JACQUES HADAMARD

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1. Jacques Hadamard died on 17 October, 1963, at the age of 98. He published his first mathematical paper of importance in 1888, and continued working until he was over 90, covering an immense range of mathematical subjects including educational, philosophical and psychological aspects of mathematics. To classical analysts his name has been well known as the author of the Hadamard gap theorem, the Hadamard three circle theorem, the Hadamard factorization theorem for integral functions and others results on Taylor series published before 1900, but perhaps it was his proof of the prime number theorem in 1896 more than anything else which made Hardy describe him in 1944 as the " living legend " in mathematics. To some pure mathematicians he may be better known by his theorem on the modulus of a determinant which plays such an essential part in the Fredholm theory of integral equations, or because he invented the name for functional analysis.

His work on the theory of propagation of waves and partial differential equations was no less significant, but it is less easy to pinpoint particular results. It is at the base of the modern theory of both subjects; it includes much work on the Cauchy problem and on the technique of the finite parts of integrals which, although superseded by the theory of distributions, proved to be very useful. He was no physicist, but he helped to lay the foundations of the modern theory of shock waves.

No one person could do justice to such an enormous range of mathematical activity which was matched by wide interests outside mathematics, and what I have to say owes much to the assistance of others. In particular Mlle Jacqueline Hadamard has supplied me with much information and copies of articles and speeches about her father, but, owing to the fact that all their belongings were stolen by the Germans during the war, and also because, as she admitted, " he had so little order and method, keeping everything about others and so little about himself ", the information, and in particular the list of published work, is probably incomplete, and the latter is certainly difficult to check. There are over 300 items, many of them published long ago in periodicals not easily traced from the brief descriptions given in the list in the *Selecta* [**284**]. I have drawn largely from speeches made at his Scientific Jubilee [**284a**] in 1936, from articles by Lévy, Mandelbrojt, Fréchet, Julia and Kahane, and from Hadamard's own accounts of events and influences extracted from various articles, books and speeches. I am also particularly grateful to Prof. Fréchet, Prof. Leray, Prof. Kahane, Prof. Temple, Prof. G. N. Ward, Mr. A. E. Ingham, Prof. Truesdell, Dr. F. G. Friedlander, Dr. L. S. Bosanquet and Sir Edward Collingwood for their help.

2. Jacques Salomon Hadamard was born at Versailles (S. et O.) on 8 December, 1865 (he seems to have dropped the second name early in his career); his father, Amadée Hadamard, was professor of Latin at the Lycée Louis-le-Grand, where Hadamard himself was a pupil, and his mother's name was Claire-Marie-Jeanne Picard. She taught the piano and had Paul Dukas among her pupils. Hadamard won prizes at the Concours Général for Latin and Greek, and one of his silver medals was brought back to him after the war by a general who had been one of his students at the Ecole Polytechnique. The general had found it in a shop, probably sold by a German during the war. In replying to the speeches at his Scientific Jubilee, Hadamard said: " Comment me suis-je consacré aux Mathématiques? Il y a des moments où je me le demande. On aurait bien étonné ma famille et moi-même en me le prédisant dans mes premieres années de Lycée. Je puis dédier mon exemple aux parents que désespère l'inaptitude de leurs enfants à triompher des premiers problèmes d'arithmétique; car en Arithmétique, jusque et y compris la cinquième, j'était le dernier, ou à bien peu près. Je me vois encore posant à mon père, lui-même universitaire et dont la conversation m'orientait naturellement vers les choses de l'esprit, une question sur l'Ecole Normale : 'Y fait-on des Mathématiques?'. 'Oui, m'a-t-il répondu, à l'Ecole Normale, Section des Sciences, on fait des Mathématiques " Et moi de tirer tout de suite la conclusion ' Oh alors, ce n'est pas là-dedans que j'irai '. Comme quoi il est bien imprudent de vouloir prévoir l'avenir.

"Je n'avai pas rencontré à ce moment-là, dans cette classe de Cinquième le maître qui devait me faire apercevoir d'autres horizons. Il y a peu d'années qu'a disparu M. Launay à qui revient l'hommage de mon premier et enfantin éveil à la beauté des choses scientifiques.

"Je suis donc, en dépit de ma prophétie, entré à l'Ecole Normale, et à la Section des Sciences." His father did not approve of his son's interest in mathematics.

He shared with Darboux (1864) and Borel (1889) the distinction of being first in the competition for entry to both the Ecole Normale and the Ecole Polytechnique in 1884, and with a total of points beating all previous records he entered the Ecole Normale. As Lévy wrote : " Il put ainsi se spécialiser plus vite dans les mathématiques, ce qui ne l'empêcha pas, comme nous le verrons, de s'intéresser aux questions les plus diverses." Vessiot, who entered the Ecole Normale at the same time, speaking at the Jubilee on behalf of the Association Amicale des Anciens Elèves de l'Ecole Normale Supérieure, describing the situation at the time, said : " C'était en 1884. Jules Tannery venait d'y prendre la direction des études scientifiques. Par la finesse exquise de son esprit, par sa délicate bonté, il eut sur nous une grande influence. Loin de tout dogmatisme, au hasard des interrogations et des problèmes, son enseignement nous apprenait à penser. Nous eûmes les dernières leçons de Bouquet, rigoureux apôtre de la précision, mais dont les cours nous ouvraient peu d'horizons nouveaux. M. Emile Picard, jeune et déjà illustre maître, lui succéda. C'est à son enseignement riche de faits et d'aperçus suggestifs, aux leçons d'Hermite animées d'enthousiasme, aux cours élégants de Darboux, à des conférences de Paul Appell et de M. Goursat que nous avons dû notre première initiation aux mathématiques modernes et vivantes."

