OBITUARY

R. L. GOODSTEIN

Reuben Louis Goodstein was born in London on 15 December 1912, the second son of Alexander and Sophia Goodstein. His family was of Russian origin and for a while in the twenties they lived in Danzig. His father was a cigarette manufacturer with factories in several parts of Europe. A number of these were in Germany and they were confiscated when the Third Reich came to power, as a consequence of which the family's fortunes dropped dramatically after 1933.

Goodstein was educated at St. Paul's School, London, where he received a Foundation Scholarship in 1927, a Senior Scholarship in 1928, and the Sir James Jeans Prize in 1931. This latter award was for an essay on Divergent Series won in open competition amongst pupils in London schools. In 1931 he entered Magdalene College, Cambridge with a Major Open Scholarship in mathematics. He obtained first class marks in the Mathematical Tripos both in Part I in 1932, and in Part II in 1933; his special subject (schedule B) was analysis. From 1933 to 1935 he did research under Professor J. E. Littlewood in the field of transfinite numbers, and he received Research Scholarships in 1934 and 1935.

He left Cambridge in 1935 with an M.Sc. to take up an appointment as lecturer in pure and applied mathematics at the University of Reading, a position he held until December 1947. Here he lectured on a wide range of mathematical topics, especially during the Second World War when he was lecturer in charge of applied mathematics as well as analysis and group theory for honours students. He also taught engineering apprentices in evening classes and was an examiner for the National Certificate in Mechanical Engineering. His own research ideas were developing at this time and for this work he received from the University of London the degrees of Ph.D. in 1946 and D.Litt. in 1950.

In January 1948 he was appointed Professor of Mathematics at University College, Leicester (University of Leicester from 1957 onwards); a position he held until his retirement in September 1977. At Leicester he had a major influence on the development of that institution from a small college whose students took London University degrees to a fully fledged university with over 3,000 students. He held many academic and administrative appointments including Dean of Science from 1954 to 1957 and Pro-Vice-Chancellor from 1966 to 1969. A colleague said of his time at Leicester that it combined to a rare and happy extent the aesthetic outlook and severe standards of the pure mathematician with a first hand knowledge of the compromises and approximations of the practical scientist and administrator. He suffered a stroke in 1976 which unfortunately put an end to his active research work. In his last years he lived quietly with his wife Louba and died on 8 March 1985 at the age of 72.

Apart from his sterling work at Leicester for nearly thirty years, Goodstein made valuable contributions in two further directions. The first concerns his influence on the early development of mathematical logic in the United Kingdom. He was the first person whose main interests were in mathematical logic to hold a chair in a British university. In 1958 he expressed the opinion that there would not be another in his lifetime. Fortunately he was wrong and his influence has had a considerable bearing on this more healthy state of affairs. Many British mathematical logicians have at some time either worked with him or been influenced by him. The following did research under his guidance: R. Beazer (now at Glasgow University), A. Bundy (now in the Artificial Intelligence department at Edinburgh), R. A. Cunninghame-Green (now Professor in Operational Research at Birmingham), J. Hooley, R. D. Lee (now at Essex University), M. H. Löb (now Professor of Mathematical Logic at Amsterdam), M. T. Partis (now at the University of Western Australia), H. E. Rose (Bristol), G. Rousseau (now at Leicester University), P. Schofield, K. Stewart (sometime head of department at Hatfield Polytechnic), and H. P. Williams (sometime Professor of Business Studies at Southampton). He was a member of the council of the Association for Symbolic Logic 1965–69 and received an Sc.D. from the University of Cambridge in 1975 for his work in mathematical logic. He was elected to membership of the Society in 1944.

His second main contribution concerns his work for the Mathematical Association. He was an excellent teacher, always able to communicate his enthusiasm for his subject whether to an elementary extra-mural class or in postgraduate lectures. This led him to take a great interest in the Association. He was instrumental in the moves of both the Association's library to the University of Leicester and the Association's headquarters to London Road, Leicester, which has been described as its first real home. He published over sixty articles and notes in the Mathematical Gazette of which he was editor from 1956 to 1962, and he was the Association's librarian for many years and its president in 1975–76.

Louis Goodstein was an intensely private person with what today might be considered rather traditional views – for instance, he was not keen on student participation in university committees. But all his colleagues and students will remember his quiet courtesy, his willingness to assist with any administrative or mathematical problem, and his great enthusiasm for his chosen subject.

