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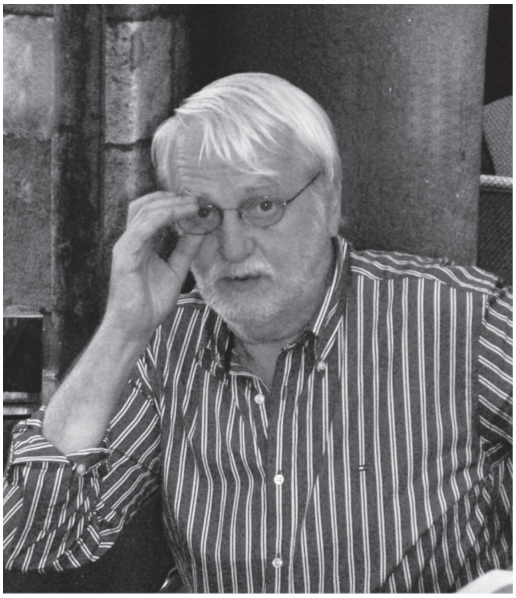
OBITUARY

Fritz Grunewald, 1949–2010



1. *Introduction*

Fritz Grunewald was an inspirational mathematician. The breadth of his knowledge was remarkable, and is reflected (though by no means fully represented) in the diverse range of areas where he made important contributions. Mathematics for him was a communal enterprise: all but two of his numerous publications were joint work. He brought a unique vision to these collaborations, and following up his ideas will keep many mathematicians busy for a long time to come. Of his 48 co-authors, those I have spoken to testify to the profound influence working with him had on their own mathematics; this is certainly true in my case.

Although his first research was in group theory, he was from the start a mathematical universalist. His most distinctive contributions to group theory came from approaching it via number theory and geometry; his work in number theory was centred round arithmetic groups. First and foremost he was an indefatigable calculator, both by hand and by computer. Theoretical insights were inspired by hands-on familiarity with the data. Although I do not think he had strong views about the philosophy of mathematics, he was in practice very much a constructivist: to understand a theory, for him, meant to be able to calculate explicit examples. This led on the one hand to insightful conjectures in number theory, and on the other hand to some fundamental decidability theorems in algebra.

As a person he was extraordinarily warm, generous and open. He shared ideas on an equal footing with students, postdocs and professors. Wherever he went (and he went all over the world) he was surrounded by mathematicians who wanted to talk to him, and he had time for everyone. His sudden and unexpected death (due probably to a pulmonary embolism) on 21 March 2010 was a devastating blow to many dozens of devoted followers and co-workers.

2. *Personal history*

Fritz's mother Helene Weisbrod was brought up on a large farm, 'der grösste Hof in Lambsheim', as she proudly described it, owned by her parents in the Pfalz region near Mannheim. After the Second World War she married Friedrich Grunewald, a dentist, and settled in the nearby town of Bad Kreuznach, where their only child Fritz Alfred Joachim was born in 1949. The marriage was sadly short-lived, however; when Fritz was about two years old his parents separated and his mother returned to the Lambsheim family home, where Fritz grew up. His maternal grandfather died when Fritz was still quite young, and one has the impression of life on the farm dominated by two powerful women, Helene and her mother (still an impressive figure when I met them both in the late 1970s).

Helene remarried quite soon, remaining on the farm; Fritz's stepfather Kurt Kinkel was also a rather dominating personality, and relations between him and the young Fritz were difficult. Kinkel disapproved of Grunewald senior, while Fritz naturally wanted to maintain contact with his father. He felt the resulting tension throughout his childhood; by the time he was 16 the tension proved excessive and Fritz and his father broke off all contact. By this time Grunewald senior had remarried; his daughter Ulrike has fond memories of her older half-brother, and recalls the pain of losing touch with him when she was only seven. It took twenty years for Fritz to re-establish contact with his father and with Ulrike; the long-separated siblings became close friends.

Friedrich Grunewald was a national-level hockey player, and the sporting gene was passed on to Fritz and then to Fritz's two sons. His younger son Andreas is a keen footballer, while Fritz excelled in table tennis, representing a Bonn club in regional competitions for many years.

Fritz went to school at the Albert–Einstein–Gymnasium in Frankenthal, a small town not far from Lambsheim. He did well at school, and was a keen mathematician from an early age; his sister recalls how he would retreat to his room and bury himself in calculations as a refuge from domestic tensions. This will resonate with the many mathematicians who collaborated with him during his adult life: his instinct when faced with a mathematical problem was always to go away and do more calculations.

