings of finite Abelian groups', *Mathematika* 16 (1969) 143–152.

51. 'On the K-theory of unimodular forms over rings of algebraic integers', *Quart. J. Math. Oxford* (2) 22 (1971) 401–423.
52. (with C. T. C. Wall) 'Equivariant Brauer groups in algebraic number theory', *Bull. Soc. Math. France Mém.* 25 (1971) 91–96.
53. 'Orthogonal and symplectic representations of groups', *Proc. London Math. Soc.* (3) 24 (1972) 470–506.
54. 'Artin root numbers and normal integral bases for quaternion fields', *Invent. Math.* 17 (1972) 143–166.
55. 'On the classgroup of integral group rings of finite Abelian groups II', *Mathematika* 19 (1972) 51–56.
56. (with J. Queyrut) 'On the functional equation of the Artin L-function for characters of real representations', *Invent. Math.* 20 (1973) 125–138.
57. 'The Picard group of non-commutative rings, in particular of orders', *Trans. Amer. Math. Soc.* 180 (1973) 1–45.
58. 'Artin root numbers, conductors and representations for generalised quaternion groups', *Proc. London Math. Soc.* (3) 28 (1974) 402–438.
59. (with M. Keating and S. Wilson) 'The classgroup of quaternion and dihedral 2-groups', *Mathematika* 21 (1974) 64–71.
60. 'Module invariants and root numbers for quaternion fields of degree $4l^r$', *Math. Proc. Cambr. Phil. Soc.* 76 (1974) 393–399.
61. (with C. T. C. Wall) 'Graded monoidal categories', *Compositio Math.* 28 (1974) 229–285.
62. 'The Galois module structure of algebraic integer rings in fields with generalised quaternion group', *Bull. Soc. Math. France, Mémoire* 37 (1974) 81–85.
63. (with I. Reiner and S. Ullom) 'Class groups and Picard groups of orders', *Proc. London Math. Soc.* (3) 29 (1974) 405–434.
64. 'Locally free modules over arithmetic orders', *J. Reine Angew. Math.* 274/75 (1975) 112–138.
65. 'Galois module structure and Artin L-functions', *Astérisque* 24–25 (1975) 9–13.
66. 'Artin root numbers for quaternion characters', *Symposia Math.* 15 (1975) 353–363.
67. 'Resolvents and trace form', *Math. Proc. Cambridge Philos. Soc.* 78 (1975) 185–210.
68. 'Module conductors and module resolvents', *Proc. London Math. Soc.* (3) 32 (1976) 279–321.
69. 'A normal integral basis theorem', *J. Algebra* 39 (1976) 131–137.
70. 'Galois module structure and Artin L-functions', *Proc. Int. Congress 1974, Vancouver*, Vol. 1 (ed. R. D. James; Canadian Mathematical Congress, 1975) 351–356.
71. (with J. W. S. Cassels) 'H. A. Heilbronn obituary', *Biographical Memoirs of the Fellows of the Royal Society* 22 (1976) 119–135.
72. 'Arithmetic and Galois module structure for tame extensions', *J. Reine Angew. Math.* 285/297 (1975) 380–440.
73. 'Local Hermitian group modules', *Conference on quadratic forms*, Queens papers on pure and applied mathematics 46 (Queens University, Kingston, Ontario, 1977).
74. *L-functions and Galois properties* (editor) Proceedings, Durham Conference (Academic Press, 1977).
75. 'Galois module structure', [**74**] 133–191.
76. 'Stickelberger without Gauss sums', [**74**] 589–707.
77. 'Symplectic local constants and Hermitian Galois module structure', *International Symposium, Kyoto 1967* (ed. S. Iyanaga; Japan Society for the Promotion of Science, Tokyo, 1977) 25–42.
78. 'Rings of integers in tame extensions as Galois modules', *Astérisque* 41–42 (1977) 31–33.
79. 'Non-Abelian Jacobi sums', *Number theory and algebra* (Academic Press, 1977) 71–75.
80. 'Some problems of Galois module structure for wild extensions', *Proc. London Math. Soc.* (3) 37 (1978) 193–212.
81. *On parity problems*, Séminaire de la Théorie des Nombres, Bordeaux (1978–79) 21.
82. (with M. J. Taylor) 'The arithmetic theory of local Galois Gauss sums for tame characters', *Trans. Royal Soc.* A 298 (1980) 141–181.
83. 'Galois module structure and root numbers for quaternion extensions of degree 2^n', *J. Number Theory* 12 (1980) 499–518.
84. 'The Hermitian classgroup', *Integral representations and applications*, Proc. Oberwolfach, Lecture Notes in Math. 882 (ed. K. W. Roggenkamp; Springer, 1981) 191–206.
85. 'Emmy Noether's contributions to algebraic number theory', *Emmy Noether memorial volume* (ed. J. Brewer and M. Smith; Marcel Dekker, 1981).
86. 'Value distributions of symplectic root numbers', *Proc. London Math. Soc.* (3) 46 (1983) 83–99.
87. *Galois module structure of algebraic integers*, Ergeb. Math. Grenzgeb. 3, Vol. 1 (Springer, 1983) 1–258.
88. (with C. J. Bushnell) *Gauss sums and p-adic division algebras*, Lecture Notes in Math. 987 (1983) 1–186.