During the preparation of this obituary I have received considerable assistance from Roy Davies, Peter Goodstein, Alan Hayes, Georg Kreisel, and an anonymous compiler of biographical data on the academic staff at Leicester. Their help has been invaluable and I am most grateful.

Goodstein's mathematical publications

Goodstein's published work will be considered under five headings (a) the restricted ordinal theorem, (b) primitive recursive arithmetic and analysis, (c) philosophy of mathematics, (d) textbooks, and (e) Mathematical Association notes. We shall not consider (d) or (e) in detail. His textbooks were characterised by their clear style and ingenious methods to elucidate difficult points. He was disappointed that his mathematical analysis text [**1**], which presented a novel approach to elementary differential and integral calculus (the 'uniform calculus'), did not find favour. Also, many of his Mathematical Association notes (66 in all) and notes for other journals gave valuable points for the teaching of mathematics both in school and in the university.

(a) *The restricted ordinal theorem.* Although only two papers [**26**, **40**] are involved and the work was carried out early in his career, this is one of his most important contributions.

Given a positive integer n, another integer $P_x(n)$ can be constructed as follows. Write n in radix form with base x ($x > 1$), that is

$$n = a_k x^k + a_{k-1} x^{k-1} + \ldots + a_0$$

where $0 \leqslant a_i < x$ for $i = 0, \ldots, k$. Further, write each exponent of x in this expression in radix form with base x, and continue this process until all exponents, exponents of exponents, etc., are in radix form. Now define a new integer $P_x(n)$ by subtracting 1 and replacing each occurrence of x in this radix form by $x+1$. So for example if $n = 529$ and $x = 2$ then

$$529 = 2^9 + 2^4 + 1 = 2^{2^3+1} + 2^{2^2} + 1 = 2^{2^{2+1}+1} + 2^{2^2} + 1,$$

and

$$P_2(529) = 3^{3^{3+1}+1} + 3^{3^3} \approx 1.33 \times 10^{39}.$$

The sequence $n, P_x(n), P_{x+1} P_x(n), \ldots$ is now called a *Goodstein sequence*. The *restricted ordinal theorem* states that for all positive integers n and bases x, the Goodstein sequence

$$n, P_x(n), P_{x+1} P_x(n), P_{x+2} P_{x+1} P_x(n), \ldots$$

terminates with zero in a finite number of steps. This result is a number theoretic analogue of the fact that all strictly decreasing sequences of transfinite ordinals are finite. Goodstein gave a proof of the restricted ordinal theorem in [**26**] using transfinite induction I_{ε_0} for ordinals less than ε_0, and he noted the connection with Gentzen's proof ⟨**2**⟩ of the consistency of arithmetic which also uses I_{ε_0}. His interest in the finitist programme was clearly developing at this time for he described I_{ε_0} as 'a minimum deviation from the previously accepted field of finitist processes'.

The importance of this result only became apparent in 1982 when Kirby and Paris ⟨**3**⟩ showed that it provided a straightforward number theoretic property which is not provable in first order arithmetic. The Kirby–Paris result grew out of the Paris–Harrington independence results and uses the subrecursive hierarchy as well as Goodstein's material. The first two of these topics have only been developed recently and this explains the delay in the recognition of the significance of Goodstein's 1944 paper.

(b) *Primitive recursive arithmetic and analysis.* The full potential of what is now called primitive recursive arithmetic and analysis was developed over a thirty-year period in two books [**5**, **6**] and a long series of papers beginning with [**30**] published in 1945. Goodstein took an extreme finitist view of mathematics (see (c)) and this led him to investigate those concepts and theorems from arithmetic and analysis which can be interpreted primitive recursively. The first discovery (in [**30**]) enabled the formalisation of primitive recursive arithmetic to be given as a 'logic-free equation calculus'; that is, it is not necessary to include any reference to propositional connectives or rules in the basic formalisation. (This was also discovered, independently, by H. B. Curry; see ⟨**1**⟩.) Primitive recursive arithmetic has an axiomatisation in which all propositions are equations of the form $A = B$ where A and B are primitive recursive functions or terms, and whose rules are standard substitution and uniqueness (induction) rules. (He attributed the replacement of induction by uniqueness to Wittgenstein; see [**119**].) The propositional connectives (and, or, not, implies) and the bounded quantifiers can be introduced arithmetically. For example, as only natural numbers are involved,

$$A = B \quad \text{and} \quad C = D$$

is equivalent to

$$|A, B| + |C, D| = 0.$$

The number theoretic aspects of primitive recursive arithmetic were presented in [**5**]. It should perhaps be noted that no consideration was given to other systems in what is now known as the subrecursive hierarchy, although he did consider multiple recursions in some papers.