Having completed his Abitur (with the top grade for his year in the Gymnasium) he began his university studies at Göttingen, in Mathematics and Physics, in 1969. Among his teachers was Professor Jens Mennicke.

Mennicke had recently achieved prominence with his solution of the congruence subgroup problem for $\mathrm{SL}_n(\mathbb{Z})$. This establishes deep connections between algebra and arithmetic via matrix groups over the integers, a theme that clearly inspired the young Fritz and one that he continued to explore throughout his life. In 1971, Mennicke took up a chair at the new university of Bielefeld, joining a brilliant mathematics department where the new appointments included Bernd Fischer, Friedhelm Waldhausen and Andreas Dress. Fritz followed him to Bielefeld, where he wrote a Diplomarbeit on Hecke rings of finite groups under Dress's supervision. By 1973 he had completed his doctorate, with Mennicke as supervisor: this was his first major research undertaking (described below), and marked the beginning of an intense and long-lasting collaboration between the two, resulting ultimately in 27 joint publications including Fritz's only major book.

Fritz spent the following year 1973–1974 as a postdoc at Queen Mary College, London. We shared an office and became lifelong friends; more importantly (for mathematics) this resulted in Fritz's second long-term collaboration, with me (resulting in twenty joint papers). His girlfriend Barbara, whom he had met in Bielefeld in 1970, joined him in London; they married in 1974.

Returning to Bielefeld, he worked as a 'Wissenschaftlicher Assistent' until 1977, held a Habilitationsstipendum for the next two years and was awarded the Habilitation in 1979.

By this time, Fritz was ready to move on. The most vibrant centre of German mathematics was Bonn, where the famous geometer Friedrich Hirzebruch hosted an annual Arbeitstagung, and was shortly to establish the first Max-Planck-Institut für Mathematik. Fritz was awarded a Heisenberg Stipendium (similar to the Royal Society University Research Fellowships in the UK) and moved to Bonn with his young family, now including Natalie (born 1976) and Joachim (born 1978). In 1981, he became a C3 Professor (associate professor) at the University of Bonn.

His final career move was to a full professorship at the University of Düsseldorf, in 1992; he served as Chairman of the Mathematics Institute there during 2000–2002.

Fritz and Barbara had a third child, Andreas, in 1986. Barbara's distinguished career as a law professor ran alongside his own, and Fritz always took his share of domestic and parental responsibilities, arranging his work around the imperatives of childcare, shopping and cooking. In those days this was unusual among (male) German professors, and occasionally led to raised eyebrows among some colleagues in the department; but his priorities were clear.

Fritz's gentle personality made him a favourite with students. Loth to turn anyone down, he was usually supervising several Diplom- and doctoral students at the same time. Each Diplom was in effect a new research project, so in addition to his more visible published work he was in effect involved in a large number of smaller scale research collaborations throughout his career; he took these seriously and they took up a large part of his time. At a rough estimate he supervised about 70 Diploms, as well as 31 completed doctorates. Many of his students went on to academic careers.

He was involved in several long-term collaborations, with groups in Belgium, Germany, Brazil, Argentina and Israel (at least). As well as frequently travelling to these places, he co-organized many international conferences, and four Oberwolfach workshops.

Although he was widely consulted and universally respected, Fritz received perhaps fewer marks of formal recognition than he deserved. Notable exceptions were the award of the Reinhard– und Emmi–Heynen–Preis by the University of Düsseldorf in 2001, and an invitation to speak at the International Congress of Mathematicians in Madrid in 2006.

In 2009, a special conference on 'Group theory, number theory and geometry' was held in Oxford to mark his 60th birthday. This lasted five days and drew over 100 participants.

3. *Mathematical work*

Group theory

$B(2,8)$. Fritz's first major research project, the topic of his 1973 doctorate, was an attack on the notorious Burnside Problem for groups of exponent 8. The problem remains unsolved to this day. What Fritz managed to establish, by truly heroic hand calculations, was the finiteness of certain finitely presented exponent-8 groups which it was hoped are not too far removed from the actual free Burnside group. When this was submitted to a journal (as part of a joint work with his supervisor Jens Mennicke), the referee remarked 'this belongs in the Guinness book of records, not a mathematical journal!' The Australian mathematicians George Havas and Mike Newman visited Bielefeld for a workshop on the topic in 1977; they were approaching the same problem using computer methods, and their combined work was reported in [**8**, **17**], together with a briefer account of the Grunewald–Mennicke approach [**18**].