Hadamard said of Jules Tannery : " Une lumière éclaira ma vie, comme elle a éclairé celle de tous normaliens de mon âge. . . . Pour nous ce fut le guide scientifique, intellectuel, moral. Quant à moi, je n'oublierai jamais l'entrevue où, dès les premières échangées, j'eus la révélation de sereine et, à la fois, humaine supériorité de l'homme que je devais admirer et aimer toute ma vie. Tout ce que nous avons pu faire les uns et les autres est un peu son oeuvre, car il a laissé quelque chose de sa personne et de son âme dans la personne et dans l'âme de chacun entre nous." Hadamard also said : " Je ne crois pas que aux qui ne l'ont pas entendu puissent se rendre compte de ce qu'était de magnifigue l'enseignement de Hermite, débordant d'enthousiasme pour la Science, qui semblait prendre vie à sa voix et dont il ne se laissait pas de nous faire sentir la beauté, tant il la ressentait lui-même jusqu'au plus profond de son être." However, later he wrote : " In his lectures (Hermite) liked to begin his argument by ' Let us start from the identity ', and here he was writing a formula the accuracy of which was certain, but whose origin in his brain and way of discovering he did not explain and we could not guess." Referring to his good fortune to hear Darboux, one of the founders of differential geometry, he said of Darboux's voice " dont la douceur musicale m'est toujours rester dans la mémoire ". In the second year Picard lectured to Hadamard, and at the Jubilee Picard thought that Hadamard might not remember his lectures, " Nous faisions de modestes problèmes de Mécanique rationnelle ", but, on the contrary, Hadamard said : " Je puis même, à cet égard, compléter vos souvenirs. Il est parfaitement vrai que vous aviez assumé la tâche—devrai-je dire la corvée—de nous entraîner à cet exercice artificiel et lamentable monotone qu'est la problème de Mécanique pour la licence. Vous aviez pu le rendre presque intéressant; je me suis toujours demandé depuis comment vous vous y étiez pris, car je n'y suis jamais arrivé quand ce fut mon tour." However, Picard also introduced him to hydrodynamics, and other parts of mathematical physics and differential geometry : " tout cela dans cet enseignement, le plus magistrale à mon avis, que j'aie entendu, où, il n'y a pas un mot de trop ni un de manque, où, tous les détails accessoires étant à la fois rigoureusement traités et rélégués à leur place, l'essentiel de la difficulté et du moyen employé pour en triompher apparaît en pleine lumière." This kind of perfection was very far from being achieved by Hadamard in his own lectures, but, as we shall see later, it was hardly his aim in teaching.

Poincaré's work seemed a quarter of a century in advance and too remote to exercise an important influence on students until Painlevé showed the way to continue Poincaré's work. On the other hand, the physicist Duhem in the year above Hadamard, was an important influence then and later : " Duhem, qui avait tout de suite adopté le jeune conscrit tout fraîchement débarqué à l'Ecole, et noué avec lui une amitié qui nous a liés pour notre vie." Neither at the Ecole nor at any later period of his life did Hadamard confine his attention to mathematics, but as Vessiot said : " . . . avec cette curiosité d'esprit toujours en éveil à laquelle rien n'échappe qui ait valeur d'information ou de suggestion, tu tirais profit des conversations avec les littéraires comme avec les scientifiques, avec les physiciens et les naturalistes comme avec les mathématiciens. Et sans doute répondrai-je à ta pensée en évoquant ici la mémoire de notre aîné, le physicien Pierre Duhem, qui fut alors, je crois, plus que tout autre, ton ami. N'est-ce pas auprès de lui que tu pris le goût et le sens de la physique mathématique, à laquelle devaient se rapporter plus tard beaucoup de tes plus beaux travaux? "

After agrégation, as a fourth-year student, then as " boursier d'études ", he began the researches which in a few years made him famous. From 1890 to 1893 he was also professor at the Lycée Buffon. After preliminary announcements of results in *Comptes Rendus* in 1888 and 1889, his famous thesis was published in 1892 and in that year he obtained the Grand Prix des Sciences mathématiques for a memoir applying the results of his thesis to the Riemann zeta function.

In 1892 he married Mlle Louise Anna Tremel : as Fréchet said [**282**], " femme intelligente et de grand coeur qui l'a soutenu et aidé efficacement dans toute sa carrière ". She died in 1960, a great loss from which he never really recovered. They had three sons, Pierre, Etienne, Matthieu, and two daughters, Cécile and Jacqueline; the first two sons were most promising mathematicians, one admitted at the Ecole Polytechnique, the other at the Ecole Centrale. They became officers and were killed in the first world war within an interval of less than two months. Hadamard said [**284a**]: " Quand je jette en arrière un regard sur une existence plus longue que la leur, j'y vois le rare bonheur d'une suite d'années magnifiquement épanouies depuis l'année 1892 a partir de laquelle j'ai connu la beauté de la vie, jusqu'en 1916, à partir de laquelle nulle joie n'a plus être pour moi vraiment pure." The third son was killed in North Africa in the last war; he had two sons who are now married and have two children each. Cécile also married and had three children; there are four grand-children and six great-grand-children living. None of those surviving have yet shown any special mathematical ability. Mlle Jacqueline Hadamard remained with her father until the end, helped by her sister and her sister-in-law.