From early in his career Goodstein was interested in characterising those parts of analysis which are acceptable to the finitist. For example, as unbounded quantifiers cannot be used, such basic results as the Weierstrass theorem (all monotone bounded sequences have a limit) are unavailable; but it is surprising how much can be saved.

A primitive recursive real number is a primitive recursive function $f: \mathbb{N} \to \mathbb{Q}$ (that is, a sequence of rational numbers) such that another primitive recursive function $g: \mathbb{N} \to \mathbb{N}$, which is strictly monotonic, can be found to satisfy

$$n_1 > n_2 \geqslant g(k) \quad \text{implies} \quad |f(n_1) - f(n_2)| < 10^{-k}.$$

(Note that there are only countably many primitive recursive reals.) Similarly, many of the basic definitions of analysis are presented as usual except that each existential quantifier is replaced by an actual primitive recursive function. Goodstein showed that these restrictions are not as formidable as they appear at first sight; in a long series of papers he was able to present many analytic topics from his strictly finitist viewpoint. For example, the theory of the exponential, logarithmic and circular functions was given in [**65**], a constructive theory of plane curves was given in [**70**], and the recursive transcendence of π was proved in [**87**]. His book [**6**] gave a good introduction to this work. One result that eluded him for many years was the so-called fundamental theorem of algebra. But finally, in [**112**], he was able to show that Gauss's second proof of this result can be rewritten in finitist form.

(c) *Philosophy of mathematics.* Over the period of his working life Goodstein wrote a number of essays on the nature and structure of mathematics. As he stated in [**81**], his purpose was to study:

> '...the problem of the nature of the entities of mathematics [which] continues to be, as it has been for the past hundred years, one of the central questions in foundation researches. Whether it is considered in its full generality or in the limited aspect of the existence problem, the question leads immediately to the heart of the controversy between formalism and finitism, realism and platonism.'

He expressed doubts about both the formalist and intuitionist (finitist) approaches to mathematics, and this led him to take up an extreme finitist position which, roughly speaking, consisted of those parts of mathematics that are accepted in all schools of thought. He was greatly influenced by Wittgenstein whose lectures he attended in Cambridge between 1931 and 1935.

With Brouwer he rejected the formalist approach which claims that mathematics has no 'meaning' and should be treated as a 'game' played with axioms and rules. For him axiomatisation was only a tool which could be used to characterise parts of mathematics. As a consequence, he rejected the *tertium non datur* when it is used to assert existence. He often quoted the following example based on Goldbach's conjecture. Let $P(n)$ denote the proposition '$n > 1$ and $2n$ is a sum of two primes'. (Note that, for each n, the validity of $P(n)$ can be checked in $O(n^2)$ steps.) Now let $p(n) = 0$ if, for all $m \leqslant n, P(m)$ holds, and let $p(n) = 1$ if there is an $m \leqslant n$ such that

$P(m)$ is false. The sequence $\{p(n)\}$ is bounded and monotonic and so, by Weierstrass's theorem, it has a limit if we accept the tertium non datur. Clearly this limit is 0 or 1 but the proof gives no method for deciding between these two possibilities. He felt that this situation was unacceptable, and hence that this, and similar, methods of proof should be discarded and replaced by more constructive ones. As he wrote in [**81**]: 'It is not whether or not a certain entity exists which is in dispute but how the terms should be used.' A proof of the existence of an entity T should only be accepted if it provides, at least in principle, a method for constructing T. This is an ideal to which many mathematicians aspire with varying degrees of success.