Finite presentability. Fritz spent the academic year 1973–1974 as a postdoc at Queen Mary College, London. When he arrived, the head of department Karl Gruenberg took one look at his long hair and hippyish attire and decided he should share a room with me. We immediately became firm friends and began a 37-year long (intermittent) collaboration. At this stage, Fritz was working on the first of his two single-authored papers [6]; here he gives necessary and sufficient conditions for finite presentability of certain group extensions involving relation modules, and of certain subdirect products of free groups. Though not much noticed at the time, this note ultimately became one of his most cited papers: it was a first step in what proved to be a very rich field of research, culminating in the recent paper ⟨4⟩ of Bridson, Howie, Miller and Short.

Polycyclic groups. He also began a long-running project with me on the topic of polycyclic groups. Earlier work of Baer and more recent work of Remeslennikov indicated that these needed to be studied through algebraic number theory; still more recent results of L. Auslander, Borel and Fred Pickel showed that linear algebraic groups were also a key part of the picture. Fritz (unlike me at the time) had a good grounding in all these subjects. Fritz's first published paper [1] used some elementary algebraic number theory to construct interesting examples of polycyclic groups. After his return to Bielefeld, he encouraged Professor Mennicke to offer me an Assistent position, and I joined him in 1976. Together we proved that polycyclic groups are 'subgroup conjugacy separable' [4], and in a series of papers (some with Pickel) established the 'finite genus' property for these groups [3, 5, 10, 16, 29]. This says that a family of polycyclic groups all with the same profinite completion contains only finitely many isomorphism types; this had recently been established for nilpotent groups by Pickel, and was at the time one of the two main open problems in the area. One of Fritz's essential insights (for this project) was a result in algebraic number theory: we knew that Chevalley's theorem on the congruence subgroup property of groups of algebraic integer units had to be a key step, but it was not sharp enough; Fritz saw how it could be generalized in the relevant manner [14].

This was typical of the experience of collaborating with Fritz. After much technical work we would come to a sticking point; he would go away, and then after a while come back and say 'I can see a way to do X'. He did not have all the technical details, but as often as not his vision was sound and we were able in due course to cook up the proof.

The issue of 'genus' (what it means for groups to have isomorphic profinite completions) is one that he returned to several times throughout his career.

The most approachable class of polycyclic groups is the finitely generated nilpotent groups. Fritz wrote several works on the classification of these, using ideas from algebraic geometry, number theory and algebraic groups: [9] (with R. Scharlau), [26] (with D. Segal and L.S. Sterling), [34] (with D.S.) and [35] (with J. O'Halloran).

His most important result on this topic was the solution (with D.S.) of the *isomorphism problem* [20], the other main open problem in the area (in fact, the general expectation had been for a *negative* solution)†. That is, the construction of an algorithm to decide whether two nilpotent groups, given by finite presentations, are isomorphic (Adian and Rabin had famously proved that no such algorithm exists for finitely presented groups in general). It had become clear that this problem should be construed as a question about orbits of an arithmetic group: Pickel had settled the genus problem for nilpotent groups by reducing it to a 'local–global' finiteness theorem for such orbits, and the result of [29] was in the same spirit; what was needed now was a *decidability* theorem.

Decidability. The second of Fritz's single-authored papers [15] presents an explicit algorithm for solving the *conjugacy problem* in $\mathrm{GL}_n(\mathbb{Z})$, the first general result of this type for arithmetic groups. There was no doubt in his mind that something similar should be possible

† The solvability of the isomorphism problem for all polycyclic groups was eventually established in ⟨12⟩, when Fritz had moved on to other things. That paper acknowledges a fundamental debt to Fritz.

in much greater generality, and, as often, his intuition was correct. Using more sophisticated tools (Borel and Harish-Chandra's reduction theory), we established the decidability of the 'orbit problem' for any rational action of any arithmetic group on a $\mathbb{Z}$-lattice, and along the way gave an effective procedure for finding finite generating sets for arithmetic groups [**12**, **19**]. Subsequently, this was all generalized to the case of S-arithmetic groups [**28**, **37**]; the extra ingredient needed here was the theory of Bruhat–Tits buildings (which, like many other things, Fritz had to explain to me). These results had many implications: roughly speaking, any Diophantine problem involving a linear group action should be decidable. As well as enabling the solution of the isomorphism problem for nilpotent groups, mentioned above, it solves the conjugacy problem in all S-arithmetic groups, and shows the decidability of the equivalence of forms over a ring of algebraic integers (or S-integers). A different Diophantine application concerns the solvability in integers of quadratic equations (in arbitrarily many unknowns). Siegel, in a rather difficult paper, had shown that this is a decidable question. The note [**23**] gives a simple algorithm based on the effective constructibilty of finite generating sets for integral orthogonal groups, a special case of results from [**19**]. Twenty-four years later, the logician Harvey Friedman asked Fritz if one could decide the solvability in *positive* integers: this (more delicate) question is settled positively in [**82**]; a key insight here was Fritz's intuition that a Fuchsian group has 'dense orbits at infinity', which lies behind a sharpened Strong Approximation Theorem for orthogonal groups established and used in that paper.