In 1893 Hadamard was called to Bordeaux as maître de conférences; Duhem had left Paris in 1887 and after short periods at Lille and Rennes he also became maître de conférences de physique at Bordeaux in 1893, and the two friends resumed their conversation at the point where they had left off in 1887. Duhem's Lille lectures on hydrodynamics, elasticity and acoustics had been published in 1891, and Hadamard had naturally read them. In the volume of *Mémoires de la Société des Sciences Physiques et Naturelles de Bordeaux* dedicated to the life and work of Duhem [**227**] Hadamard wrote : " Notre réunion à la Faculté des Sciences de Bordeaux me procura la rare fortune d'en compléter la lecture par de précieux et continuels échanges de vues. A cette lecture, à ces échanges de vues, je dois la plus grande partie de mes travaux ultérieurs, presque tous consacré au Calcul des Variations, à la théorie d'Hugoniot, aux équations aux dérivées partielles hyperboliques, au principe de Huyghens." Hadamard only remained at Bordeaux until 1897, by which time all his most famous papers on classical analysis, culminating in the proof of the prime number theorem in 1896 had been published. While at Bordeaux he published several papers on dynamics, in particular the one for which he received the Prix Bordin of the Academy. In fact the prize was for work on geodesics, and the memoir was mainly concerned with geodesics, which also enter explicitly or implicitly into much of the later work which stems, as we have seen, from his contact with Duhem. Hadamard was awarded the Prix Pétit d'Ormoy in 1903 for the whole set of his mathematical work, and in 1907 he shared the Prix Vaillant with three others.

Returning to Paris as maître de conférences at the Sorbonne in October 1897, he became professeur adjoint in February 1900; in November 1897 he also became suppléant to Maurice Lévy in the Chair of Mécanique Analytique et Mécanique Céleste at the Collège de France.

About this time Hadamard was much concerned about the Dreyfus case, which began in 1894. Dreyfus' wife was the daughter of David Hadamard, a cousin of Jacques Hadamard's father. It is said that Hadamard confided to his friend Painlevé that he had only seen Dreyfus once and that " sa figure ne lui avait guère plu ". Painlevé gossiped, and some days later people read in the newspapers " même la famille de Dreyfus ne croît pas à son innocence ". At once Hadamard involved Painlevé in the action in favour of Dreyfus. The agitations involved Zola, and led to a fresh trial in 1899 with extraordinary scenes after which Dreyfus was pardoned, but he was not satisfactorily cleared until much later. During the Zola trial in 1898 the idea was launched which led to the founding of the Ligue des Droits de l'Homme to rally "all those who without distinction of religious belief wish for a sincere union between all Frenchmen and are convinced that every kind of arbitrary action or intolerance threatens civil disturbance and is a menace to civilization and progress ". Hadamard sat on the Central Committee of the Ligue from its foundation until less than a year before he died, when Mlle Jacqueline Hadamard took his place. In the 1920's he made several contributions to the *Cahiers* of the Ligue. It was never as a relative of Dreyfus that he was concerned, but for justice itself, and throughout his busy life, however many honours were showered upon him, he always had the preoccupations of " l'homme de bien, à qui aucune inquiétude humaine n'a paru étrangère", as his pupils and friends wrote in their dedication of the *Selecta*.

The Collège de France is very proud of its independence, an independence and freedom not enjoyed elsewhere in France. As Hadamard himself said, " Suivre la fantaisie scientifique en toute liberté c'est la tradition et l'honneur du Collège de France ". All the titles of the courses had to be approved by the Assemblée des professeurs (even though only two or three colleagues understood them), and the rule was to facilitate the discoveries of tomorrow and to that end to teach new knowledge and also knowledge forgotten or not appreciated and, above all, not to repeat oneself. The title of the chair was of little importance; Hadamard succeeded Maurice Lévy in the same chair in 1909, and, although a chair with a more suitable title could have been arranged, he obstinately retained it until he retired in 1935. It was considered that he did no more than his duty in lecturing on prime numbers (including his own recent work), a subject not then taught elsewhere in France. But perhaps the title had some influence on his choice of topic in the case of the courses given in 1898–99 and 1899–1900, which formed the basis of his book *Leçons sur la propagation des ondes*. A later course of " leçons " " recuilliées " by Fréchet formed the preliminary stage of his book on the calculus of variations. Fréchet had been at the Lycée Buffon, where Hadamard noticed his interest in mathematics and continued to encourage the boy all through the Bordeaux period. Some professors used to use great ingenuity in thinking of new titles, but Hadamard used few titles and repeatedly considered a subject from a new point of view under an old title.

It is perhaps worth saying something at this stage about Hadamard's views on the teaching of mathematics and his activities in support of those views. We have seen that he taught at the Lycée Buffon for three years, and Darboux asked him to write on geometry for his series " Cours complet de Mathématique élémentaire ". Two substantial volumes appeared, that on plane geometry [**59**] in 1898 and that on solid geometry [**71**] in 1899. Lebesgue described it as " chef d'oeuvre inégalé ", and Julia recorded how in the holidays after he had received the book on plane geometry as a prize it became a companion to him and helped to turn his thoughts towards becoming a mathematician.

The content of the first volume ranges from Euclid's theorem on adjacent angles, with which English text books used to begin, to inversion and pole and polar. There are some interesting " Notes " at the end, and part of the Note " Sur la méthode en géométrie " was reprinted in the *Selecta*. It is a detailed description of how to set about trying to prove a geometrical proposition on lines now familiar from Pólya's use of it in *How to solve it*. For instance : " il faut intervenire dans le raisonnement l'hypothèse et même en général toute l'hypothèse." The second volume begins with lines and planes; among other topics it includes polyhedra, and conics, both as plane curves and as sections of a cone; there is a chapter on surveying and something about developable surfaces and about rotation groups. The order and emphasis must have been very unusual for those days, and I should think that the level of difficulty was too uneven for it to be generally useful.