Goodstein went further than Brouwer and the intuitionists arguing by analogue that if we reject the tertium non datur

$$\text{not}(\forall x)\,R(x) \;\text{ implies }\; (\exists x)\,\text{not}R(x),$$

then we should also reject the complementary implication

$$\text{not}(\exists x)\,R(x) \;\text{ implies }\; (\forall x)\,\text{not}R(x).$$

This led him to a complete rejection of quantification theory and thus to the development of his equation calculus (see (b)). As a consequence he was also forced to question the acceptability of the standard definition of a recursive function. A function is recursive (computable) if a machine exists to evaluate it; although, in general, no bound can be given in advance on the length of the computation. He argued that the notion of a computable function was not captured by this definition as it allowed arbitrarily long calculations. Hence he put forward primitive recursive arithmetic as an acceptable basis for mathematics because it is 'logic-free' and all its functions have simply bounded computations. The main criticism of this approach is that wide areas of mathematics are completely ignored.

Apart from his negative approach, Goodstein also attempted to justify his position positively. When asked how to define an entity in mathematics he would (following Wittgenstein) turn the question around and ask how does it fit into the structure as a whole; that is, what role does it play? He introduced the natural numbers by stating [**5**] that 'the object of our study is not number itself but the transformation rules of the number signs'. He used an analogue with the game of chess. The 'king' in chess is not a piece of wood or ivory but an entity that can make certain moves. So in mathematics the natural numbers obey the rules of primitive recursive arithmetic. He realised that his description of the properties of the natural numbers was very incomplete and expressed the hope that in time more of the rules would be discovered. As he wrote in [**19**]:

> 'It is not a new foundation of mathematics that is needed but a close examination of its skeletal structure and of its ornamental coverings. Mathematics is like a city of fine buildings, filled with precious gems, but buried deep in the mud and sand of a desert. The task of digging up these treasures is a slow and arduous one; some progress has been made....'

References

⟨1⟩. H. B. CURRY, 'A formalisation of recursive arithmetic', *Amer. J. Math.* 63 (1941) 263–82.
⟨2⟩. G. GENTZEN, 'Die Widerspruchsfreiheit der reinen Zahlentheorie', *Math. Ann.* 112 (1936) 493–565.
⟨3⟩. L. KIRBY and J. PARIS, 'Accessible independence results in Peano arithmetic', *Bull. London Math. Soc.* 14 (1982) 285–93.

Publications of R. L. Goodstein

Books

1. *Mathematical analysis. The uniform calculus and its applications* (Clarendon Press, Oxford, 1948; reprinted 1949).
2. *Constructive formalism* (Leicester University Press, 1951; 2nd. edition 1965).
3. (with E. J. PRIMROSE) *Axiomatic projective geometry* (Leicester University Press, 1953; 2nd. edition 1962).
4. *Mathematical logic* (Leicester University Press, 1957; 2nd. edition 1961, Russian edition 1961, 3rd. edition 1965).
5. *Recursive number theory* (North-Holland, Amsterdam, 1957; Russian edition 1970).
6. *Recursive analysis* (North-Holland, Amsterdam, 1961; Russian edition 1970).
7. *Fundamental concepts in mathematics* (Pergamon Press, Oxford, 1962; reprinted 1964, Dutch edition 1966, Japanese edition 1968).
8. *Boolean algebra* (Pergamon Press, Oxford, 1963; Japanese edition 1971).
9. *Essays in the philosophy of mathematics* (Leicester University Press, 1965).
10. *Complex functions* (McGraw Hill, London, 1965; Spanish edition 1975).
11. *Development of mathematical logic* (Logos Press, London, 1971; Japanese edition 1978.