Affine crystallographic groups. L. Auslander had conjectured in 1964 that the fundamental group of any compact complete affinely flat manifold is virtually polycyclic. Slightly more generally, an *affine crystallographic group* (ACG) is any properly discontinuous group Γ of affine transformations on a real vector space V such that the quotient space $\Gamma\backslash V$ is compact. In the paper [**50**] (with Margulis) Fritz proved that Γ must be virtually polycyclic if the linear part of the action sits inside an orthogonal group of type $\mathrm{O}(n,1)$. Margulis and his co-workers (Abels and Soifer) have since generalized this to several other cases, $\mathrm{O}(p,q)$ for various p and q, but the full conjecture is still open. The paper [**60**] (with D.S.) goes in another direction: it develops analogues of Bieberbach's finiteness theorems (which classify the *Euclidean* crystallographic groups) that apply to all virtually polycyclic ACGs. If the Auslander conjecture is true, this of course means *all* ACGs; the original Bieberbach groups (being virtually abelian) are a special case. The analogy with the Euclidean case is not complete, however: Fritz also constructs an infinite family of commensurable but pairwise non-conjugate ACGs. He continued to work on the topic, with Paul Igodt and others [**75**].

The other main problem in this area was the conjecture, arising from work of Milnor, that every torsion-free polycyclic group is an ACG. In the special case of a *nilpotent* group, this can be translated via the Mal'cev correspondence into a question about rational nilpotent Lie algebras: does every such Lie algebra have the structure of a *left symmetric algebra*? Fritz and I spent some time trying to prove this; eventually, to general astonishment, Yves Benoist showed in 1995 that it is *false* ⟨**2**⟩, by constructing a counterexample: an eleven-dimensional algebra having no twelve-dimensional faithful module. In [**62**] (with Dietrich Burde) Fritz produced a general framework for many such examples; this was subsequently extended in several papers by Burde.

Arithmetic groups. Perhaps the dominant theme in the Grunewald *oeuvre* is the arithmetic and geometry associated to arithmetic subgroups of SL_2. Much of his major book [**65**] is devoted to this topic, which will be discussed in a later section. He was also interested in algebraic properties of arithmetic groups in general: these played a large part in many of the works described above, and he came back to them in a series of papers with Platonov.

The fundamental papers [**71**, **73**] study three related issues: *rigidity* (in the sense of Mostow and Margulis), finite extensions of arithmetic groups and non-abelian cohomology. The main result determines conditions on the radical of an algebraic $\mathbb{Q}$-group that imply rigidity (this means that an isomorphism between two arithmetic groups extends, possibly with some twisting, to a rational isomorphism between the Zariski-closures). This is used to give conditions under which every finite extension of an arithmetic group Γ is arithmetic; this holds, for example, whenever Γ is soluble. It is also shown that in any such finite extension, the finite subgroups lie in finitely many conjugacy classes; this generalizes a famous theorem of Borel and Harish-Chandra about arithmetic groups. A remarkable consequence is the theorem that the cohomology set $H^1(G, \Delta)$ is finite whenever G is a finite group acting by automorphisms on an arithmetic group Δ (a very special case of this had been proved by Borel and Serre in their work on finiteness properties of Galois cohomology).

Building on [**71**] and earlier work in [**60**], Fritz and Oliver Baues produced in [**91**] a profound analysis of the automorphism group of a virtually polycyclic group, showing clearly how it is built out of arithmetic pieces. One remarkable result here is that the *outer* automorphism group is always an arithmetic group.