89. *Central extensions, Galois groups and ideal class groups of number fields*, Contemp. Math. 24 (Amer. Math. Soc., Providence, RI, 1983) 1–86.

90. *Classgroups and Hermitian modules*, Progr. Math. 48 (Birkhäuser, 1984) 1–225.

91. *Gauss'sche Summen* (notes of lectures, taken by Kleinert, Denninger and Everest), Vols 1 and 2. (Math. Inst. Köln, 1983).

92. *Gauss sums*, Sémin. Théor. Nombres de Paris, 1982–83 (Birkhäuser, 1984) 91–101.

93. 'Tame representations, local principal orders and local Weil groups', *Orders and their applications*, Proc. Conf., Oberwolfach 1984, Lecture Notes in Math. 1142 (Springer, 1985) 126–149.

94. (with C. J. Bushnell) 'Non-Abelian congruence Gauss sums and p-adic simple algebras', Proc. London Math. Soc. (3) 50 (1985) 207–264.

95. 'Orthogonal representations of Galois groups, Stiefel–Whitney classes and Hasse–Witt invariants', *J. Reine Angew. Math.* 360 (1985) 84–123.

96. 'Principal orders and embedding of local fields in algebras', *Proc. London Math. Soc.* (3) 54 (1987) 127–266.

97. *Tame representations of local Weil groups and chain groups of local principal orders* (Heidelberger Akademie der Wissenschaften, 1986) 1–100.

98. 'Problems in representation theory arising from Galois modules', *Proc. Symp. Pure Math.* 47 (1987) 81–85.

99. *L-functions at* $s = 0$ *and Galois modules*, Sémin. Théor. Nombres Bordeaux (1986–87) 32-01 to 32-20.

100. 'Tame representations in base change', *Contemp. Math.* 86 (1989) 207–213.

101. 'The papers on discriminants of cubic fields', Comments in *Heilbronn's collected papers*.

102. 'Module defect and factorisability', *Illinois J. Math.* 32 (1988) 407–421.

103. 'L-values at zero and multiplicative Galois module structure (also Galois Gauss sums and additive Galois module structure)', *J. Reine Angew. Math.* 397 (1989) 42–99.

104. *The genus class group*, Sémin. Théor Nombres Bordeaux (1992).

105. (with M. J. Taylor) *Algebraic number theory*, Grad. Texts in Math. 27 (Cambridge Univ. Press, 1991).

106. 'Units in real Abelian fields', *J. Reine Angew. Math.* 429 (1992) 191–217.

107. 'L-functions and Galois modules' (notes by D. Burns and N. B. Byott), *Proc. Durham Symposium July 1989*, Lond. Math. Soc. Lect. Note Ser. 153 (ed. J. Coates and M. J. Taylor; 1991) 77–87 and 95–110.

108. 'Strict factorisability and shadows', *Algebra and number theory*, Proc. Essen Conference 1992 (ed. G. Frey and J. Ritter, De Gruyter, Berlin, 1994) 1–26.

109. 'Some remarks on wild extensions of number fields', *J. Reine Angew. Math.* 495 (1998) 29–33.

110. (with C. T. C. Wall) 'Equivariant Brauer groups', *Quadratic forms and their applications* (Dublin, 1999), Contemp. Math. 272 (ed. E. Bayer-Fluckiger, *et al.*; Amer. Math. Soc., Providence, RI, 2000) 57–71.

111. (with J. W. S. Cassels) 'Hans Arnold Heilbronn', *Bull. London Math. Soc.* 9 (1977) 219–232. (This obituary also appeared in the *Biographical Notices of the Royal Society*, from which it was reproduced by permission.)

School of Mathematics
The University of Manchester
Sackville Street
Manchester M60 1QD
United Kingdom

M. J. Taylor