Papers and notes

12. 'Geometry in modern dress', *Math. Gaz.* 22 (1938) 217–224.
13. 'A class of diophantine equations', *Math. Gaz.* 22 (1938) 338–341.
14. 'A note on projective geometry', *Math. Gaz.* 23 (1939) 86–88.
15. 'How unlikely', *Math. Gaz.* 23 (1939) 205–206.
16. 'Ways and means', *Math. Gaz.* 23 (1939) 206.
17. 'A logical point in dynamics', *Math. Gaz.* 23 (1939) 215–216.
18. 'Rational triangles', *Math. Gaz.* 23 (1939) 264–267.
19. 'Mathematical systems', *Mind* 48 (1939) 58–73 (reprinted in [9]).
20. 'On chords of a conic which touch another conic', *Math. Gaz.* 24 (1940) 103–105.
21. 'On von Mises' theory of probability', *Mind* 49 (1940) 58–62.
22. 'Teaching the calculus', *Math. Gaz.* 25 (1941) 208–217.
23. 'Recurring digits in irrational decimals', *Math. Gaz.* 25 (1941) 273–278.
24. 'Lottery chances', *Math. Gaz.* 26 (1942) 184.
25. 'The limit of $\{\Delta^n f(x)\}/(\Delta x)^n$', *Math. Gaz.* 26 (1942) 220–222.
26. 'On the restricted ordinal theorem', *J. Symbolic Logic* 9 (1944) 33–41.
27. 'The equation $a^b = b^a$', *Math. Gaz.* 28 (1944) 76.
29. 'Ferry puzzles', *Math. Gaz.* 28 (1944) 202–204.
30. 'Function theory in an axiom-free equation calculus', *Proc. London Math. Soc.* (2) 48 (1945) 401–434.
31. 'Transport problems', *Math. Gaz.* 29 (1945) 16–17.
32. 'Mental multiplication', *Math. Gaz.* 29 (1945) 71–72.
33. 'Examination questions', *Math. Gaz.* 29 (1945) 125–128.
34. 'Find the penny', *Math. Gaz.* 29 (1945) 227–229.
35. 'Integration by substitution', *Math. Gaz.* 30 (1946) 155–156.
36. 'Linkage puzzles', *Math. Gaz.* 30 (1946) 156–157.
37. 'A theorem on uniform convergence', *Math. Gaz.* 30 (1946) 287–290.
38. (With T. A. A. BROADBENT) 'The convergence of iterative processes', *J. London Math. Soc.* 22 (1947) 168–171.
39. 'The strong convergence of the exponential function', *J. London Math. Soc.* 22 (1947) 200–205.
40. 'Transfinite ordinals in recursive number theory', *J. Symbolic Logic* 12 (1947) 123–129.
41. (With M. RUMNEY) 'Some new theorems on divisibility', *Math. Gaz.* 31 (1947) 90–92.
42. 'Note on continuity', *Math. Gaz.* 31 (1947) 164–165.
43. 'A note on identities', *Math. Gaz.* 31 (1947) 165–166.
44. 'Commutative involutions', *Math. Gaz.* 31 (1947) 224–226.
45. 'Proof by reductio ad absurdum', *Math. Gaz.* 32 (1948) 198–201 (reprinted in [9]).
46. 'A necessary and sufficient condition for differentiability', *Edinburgh Math. Notes* 37 (1949) 13–16.
47. 'On the evaluation of Planck's integral', *Edinburgh Math. Notes* 37 (1949) 17–20.
48. 'Missing value theorems', *Math. Gaz.* 33 (1949) 19–25.
49. 'Simultaneous differential equations', *Math. Gaz.* 33 (1949) 30–32.
50. 'Discriminants and resolvents', *Math. Gaz.* 33 (1949) 296–297.
51. 'The formal structure of a denumerable system', *Trans. Amer. Math. Soc.* 68 (1950) 174–182.