Group-theoretic zeta functions. Fritz's two most cited papers are [**47**] (with D.S. and Geoff Smith) and [**77**] (with Marcus du Sautoy). A number theorist at heart, Fritz was always interested in arithmetical sequences that can be associated to infinite groups. By analogy with the Dedekind zeta function of a number field, one defines the *zeta function* of a group G to be

$$\zeta_G(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

where a_n is the number of subgroups of index n in G. When G is finitely generated and nilpotent, ζ_G splits as an Euler product, the local factor at a prime p being

$$\zeta_{G,p}(s) = \sum_{n=0}^{\infty} a_{p^n} p^{-ns}.$$

The main result of [**47**] is that, for each prime p, $\zeta_{G,p}(s)$ represents a *rational function* $Q_p(p^{-s})$ of p^{-s}. The nature of these rational functions was, and to some extent still is, deeply mysterious; several explicit examples are computed in [**47**], and several conjectures are formulated. These have generated much work by several authors (notably du Sautoy and Voll) concerning the variation of Q_p as p varies over all primes, the existence of 'local functional equations', and the analytic properties of the global function ζ_G. Another, related, kind of zeta function, also discussed in [**47**], is the Igusa zeta function $\zeta_{\mathfrak{G}}$ associated to a linear algebraic group $\mathfrak{G}$ (when $\mathfrak{G}$ is the algebraic group associated to $\mathrm{Aut}(G)$, $\zeta_{\mathfrak{G}}$ is the generating function for the number of subgroups H of index n in G such that $\widehat{H} \cong \widehat{G}$).

These global zeta functions are not in general well-behaved compared with the classical zeta functions of number theory; but they are tantalizingly similar to them, and Fritz spent several years exploring their behaviour. On the basis of extensive computations, he came to formulate the concept of a 'ghost zeta function', a 'nice' function that ζ_G or $\zeta_{\mathfrak{G}}$ is in a sense trying to be. The papers [**68**, **79**] (with du Sautoy) set up the theory of these 'ghosts' and determine them explicitly for Chevalley groups $\mathfrak{G}$.

Classically, analytic number theory uses the analytic properties of zeta functions to deduce growth properties of numerical sequences. The ground-breaking paper [**77**] takes this approach to the 'subgroup growth' sequence (s_n) of a finitely generated nilpotent group G (here $s_n = \sum_{j=1}^{n} a_j$). The key idea of this paper is the concept of a *cone integral*, a kind of p-adic integral defined by polynomial data. It is shown that each local factor $\zeta_{G,p}$ is a cone integral. By analysing these and applying Lang–Weil type estimates, Fritz and du Sautoy show first that the abscissa of convergence α_G of the Dirichlet series ζ_G is a rational number; using more algebraic geometry they go on to show that ζ_G can be approximated by a suitable Artin L-function, and deduce that the complex function $\zeta_G(s)$ has a meromorphic analytic continuation a little to the left of the line $\mathrm{Re}(s) = \alpha_G$. Standard methods of analytic number theory (Tauberian theorems) allow them to infer the asymptotic estimate

$$s_n \sim cn^{\alpha_G}(\log n)^{\beta},$$

where $c \in \mathbb{R}$ and β is a non-negative integer.

As Lubotzky wrote in *Math Reviews:* 'With the current paper . . . the authors do more than just solve some major problems: by a detailed analysis of the "p-adic cone integrals" . . . the authors give quite an explicit method to compute $\zeta_{G,p}(s)$, by transforming the problem to the understanding of various explicitly presented rational varieties taken mod p . This gives new insight for future research and connections between subgroup growth, nilpotent groups and algebraic geometry over finite fields'.

Profinite completions. From the beginning of his professional career, Fritz was intrigued by the question of what it means for groups to have the same profinite completion. Grothendieck had famously asked in 1970 whether a homomorphism between two residually finite, finitely presented groups that induces an isomorphism between their profinite completions is necessarily an isomorphism. The 2004 paper [**84**] (with Martin Bridson) constructs examples to show that the answer is negative; these examples also answer another question of Grothendieck: they have infinite index in their Tannaka duals.

A group G is said to be 'good' (a concept introduced by Serre) if the natural map $G \to \widehat{G}$ induces an isomorphism $H^n(\widehat{G}, M) \to H^n(G, M)$ for every n and every finite G-module M. In [**97**] (with Andrei Jaikin-Zapirain and Pavel Zalesskii) Fritz shows that his favourite Bianchi groups, as well as all 'limit groups', are good. The proof for Bianchi groups is very much 'a 3-manifolds proof', using Haken hierarchies.