In the preface to the first volume Hadamard wrote, apropos of the note on method : " J'ai voulu résumer les premiers principes de la méthode mathématique : principes dont les commençants devraient êtres pénétrés dès la première année d'enseignement et que, cependant, l'on voit trop souvent méconnus par les élèves même de nos écoles supérieures. La forme dogmatique qui j'ai dû adopter n'est pas, il faut l'avouer, celle qui convient le mieux dans l'espèce: un tel sujet doit s'enseigner par une sorte de dialogue dans lequel chaque règle intervient au moment même où sa nécessité apparaît." At that time the heuristic method was being used in Germany, but was being strongly opposed in France. Hadamard wrote [**108**] that some of the objections were full of irony; for instance, some said that " faire travailler les élèves et les exercer à l'observation ' c'est moins fatiguant que de faire un cours '." Other objections, such as loss of time, confusion of ideas and the risk of losing sight of an objective, he admitted as sensible, and with the multitude of ideas which crowded into his mind his own courses were not always free from such defects. Years later in the preface to his *Cours d'Analyse* [**221**, **250**], given at the Ecole Polytechnique, he complained of " des exigences multiples et même quelque peu contradictoire qui s'imposent . . . par la condensation, la rapidité presque télégraphique d'un cours au quel le temps . . . est si parcimonieusement mésuré ", of the necessity for the course to be " d'une culture élevée ", and, also since it was for future engineers, not to lose sight of practical applications.

According to M. Roustan, Minister of Education in 1936, " il a contribuée puissament à élargir la place des sciences mathématiques et expérimentales dans la formation du lycéen moderne", and at the end of the Jubilee ceremony, in the name of the Government and of the Université française, Hadamard was awarded the grade of Commander in the Order of the Légion d'honneur. Hadamard was a member of Council, Vice-President and President of the Société Mathématique de France, and frequently attended sessions, taking an active part and, moreover, according to Fréchet he was unequalled as a " sergent recruteur " for the society.

His intense interest in mathematical method was part of his concern with the philosophical and psychological problems of how great mathematical results are obtained. He lectured and wrote on these topics from the time of the Philosophical Congress of 1900 until 1959, when he revised the French translation of his book, *Psychology of invention in the mathematical field* [**295**, p. 322]. This book contains much interesting information about his thought processes, and the circumstances and methods of approach to his most important results, some of which I shall quote later.

When Hadamard was elected to the Chair at the Collège de France in 1909 he ceased to lecture at the Sorbonne, but he was entrance examiner at the Ecole Polytechnique in 1910 and 1911, and was elected to succeed Jordan in the Chair of Analysis there in 1912. At the Jubilee, Général Hachette, Commandant l'Ecole Polytechnique, recalled the appreciations of Humbert, Painlevé, Tannery and Poincaré at the time of the election : Humbert mentioned partial differential equations and Hadamard's interest in applications to engineering, and Poincaré spoke of " des travaux considérables et de premier ordre sur de nombreuses branches des mathématiques, en particulier sur les ondes élastiques ". Hadamard's entry was triumphal, and was followed in a few months by his election to the Academy on the death of Poincaré, and in 1920 he succeeded Appell in the chair of mathematical analysis at the Ecole Centrale des Arts et Manufactures.

In 1913 he began what became his famous seminar, the first in France on mathematics, although Marcel Brillouin had conducted one on physics which Hadamard had enjoyed enormously in 1899. Hadamard's seminar took place at the Collège de France, where the smaller numbers made possible a type of contact which was almost impossible at the Ecole Polytechnique. Paul Lévy describes Hadamard's system thus : " Au début de l'année, il avait établi une liste des livres ou mémoires qu'il désirait voir analyser, et ce programme comprenait ainsi, suivant sa propre expression, tout ce qui l'amusait; cela allait de la géométrie à la théorie des fonctions, et des problèmes concrets relatifs à des équations particulières aux problèmes les plus abstraits de l'analyse fonctionnelle. Bref, presque tout, dans le domaine mathématique, l'intéressait. Avant la rentrée, il réunissait ses collaborateurs, et on se partageait la tâche. Pendant les séances du séminaire, il m'arrivait de ne pas suivre un exposé trop ardu, et j'admirais que, pour lui, rien ne parut difficile. Toujours attentif, il intervenait souvent pour préciser un point mal expliqué par l'orateur, et si, par hasard, quelque chose lui échappait, aucune fausse honte ne l'empêchait de le dire et de demander des explications complémentaires." The war interrupted the seminar, but he soon revived it, and at a time when French mathematicians were inclined to fall back on their own resources it was in Dieudonné's words " l'unique porte restée ouverte sur l'extérieur ". The work of preparation for the seminar was immense, as Lebesgue said : " il faut choisir les mémoires, s'en procurer des exemplaires à repartir entre les collaborateurs et surtout recruter ceux-ci. Il faudra se tenir au courant, jour par jour, de l'état d'avancement de leur travail pour établir l'ordre du jour des séances en s'assurant la présence de telles personne dont la compétence permettra de tirer tout le profit de l'étude entreprise; bien d'autres soins encore sont indispensables pour obtenir de tous une collaboration fructueuse et sans heurts." Every week a mathematician, sometimes a well-known foreign mathematician, used to speak about his own work or about a paper recently published in the subject he knew best. That is, as Mandelbrojt explained : " the subject they knew better than any other subject; but usually that same subject, whatever it was, and whoever spoke, was still better known, or at least better understood, by the Master, Professor Hadamard." The seminar continued to meet twice a week for more than 20 years, and there " étudiants et professeurs, français et étrangers, s'initient fraternellement, confidentiellement à la science qui se fait ".