52. 'The Gauss test for relative convergence', *Amer. J. Math.* 72 (1950) 217–228.
53. 'Mean value theorems in recursive function theory. Part 1. Differential mean value theorems', *Proc. London Math. Soc.* (2) 52 (1950) 81–106.
54. 'On the multiplication of series', *Math. Gaz.* 34 (1950) 16–18.
55. 'The foundations of mathematics', Inaugural lecture, University College, Leicester (1951, reprinted 1959).
56. 'An introduction to the theory of continuous groups', *Math. Gaz.* 35 (1951) 91–96.
57. 'Some criteria for the infinite oscillation of alternating series', *Math. Gaz.* 36 (1952) 126–127.
58. 'On the limit of the ratio of $\sin x$ to x', *Math. Gaz.* 36 (1952) 189–192.
59. 'A problem in recursive function theory', *J. Symbolic Logic* 18 (1953) 225–232.
60. 'Permutation in recursive arithmetic', *Math. Scand.* 1 (1953) 222–226.
61. 'Euclidean geometry and the rigid motion group', *Math. Gaz.* 37 (1953) 117–118.
62. 'A free variable function theory', *Proc. Internat. Math. Congress, Amsterdam* (1954) Vol. 2, pp. 401–402.
63. 'The recursive irrationality of π', *J. Symbolic Logic* 19 (1954) 267–274.
64. 'Logic free formalisations of recursive arithmetic', *Math. Scand.* 2 (1954) 247–261.
65. 'The relative exponential, logarithmic and circular functions in recursive function theory', *Acta Math.* 92 (1954) 171–190.
66. 'A note on convergence', *Math. Gaz.* 38 (1954) 140–141.
67. 'On functions of the roots of a quadratic', *Math. Gaz.* 38 (1954) 202–204.
68. 'A problem in linear differential equations', *Math. Gaz.* 38 (1954) 275–277.
69. 'On non-constructive theorems of analysis and the decision theorem', *Math. Scand.* 3 (1955) 261–263.
70. 'A constructivist theory of plane curves', *Fundamenta Math.* 43 (1955) 23–35.
71. 'The Arabic numerals, numbers and the definition of counting', *Math. Gaz.* 40 (1956) 114–129 (reprinted in [9]).
72. 'A geometrical interpretation of the formula for the derivative of a function of a function', *Math. Gaz.* 40 (1956) 46.
73. 'A permutation problem', *Math. Gaz.* 40 (1956) 46–47.
74. 'The expansions of the elementary functions', *Math. Gaz.* 40 (1956) 48–51.
75. 'An inequality for Euler's function $\phi(n)$', *Math. Gaz.* 40 (1956) 131.
76. 'Digit transfers', *Math. Gaz.* 40 (1956) 131–132.
77. 'The decision problem', *Math. Gaz.* 41 (1957) 29–37 (reprinted in [9], German translation in *Grundlagen der Modernen Math.* 1972).
78. 'A generalisation of Taylor's theorem', *Math. Gaz.* 41 (1957) 133–134.
79. 'The definition of number', *Math. Gaz.* 41 (1957) 180–186 (reprinted in [9]).
80. 'Models of propositional calculi in recursive arithmetic', *Math. Scand.* 6 (1958) 293–296.
81. 'On the nature of mathematical systems', *Dialectica* 12 (1958) 296–316 (reprinted in [9], German translation in *Grundlagen der Modernen Math.* 1972).
82. 'On the formalisation of indirect discourse', *J. Symbolic Logic* 23 (1958) 417–419.
83. 'The system of equations $b^x = x_1, b^{x_1} = x_2, \dots, b^{x_n} = x$', *Math. Gaz.* 42 (1958) 296–299.
84. 'Recursive analysis', *Constructivity in mathematics* (ed. A. Heyting, North-Holland, Amsterdam, 1959), pp. 37–42 (German translation in *Grundlagen der Modernen Math.* 1972).
85. (With M. RUMNEY) 'On the resultant of two quadratic forms', *Math. Gaz.* 43 (1959) 191.
86. 'Automorphic numbers in a general scale', *Math. Gaz.* 43 (1959) 270–272.
87. (With J. HOOLEY) 'On recursive transcendence', *Notre Dame J. Formal Logic* 1 (1960) 127–137.
88. 'On the equation $ax-by = 1$', *Math. Gaz.* 44 (1960) 132–134.
89. 'The general solution in integers of the system of equations $ax_r - bx_{r+1} = 1, r = 1, 2, \dots, n, (a, b) = 1$', *Math. Gaz.* 44 (1960) 217–218.
90. 'The graphical solution of a quadratic with imaginary roots', *Math. Gaz.* 44 (1960) 289–290.
91. 'Polynomial generators over Galois fields', *J. London Math. Soc.* 36 (1961) 29–32.
92. 'The axiomatic method', *Aristotelian Soc. Suppl. Vol.* 36 (1962) 145–154 (reprinted in [9]).
93. 'Truth tables', *Math. Gaz.* 46 (1962) 18–23 (reprinted in *Hausdorff's Set theory*, Chelsea Publ. Co., New York, 1962).
94. 'Boolean algebra; Duality in mathematics; Euclidean geometry; Lie groups in differential equations; Mathematics; *Encyclopaedic Dictionary of Physics* (Pergamon Press, Oxford, 1963).
95. 'Polynomial generators over finitely generated rings', *J. London Math. Soc.* 38 (1963) 79–80.
96. 'A decidable fragment of recursive arithmetic', *Z. Math. Logik Grundlag. Math.* 9 (1963) 199–201.
97. 'The significance of incompleteness theorems', *British J. Philos. Sci.* 14 (1963) 208–220 (reprinted in [9]).
98. 'Diophantine equations in recursive difference', *Math. Scand.* 15 (1964) 43–48.
99. 'Pure and applied mathematics', *Ratio* 6 (1964) 1–11 (reprinted in [9]).
100. 'Multiple successor arithmetics', *Formal systems and recursive functions* (ed. J. N. Crossley and M. A. E. Dummett, North-Holland, Amsterdam, 1965), pp. 265–271.