In continued collaboration with Zalesskii, he returned to the question (considered in 1980 for polycyclic groups) 'how big is the genus of a group?' The *genus* of a residually finite group G is the set of isomorphism classes of residually finite groups having the same profinite completion as G. The posthumous paper [**107**] (with Zalesskii) shows that if G is a finitely generated virtually free group, then the f.g. virtually free groups in the genus of G lie in finitely many isomorphism classes, and even gives a formula for the number of such classes. This is very much a work in progress, raising a number of challenging problems.

The automorphism group of $\mathbf{F}_n$ (thanks to A. Lubotzky). Suppose that $\pi : F \to G$ is an epimorphism from an n-generator free group ($n \geqslant 2$) to a finite group G. Then

$$A(\pi) = \{\alpha \in \mathrm{Aut}(F) \mid \pi \circ \alpha = \pi\}$$

is a finite-index subgroup of $\mathrm{Aut}(F)$ preserving $R = \ker \pi$ and acting on $\overline{R} = R/[R, R]$ by G-module automorphisms. A classic result of Gaschütz shows that $\mathbb{Q} \otimes \overline{R}$ is isomorphic as $\mathbb{Q}[G]$-module to $\mathbb{Q}[G]^{n-1} \oplus \mathbb{Q}$. The paper [**102**], with Alex Lubotzky, studies the action of $A(\pi)$ on this module and shows that, under certain assumptions on π, the image of $A(\pi)$ is an arithmetic subgroup in the algebraic $\mathbb{Q}$-group of all $\mathbb{Q}[G]$-module automorphisms of $\mathbb{Q} \otimes \overline{R}$. In this way, one gets a large collection of arithmetic groups appearing as (virtual) quotients of $\mathrm{Aut}(F)$; examples include $\mathrm{SL}_{(n-1)k}(\mathbb{Z})$, $\mathrm{SL}_{n-1}(\mathbb{Z}\sqrt{-1})$ and others.

As well as having immediate applications, for example, that $\mathrm{Aut}(F_3)$ is 'large'[†] and hence does not have Kazhdan's Property T, this opens up many questions; the most natural one is whether classical results known for the Torelli group (the kernel of the natural mapping

[†] A group is *large* if some subgroup of finite index has a non-abelian free quotient.

$\mathrm{Aut}(F) \to \mathrm{GL}_n(\mathbb{Z})$), such as finite generation, also hold more generally for the analogous subgroup of $A(\pi)$.

In the last week of his life, Fritz was working with Lubotzky on extending the above ideas to mapping class groups. They developed a 'symplectic Gaschütz theory', in which the free group F is replaced by a surface group Γ. They were able to describe the algebraic groups arising in this context in terms of data from the representation theory of the finite group G; but it remains to show that the image of a corresponding subgroup of $M = \mathrm{Out}(\Gamma)$ is arithmetic, under suitable conditions; the case where G is abelian has been done by Looijenga ⟨**11**⟩.

Verbal dynamical systems (thanks to B. Kunyavskii and E. Plotkin). The paper [**88**] (with several co-authors) typifies Fritz's *modus operandi*; it initiated a lively and ongoing research programme, with Fritz (in his last few years) as the main driving force. The original motivating problem was to construct an explicit, simply defined, sequence of two-variable group words $u_n(x, y)$ with the property that a finite group G is soluble if and only if $u_n(G) = 1$ for some n (in analogy to the well-known sequence of Engel words that characterizes finite nilpotent groups in a similar way). The first stage of this project, invisible in the final publication, was the search for suitable words: Fritz always carried a powerful laptop and he kept it running for weeks, trying out candidates for the word u_1. The clever human–computer interaction involved in this procedure is described in ⟨**10**⟩. Having picked u_1, he defines the sequence recursively by

$$u_{n+1}(x, y) = [u_n(x, y)^x, u_n(x, y)^y].$$

Now the problem comes down to proving that in every minimal simple group G there exist elements $a,\ b$ such that

$$u_1(a, b) = u_2(a, b) \neq 1.$$

Considering each such G as a matrix group, this is interpreted as a system of equations in the matrix entries; the problem is thus turned into one of algebraic geometry over finite fields. Its solution is a tour de force, using highly sophisticated tools from algebraic geometry and a significant amount of computer algebra. Shortly afterwards, J. Bray, J.S. Wilson and R.A. Wilson found an alternative, simpler approach to the original problem ⟨**3**⟩; but as the authors of [**88**] point out, 'We also develop a new method to study equations in the Suzuki groups. We believe that, in addition to the main result, the method of proof is of independent interest: it involves surprisingly diverse and deep methods from algebraic and arithmetic geometry, topology, group theory, and computer algebra (Singular and Magma)'. The posthumous paper [**100**] is a further step along this new path: 'We study dynamical systems arising from word maps on simple groups. . . . These results . . . give rise to some new phenomena and concepts in the arithmetic of dynamical systems'.