Whatever Hadamard did, he did with enthusiasm; at committee meetings of the Academy and elsewhere he was always asking for the texts of communications, and explanations about proposals for prizes which were seldom discussed by others, and Kahane wrote of him sitting on the platform at meetings of democratic organizations " où nous l'avons souvent vu tapotant la table de ses mains agiles, attendre avec impatience la fin d'un exposé pour donner son avis; car ce n'est pas à titre passif qu'il accepte de participer à un movement, mais toujours avec toute son energie et sa flamme."

Honours were showered on him; he was a foreign member of the Academies of the Lincei, of Belgium, Holland, India and the U.S.S.R., of the National Academy of the U.S.A., the Royal Society of London, the Royal Society of Edinburgh and the London Mathematical Society, and many others. He gave lectures and courses all over the world, and many of them were published. For instance, those at Columbia in 1911, and at Rice Institute, Texas, in 1920, were published, and the Silliman lectures at Yale formed the basis of his book *Lectures on Cauchy's problem in linear partial differential equations*, which is his major contribution to that theory. He read papers at most of the international congresses of mathematicians from the first at Zurich in 1897 to the Harvard congress in 1950, with the exception of the 1924 congress at Toronto. He led the French delegation at Zurich in 1932, and was one of three honorary presidents of the Harvard Congress. He was President of the International Commission on the Teaching of Mathematics at the time of the Oslo congress in 1936, but was absent in China, where he was professor at the Academica Sinica and at the University Tsing-Hoa in Pekin. Unfortunately the manuscript of the course he gave there was sent to Pekin and seems to have been lost.

Hadamard collected ferns until he was well over 80, and his collection was considered to be the next best in France after that in the Museum and that of Prince Bonaparte. He was also interested in fungi. At each stop on the Trans-Siberian railway he sallied forth in search of ferns and fungi, to the alarm of Mme Hadamard, who feared that he would not be back when the train started again. When he went to Rio de Janeiro to lecture after the war, he arranged to visit the tropical forests near Natal (Brazil) and also the forests of Parana. In these travels Mme Hadamard accompanied him whenever she could and endeavoured to reduce fatigue for him. In Brazil she rode a horse for the first time in her life, being then over 60. Hadamard loved mountain climbing, and climbed Mont Blanc when he was over 60. Kahane wrote that he represented the mathematician the public imagines : " Il pense à autre chose qu'aux voitures lorsqu'il traverse une rue, il n'a jamais su faire le noeud de son cravate."

Hadamard loved music and used to have a small orchestra for amateurs in his house; Einstein played in it whenever he was in Paris; Duhamel, the writer, was the flautist; Hadamard played the violin, and Mme Hadamard played the piano, supplementing by playing the parts of the brass instruments when required.

Hadamard recognised the danger of Hitlerism very early and, although a free thinker and anti-zionist, he was against all racial discrimination and worked to help the Jews in Germany in a more enlightened way than the Israelite Consistory and Zionist circles. With Paul Langevin he schemed to get a chair created for Einstein in France.

When the second world war broke out, it was not until 1941 that Hadamard with his wife and Mlle Jacqueline Hadamard escaped to the United States. They did this with the help of Rapkine, a devoted young French scientist from Canada, who risked his life to save French scientists. At first Hadamard had a visiting professorship at Columbia, but as the Americans did not wish to use his gifts in the war effort, Rapkine arranged for the family to go to England, where they took part in Operational Research with the R.A.F. They remained in London as long as their work was useful and then returned to France. He also lectured in various British universities during that period, and at the London Mathematical Society in January 1945. I remember him as a little figure lecturing in an overcoat because of the cold, and when he tripped, instead of falling, he ran lightly down the steps. He returned to France in 1945, and on his 80th birthday he was made Grand Cross of the Légion d'honneur.

When the Feltrinelli prize was founded by the Italians in 1955 to compensate for the absence of a Nobel prize for mathematics, he was the first laureate, and in 1962, on the 50th anniversary of his election as a member of Academy of Sciences, a gold medal was struck in his honour. This was a great occasion, and the medal was brought to his house by M. Julia, M. Louis de Broglie, the Secretary of the Academy, and M. Denjoy, the President, who all made speeches in his honour, thus following the precedents created by the 50th anniversaries of the astronomer Hervé Faye and Emile Picard; the former was elected in 1847 and was a member for 55 years, and the latter was elected in 1889 and was a member for 57 years. Hadamard was elected at a later age than Picard and was a member for 51 years.

I must now attempt to evaluate Hadamard's contributions to mathematics. This is extremely difficult, partly because they are spread through so many branches of pure mathematics and even some branches of mathematical physics, partly because his influence in some branches of mathematics made itself felt more strongly through the work of others than by his own writings, and partly because, although his early work has become quite classical, the state of the subject when he wrote was so very different. For when he began to do research the dissemination of mathematical knowledge seems to have been unsystematic and inadequate in many respects. I can only discuss some of the topics involved rather briefly and try to illustrate these difficulties.

His thesis [**6**] on Taylor series for the doctor's degree began with a proof of the so-called Cauchy test for convergence of a power series, $\Sigma a_n x^n$, and included a lucid definition from first principles of an upper limit. In his book [**67**; **222**(*a*)] he attributed the precise definition of an upper limit to du Bois Reymond, but in his thesis he did not mention this and only gave very vague references to the work of Abel and Cauchy, while in the preliminary notice [**3**] he did not even mention Cauchy or Abel. I doubt whether he had read du Bois Reymond or even Cauchy when he obtained the result, and it seems possible that later definitions are based more on Hadamard than du Bois Reymond.