101. (With R. D. LEE) 'A decidable class of equations in recursive arithmetic', *Z. Math. Logik Grundlag. Math.* 12 (1966) 235–239.
102. 'The solution of equations in a lattice', *Proc. Royal Soc. Edinburgh* A67 (1967) 231–242.
103. 'On weak complements in a lattice', *Revue Roumaine Math. Pure Appl.* 12 (1967) 1059–1063.
104. 'Formulae for primes', *Math. Gaz.* 51 (1967) 35–36.
105. 'A functional equation for implication', *Math. Gaz.* 51 (1967) 38–40.
106. 'On the axiomatic method', *Math. Gaz.* 51 (1967) 318–319.
107. 'The general solution of the equation $(n+1)v^2-nu^2 = 1$', *Math. Gaz.* 51 (1967) 320–321.
108. 'Existence in mathematics', *Compositio Math.* 20 (1968) 70–82.
109. 'The irrationality of the root of a non-square integer', *Math. Gaz.* 52 (1968) 50.
110. 'Free variable axioms for groups', *Math. Gaz.* 52 (1968) 342–347.
111. 'The discriminant of a certain polynomial', *Math. Gaz.* 53 (1969) 60–61.
112. 'A constructive form of the second Gauss proof of the fundamental theorem of algebra', *Constructive aspects of the fundamental theorem of algebra* (ed. B. Dejon and P. Henrici, Wiley, London, 1969), pp. 69–76.
113. 'Empiricism in mathematics', *Dialectica* 23 (1970) 50–57.
114. 'On sums of progressions of positive integers', *Math. Gaz.* 54 (1970) 113–115.
115. 'Polynomials with computable coefficients', *Notre Dame J. Formal Logic* 11 (1970) 447–448.
116. 'The generalised Vandermonde determinant', *Math. Gaz.* 54 (1970) 264–266.
117. 'A generalised permutation problem', *Math. Gaz.* 54 (1970) 266–267.
118. 'A property of the central conic', *Math. Gaz.* 56 (1972) 117.
119. 'Wittgenstein's philosophy of mathematics', *Ludwig Wittgenstein: philosophy and language* (ed. A. Ambrose and M. Lazerowitz, Allen and Unwin, London, 1972), pp. 271–286.
120. 'A new proof of completeness', *Notre Dame J. Formal Logic* 13 (1972) 563–564.
121. 'The fundamental formula in the algebra of sets', *Math. Gaz.* 56 (1972) 199–207.
122. 'On limiting the application of the uniqueness rule in the equation calculus', *Z. Math. Logik Grundlag. Math.* 19 (1973) 115–116.
123. 'On triangular numbers', *New Zealand Math. Mag.* 10 (1973) 99.
124. 'Thomas Arthur Alan Broadbent' (obituary), *Bull. London Math. Soc.* 5 (1973) 355–357.
125. 'The mathematical association library', *British J. Hist. Sci.* 7 (1974) 100–103.
126. 'Satisfiability in a larger domain', *Notre Dame J. Formal Logic* 15 (1974) 598–600.
127. 'Representation by consecutive squares', *New Zealand Math. Mag.* 11 (1974) 78.
128. 'Cutting squares from rectangles', *Math. Gaz.* 58 (1974) 291–292.
129. 'The importance of being 259', *Math. Gaz.* 59 (1975) 1–2.
130. 'Hilbert's tenth problem and the independence of recursive difference', *J. London Math. Soc.* 10 (1975) 175–176.
131. 'A note on a property of sequences of consecutive integers', *New Zealand Math. Mag.* 12 (1975) 81–84.
132. 'Arithmetic without sets', *Math. Gaz.* 60 (1976) 165–170.

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