In continued collaboration with a subset of the authors of [**88**], Fritz turned to the problem of characterizing the soluble radical $\mathcal{R}(G)$ of an arbitrary finite group; in a series of papers culminating with [**106**], it is shown that $\mathcal{R}(G)$ consists of those $y \in G$ such that every four conjugates of y generate a soluble subgroup (a best possible result); moreover, if y has prime order at least 5, then $y \in \mathcal{R}(G)$ if and only if every *two* conjugates of y generate a soluble subgroup (results independently obtained by P. Flavell, S. Guest and R. Guralnick around the same time). This implies that a finite group G is soluble if and only if every pair of conjugate elements lie in a soluble subgroup, a common generalization of celebrated theorems of Baer/Suzuki and of Thompson.

Number theory and hyperbolic manifolds (thanks to G. Harder, A. Reid, P. Sarnak, J. Schwermer)

Fritz had a lifelong interest in the cohomology of arithmetic groups over number fields. His work touched on all facets of this subject: geometric investigations of the corresponding locally symmetric spaces, explicit constructions of fundamental domains, and relations with the theory of automorphic forms.

Much of this work was focused on arithmetically defined hyperbolic manifolds, in particular hyperbolic 3-manifolds. These originate with arithmetically defined Kleinian groups: discrete groups of orientation-preserving isometries of hyperbolic 3-space H^3. The arithmetic Kleinian lattices Γ fall naturally into two classes, according to whether H^3/Γ is compact or non-compact. Fritz's interest was mainly in the second class, comprising the *Bianchi groups* $\mathrm{PGL}_2(\mathcal{O}_d)$ and their subgroups of finite index, where $\mathcal{O}_d$ denotes the ring of algebraic integers in an imaginary quadratic field $\mathbb{Q}(\sqrt{d})$.

Fritz was among the first to relate specific cuspidal automorphic forms for Bianchi groups with elliptic curves. In the late 1970s, encouraged by G. Harder, he started to look at eigenforms for the Hecke operators acting on the cohomology of a congruence subgroup $\Gamma(\mathfrak{a})$ in $\mathrm{PGL}_2(\mathcal{O}_d)$. Given such an eigenform, the aim was to find an elliptic curve E defined over $\mathbb{Q}(\sqrt{d})$, with good reduction outside the ideal $\mathfrak{a}$ defining $\Gamma(\mathfrak{a})$, such that, for a large number of prime ideals $\mathfrak{p}$ in $\mathcal{O}_d$, the eigenvalues of the Hecke operator $T_\mathfrak{p}$ are equal to the eigenvalues of the Frobenius $\Phi_\mathfrak{p}$ on the Tate module. Fritz approached the problem experimentally and found many examples that gave strong evidence for the conjectural underlying relation between cuspidal forms and 'motives'. A first account of this work was given in [**7**]; the manuscript [**14**] documenting more of these computations unfortunately never appeared in print, though it is quoted by Cremona ⟨**6**⟩, and the results were reported in [**30**]. (Fritz's gentle manner concealed a steely determination when it came to serious computation. He was banned for a while from using the Bielefeld University mainframe after his program had monopolized the entire system: in order to carry out the heavy-duty computation required for this project, he had devised a routine that managed to bypass the automatic quota checks.)

Modularity for rational elliptic curves (Wiles, Taylor and others) has been one of the major achievements in number theory in the last few decades. At a very early stage, Fritz's work already pointed towards a possible generalization to other number fields. Some of his early conjectures were subsequently confirmed by Berger and Harcos ⟨**1**⟩, Harris, Soudry and Taylor ⟨**9**⟩ and Taylor ⟨**14**⟩. Using their results, one can associate to an automorphic form a family of compatible Galois representations; an elliptic curve also has such a family of representations. If one has an idea which elliptic curve to consider, one can check whether or not the corresponding L-series coincide ⟨**7**⟩. However, although this recent work supports the conjectured relation in the case of imaginary quadratic number fields, the result is far from established. Still missing in particular is an intrinsic construction for the elliptic curve, as envisaged by Fritz, for example, via an embedding of certain Riemann surfaces into a suitable Bianchi orbifold.