Most of Hadamard's other results on Taylor series and those obtained by him in connection with his work on Taylor series now appear in text books as theorems with his name attached, and so I shall only mention them briefly. Hadamard's gap theorem was improved later by Fabry in the sense that he showed that the circle of convergence is a line of singularities even when the gaps are smaller, but it is worth noticing that the Hadamard gaps, occurring at long intervals, imply over-convergence, as Ostrowski showed. The Hadamard criterion for polar singularities is a completely adequate, but necessarily somewhat cumbersome, answer to the problem. The last part of the thesis contained results about the order of a function in the unit circle and fractional integrals; these now seem very straightforward, but that type of work was new then, and later developments include the work of Hardy and Littlewood on mean values, fractional integrals and summation of series involving moduli of Taylor coefficients.

At the séance of the Académie des Sciences on Monday, 29 December, 1890, the subject for the Grand Prix des Sciences mathématiques for 1892 was announced; it was " Détermination du nombre des nombres premiers inférieur à une quantité donné ", but the announcement specifically mentioned that it was desirable to fill the gaps in Riemann's famous memoir by a deeper study of $\zeta(s)$. In 1885 Stieltjes had announced that he had proved the Riemann hypothesis, and at the Jubilee Lebesgue said that everyone thought that the subject would permit the Academy to *couronner* Stieltjes. However, only two memoirs were submitted: one was too elementary and was disqualified by the fact that the author signed his name instead of distinguishing it by a motto; the other, Hadamard's [**8**], was crowned. The report accepted the fact that Stieltjes had proved the Riemann hypothesis, but said that the arithmetical applications remained to be proved, and quoted Halphen: " Avant qu'on sache établir le théorème de Riemann (et il est vraisemblable que Riemann ne l'a pas su faire) il faudra de nouveaux progrès sur une notion encore bien nouvelle le genre des transcendentes entières." The report said that the author of the first memoir seemed to remember these words and devoted all his efforts to the study of the genus. " Il va au delà des espérances qu'avait pu concevoir l'Académie en mettant la question au concours." Hadamard's motto was a quotation from Pascal, " L'art de démontrer les vérités déjà trouvées et de les éclaircir de telle sorte que la preuve en soit invincible est le seul que je veux donner ", and his starting point was a result by Poincaré on the magnitude of the Taylor coefficients of a function of finite genus. Using the Taylor coefficients he obtained all the relations now well known by Hadamard's name for the number of zeros, product expression and minimum modulus of an integral function of finite order with the appropriate limits of accuracy in terms of the order of the function. He then applied them to show that the function now known as

$$\Xi(z) = \xi(\tfrac{1}{2}+iz), \quad \xi(s) = \tfrac{1}{2}s(s-1)\,\pi^{-\frac{1}{2}s}\,\zeta(s),$$

when considered as a function of z^2 is of genus 0. Some improvements in the statements of the results of this paper were given in [**34**], which also included the enunciation of his famous three circles theorem, but he does not seem to have thought it necessary to give a proof until others had given theirs, and he commented on them in the *Notice sur les travaux scientifiques de M. Hadamard* (1912) [see **284**, pp. 94–95]. The " composition " theorem on the multiplication of singularities is in [**51**].

The proof of the prime number theorem is in [**35**]; at least what is proved, as in de la Vallée Poussin's memoir of the same date, is that the sum of the logarithms of the prime numbers less than x is asymptotic to x. This result was stated by Halphen, and, assuming the Riemann hypothesis, proved by Cahen. The fact that from this result it is easy to deduce the prime number theorem, although established by Chebyschev, does not seem to have been clearly realised until later. In 1898 de la Vallée Poussin wrote that he had been aware of it " déjà longtemps ", in a letter to von Mangoldt who published the first proof of the prime number theorem [see von Mangoldt, p. 70]. Hadamard starts out by saying " Stieltjes avait démontré, conformement aux prévisions de Riemann, que ces zéros sont tous de la forme $\frac{1}{2}+it$ (le nombre t étant réel); mais sa démonstration n'a jamais été publiée, et il n'a même pas été establi que la fonction ζ n'ait pas de zéros sur la droite $R(s)=1$ ": he set out to prove the last statement, and then deduced those arithmetic consequences from it which happened to catch his imagination at the time, including in particular Halphen's statement. A full discussion by Ingham of the relation between the proofs of Hadamard and de la Vallée Poussin will be found in Burkill's obituary notice of de la Vallée Poussin.

It is hard to exaggerate the importance of these papers both for the results achieved and for the future development of the subject. Another much-quoted paper [**10**] is that in which he proved that if the modulus of each element of a determinant Δ of the n-th order is less than 1, then $|\Delta| \leqslant n^{\frac{1}{2}n}$. For this result is essential in the Fredholm theory of integral equations, but when he wrote the paper he had not developed his interest in that side of mathematics.