Fritz's approach to the cohomology of Bianchi groups was always very direct. The papers [**22**, **24**, **25**] (with Joachim Schwermer) establish the existence of so-called interior cohomology classes for almost all Bianchi groups by using a very explicit construction of geometric objects, based on a detailed knowledge of the corresponding fundamental domains. The paper [**25**] was the genesis of the 'Cuspidal Cohomology Problem'; here it is proved that there are only finitely many values of d for which the group $\mathrm{PSL}(2, \mathcal{O}_d)$ has trivial cuspidal cohomology. From a topological perspective, the interesting consequence is that only finitely many Bianchi orbifolds Q_d can admit a finite sheeted cover that is a link complement in S^3. Subsequently, it was shown (through work of many people culminating in the final result of Vogtmann) that there are just fourteen such values of d.

Fritz and his co-authors also made detailed studies of the low index subgroup structure for some Bianchi groups for some small values of d. The paper [**61**] (with U. Hirsch) illustrates Fritz's ease with topological methods, containing some wonderfully detailed constructions using handle decompositions of link complements covering the Bianchi orbifold Q_7. Also prescient was the Grunewald–Schwermer result [**22**] that the Bianchi groups are *large*. The largeness property is conjectured to hold for all lattices in $\mathrm{PSL}(2,\mathbb{C})$ and this has become one of the main focuses of recent work in 3-manifold topology.

In order to investigate these and related arithmetical problems, Fritz and his collaborators, most notably Elstrodt and Mennicke, developed the analytic and arithmetic theory of automorphic forms for hyperbolic manifolds in three and higher dimensions. The tools used range from topological and cohomological ones to spectral, analytic as well as arithmetical algebro-geometric ones. Among their many important results are ones giving bounds towards the generalized Ramanujan/Selberg conjectures for the spectra of such hyperbolic manifolds [**51**]. Their book *Groups acting on hyperbolic space: harmonic analysis and number theory* [**65**] is by now the classical text in the subject. It gives a complete treatment of the hyperbolic geometry and related topology, spectral theory of the Laplacian and related Hecke operators, as well as the number theory that is needed to study these manifolds.

Fritz's recent paper [**105**], with Finis and Tirao, gives a comprehensive account of what is known today about the cohomology of arithmetic and some non-arithmetic hyperbolic 3-manifolds, and again, through experimentation, it points to a number of new phenomena and problems that will continue to drive research in this area for years to come.

Algebraic geometry (thanks to I. Bauer and F. Catanese)

Group actions have long been an important subject of study in algebraic geometry. In particular, they have been a source for the construction of new algebraic varieties, especially exotic ones presenting 'pathological' behaviours. In the last few years, Fritz was collaborating on several related projects along these lines with Ingrid Bauer and Fabrizio Catanese.

His first important contribution to the field of algebraic surfaces, in an appendix to ⟨**5**⟩, was the construction of what are now called *Kuga–Shavel–Grunewald* surfaces. These are compact quotients of the bi-disc by a discrete subgroup of $\mathrm{PGL}_2(\mathbb{R}) \times \mathrm{PGL}_2(\mathbb{R})$ such that every commensurable subgroup acts freely: here Fritz used his knowledge of quaternion algebras to show the existence of these rigid surfaces, which constitute countably many 'QED classes'.

Other works deal with the classification and construction of surfaces in the difficult region where the geometric genus is zero. Especially interesting here is the explicit determination of fundamental groups, for which it is easy to write a presentation, but hard to obtain a precise description. An important structure theorem is based on results obtained in [**97**], where a sufficient condition for extensions of 'good' groups to be again 'good' is given.

A major ongoing project [**89**, **94**] concerns the action of the absolute Galois group on varieties defined over number fields, and the corresponding change in the fundamental groups (which preserves profinite completions). The latter paper constructs several interesting actions of the absolute Galois group on the set of connected components of the moduli space of minimal surfaces of general type, one of which is conjectured to be faithful.

Another project transforms into group-theoretical questions the investigation of certain rigid algebraic surfaces that generalize a construction due to Beauville. Given a group G, one seeks equivalence classes, called *Beauville structures*, of generating triples a, b, c and x, y, z satisfying $abc = 1$ and $xyz = 1$, and some other more complicated properties. Which finite groups admit a Beauville structure? Following preliminary results obtained by Fritz, Bauer and Catanese, the question was taken up by several other mathematicians; the conjecture [**85**] that all non-abelian finite simple groups except for A_5 admit a Beauville structure has recently been proved by R. Guralnick and G. Malle ⟨**8**⟩.

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