As may be seen from the numbering of the papers, Hadamard published many other papers while he was at Bordeaux, including several on dynamics, in particular [**21**, **23**, **24**], but the first important memoirs on a subject other than analysis were those [**30** and **53**] for which he received the Prix Bordin in 1896. The subject set was : " Perfectionner en un point important la théorie des lignes géodesiques. Le cas d'un élément linéaires à un nombre quelconques de variables n'est pas écarté par l'Académie." In spite of its dynamical title, even the first paper is largely concerned with geodesics. As we have seen, the Dreyfus case was causing very hostile feelings between different groups in Paris. Hadamard himself used to tell how about this time he went to pay the usual New Year call on Hermite, and was met with the words " Hadamard, you are a traitor ". However, before he could sort out his emotions in relation to the politics involved, Hermite continued " You have betrayed Analysis for Geometry ". It seems probable that Hermite referred to the work for the Prix Bordin. The report on the prize dissertation remarked that there were " fort peu de resultats ", but that " l'auteur avait montré une grande ingeniosité d'esprit, avait mis en avant une foule d'idées nouvelles qui, selon toute apparence, seront un jour fécondes; le temps seul lui manquée pour en tirer un plus grand parti ". It is possible that the publication of Volume III of Darboux's *Théorie des Surfaces* in 1894 which dealt with geodesics may have stimulated Hadamard's interest in the subject, but the actual results on geodesics have probably been less influential than the methods of topological dynamics which were developed in these papers, in particular those relating to recurrent motions and minimal sets. Birkhoff made extensive use of [**30**], referring to it as " un Memoire importante ", and he also used a later paper [**73**] in the same connection. The Jordan theory of circuits, as developed by Hadamard in [**53**], is essential in symbolic dynamics, and one might almost say that Hadamard, in lectures and papers, campaigned on behalf of analysis situs as topology was then called. His contributions to the subject include a paper [**141**] on Kronecker's index, recommended in Alexandroff and Hopf as giving an alternative method of some interest, and one [**88**] in which, according to Bourbaki : " En 1903, Hadamard inaugure la théorie moderne de la dualité ' topologique ', en cherchant les ' fonctionnelles ' linéaires continues les plus générales sur l'espace $C(I)$ des fonctions continues numériques dans un intervalle compact I (espace muni de la topologie de la convergence uniforme), et en les caractérisant comme limites de suites d'intégrales $x \to \int_I k_n(t)\, x(t)\, dt$."

Hadamard took comparatively little part in the development of the theory of sets of points and the theory of integration, which were occupying the minds of Borel and Lebesgue early in this century, but in the Prix Bordin paper he obtained one of the earliest examples of a perfect discontinuous set in connection with the derived sets of certain classes of geodesic. Also it was Hadamard's reaction to Borel's criticism of Zermelo's work that opened the controversy on the multiplicative axiom, and two of the famous five letters reprinted in Borel's *Théorie des Fonctions* were from Hadamard. [Black, pp. 184–185.]

The Prix Vaillant memoir of 1907 must be mentioned; the problem set was that of finding a solution of the equation

$$\Delta\Delta U \equiv \frac{\partial^4 U}{\partial x^4} + 2\frac{\partial^4 U}{\partial x^2\,\partial y^2} + \frac{\partial^4 U}{\partial y^4} = 0$$

for given values of $\dfrac{\partial U}{\partial x}$, $\dfrac{\partial U}{\partial y}$ on a contour, a problem originating in the physical problem of the behaviour of a plate clamped along its edge. Twelve memoirs were submitted; Hadamard received three-quarters of the prize, the remainder being divided between Lauricella, Korn and Boggio. As in the case of the Prix Bordin, Hadamard produced a new method without completing all the obvious applications of it, and the report said " Si riche qu'il soit en resultats acquis, ce Mémoire est plus remarkable encore par ceux qu'il fait espérer ". The report also mentioned " lémmes élégantes (dont l'interpretation physique est immédiate) ". Hadamard formed the functional derivative of Green's function of the first and second orders, that is for the equations $\Delta U = 0$ and $\Delta\Delta U = 0$. He also formulated an equation in functional derivatives analogous to a differential equation of the first order, and integration of the former led to the determination of Green's function for a general surface S when that for a particular surface S_0 was known. P. Lévy included this in his book, and in this connection he recounted the following story :—

" Un jour, après sa leçon, je lui signalai qu'il avait omis de parler d'un problème important. L'analogie avec les équations différentielles n'est pas complète; il y a ce qu'on appelle un problème d'intégralité. Il y a en effet beaucoup de manières de déformer une surface variable S pour l'amener d'une position initiale S_0 à une position finale S_1, et, s'il s'agit d'une équation dont on ne soit pas, comme dans le cas de celle de la fonction de Green, assuré à l'avance de l'existence d'une solution, la détermination de la fonction de ligne étudiée obtenue pour la surface S peut dépendre de la manière dont on a choisi les surfaces intermédiaires entre S_0 et S_1 : ' Je le sais bien,' me répondit-il. ' Je pensais ne parler de ce problème d'intégrabilité qu'après l'avoir résolu. Mais puisque vous avez mis la main dessus, je vous l'abandonne.' Ce fut l'origine de mes recherches sur l'analyse fonctionnelle."

The *Leçons sur le Calcul des Variations* [**137**] contained further developments of the theory of functional derivatives; in particular Hadamard pointed out certain limitations in the applicability of Volterra's normal functional derivative. Hadamard always spoke of Volterra very warmly as his friend, and showed an absorbing interest in Volterra's " fonctions des lignes ", which Hadamard renamed " fonctionnelles ", and so gave the name to a new branch of mathematics, functional analysis. The *Calcul des Variations* exerted a wide influence; for, as Hadamard wrote in the " Avant-Propos " : " Le Calcul des Variations n'est autre chose qu'un premier chapitre de la doctrine qu'on nomme aujourd'hui le Calcul Fonctionnel." In 1911 Hadamard gave a general survey [**149**] of Le Calcul Fonctionnel to which Fréchet referred at the Congress at Bologna in 1928 as " un mémoir prophétique ", and the following quotation from Hadamard's survey is on the first page of Fréchet's " Les Espaces Abstraits " : " Le continu fonctionnel . . . n'offre à notre esprit aucune image simple. L'intuition géométrique ne nous apprend rien, *a priori*, sur son compte. Nous sommes forcés de remédier à cette ignorance et nous ne pouvons le faire qu'analytiquement, en créant à l'usage du continu fonctionnel un chapitre de la théorie des ensembles."

I now come to my most difficult task, the discussion of Hadamard's contributions to the theory of the propagation of waves and to the theory of partial differential equations which are contained mainly in *Leçons sur* *la propagation des ondes* [**91**] and *Lectures on Cauchy's problem* [**187**, **261**], with some later papers on Huyghen's principle [**204**, **222**]. The starting point was Hugoniot's work on the rectilinear movement of a gas or rather Duhem's exposition of it in his lectures and in the conversations to which I referred earlier in the years 1893–97. It emerged from Hugoniot's posthumous memoirs that in order to express the physical situation correctly it was necessary to consider the possibility of mathematical discontinuities, and that approximations by analytic or continuous functions could not give a true representation of the physics. A thorough discussion of movements other than rectilinear movements involved mathematical difficulties far too difficult for most physicists of that time, but these difficulties positively attracted Hadamard and the physical difficulties attracted Duhem. The waves referred to are not those of the theory of vibrations and it may help to quote Hadamard's description of his concept given in a Lecture [**152**]. " Let a perturbation be produced anywhere, like sound; it is not immediately perceived at every other point. There are then points in space which the action has not reached in any given time. Therefore the wave, in that sense a surface, separates the medium into two portions (regions) : the part which is at rest, and the other which is in motion due to the initial vibration. These two portions of space are contiguous."

In the case of shock waves the velocity is discontinuous at the surface; gas passes through the shock front, and the equation for the medium behind the shock front is non-homogeneous. In other cases the movement was said to be " compatible " with that on the other side of the wave surface.

The early history of the theory of shock waves is very complicated; many of the results were obtained independently by a second or third person many years after the original discovery was published. For instance, many of Hugoniot's results had been obtained earlier, some by Riemann in 1860, who in some respects made incorrect assumptions, some by Rankine in 1869, and as late as 1910 Rayleigh felt it necessary to show that Rankine's equations were equivalent to Hugoniot's and said that Rankine's work had been neglected. Hadamard mentions the work of Riemann and Christoffel, but not that of Rankine. A full appreciation of the value of this early work on waves has only come about in comparatively recent times, during the last ten or fifteen years, say, with its application to new fields. A full account of the work of Hadamard and Duhem on the general theory of waves is given by Truesdell and Toupin. Courant and Friedrichs give an account of the theory of shock waves, but their history of the theory is greatly over-simplified for the sake of brevity. Friedlander discusses the aspects of Hadamard's work relevant to sound pulses.

Hadamard set out to give a complete mathematical formulation of the problems of wave motion, separating the kinematical from the dyamical, and thus he made the mathematics available generally for application to other physical problems to those who had sufficient mathematical knowledge and ability.

Truesdell writes : " Hadamard's contributions to general contiuum mechanics may be listed as follows :

" 1. The basic lemma (half anticipated by Maxwell) by which compatibility in general is distinguished from the compatibility of particular, kinematically defined waves.

" 2. Recognition of levels of compatibility : geometric, kinematic, dynamic, energetic, material (though only the first two are clearly developed in his book).

" 3. Classification of kinematical singular surfaces, and construction of a general theory, including an outline of higher-order conditions.

" 4. Calculation of the exact wave speeds in finite elastic strain, and proof that they are all real and non-vanishing if and only if the equations of equilibrium, for the particular strain, are strongly elliptic.

" 5. Proof that weak singular surfaces in gas dynamics do not destroy the circulation-preserving property, and in particular such waves do not invalidate the Lagrange-Cauchy velocity-potential theorem.

" 6. Proof that an oblique curved shock wave in a gas generates vorticity.

" 7. The first rigorous definition and analysis of stability in finite elastic strain, and proof that in stable equilibrium the inequality defining strong ellipticity must hold, provided ' $\geqslant$ ' be replaced by ' $>$ '.

" All this is in his *Propagation des Ondes*. No. 4 is made clear by my article in the *Archive*. No. 7 was not noticed, apparently, until 1952; the modern American literature on continuum mechanics begins to take note of it, and a new proof, filling a lacuna noted by Duhem in Hadamard's argument, is given in the second article for the *Handbuch*, now in press. When I asked Hadamard about No. 7 in 1955, on my way to Berlin to deliver a lecture in which his work on elasticity was summarized in part, he said he had forgotten it."

The mathematical discussion of the physical problem necessitated the study of partial differential equations and in particular the Cauchy problem, which may be stated as follows:

Consider the equation

$$\frac{\partial^n u}{\partial t^n} = F\left(t, x_1, x_2, \ldots, x_p, \ldots, \frac{\partial^k u}{\partial t^{k_0} \partial x_1^{k_1} \ldots \partial x_p^{k_p}}\right)$$

where $0 \leqslant k_0 \leqslant n-1$, $0 \leqslant k_s \leqslant n$ $(s=1, 2, \ldots, p)$. The problem is to find a solution for which

$$\frac{\partial^k u}{\partial t^k} = \phi_k(x_1, x_2, \ldots, x_p), \quad k=0, 1, \ldots, n-1,$$

when $t=0$. Sophie Kowaleski showed that, if F and ϕ_k are analytic in all their arguments in a certain neighbourhood of the origin, then there is one and only one solution regular near the origin. Holmgren showed that there is no other solution with continuous derivatives of the first n orders. Hadamard showed in [**91**] that the uniqueness proof for the non-linear case can be reduced to that for a linear equation with non-analytic coefficients of sufficient smoothness, thus concentrating attention of other mathematicians on the linear non-analytic case for a very long time.

In the Cauchy-Kowaleski result the data are analytic and local, but for non-analytic data there is a great difference between elliptic equations, such as (to take very simple examples)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f\left(x, y, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right), \tag{2}$$

and hyperbolic equations such as

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f\left(t, x, y, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right). \tag{3}$$

The physical problems corresponding to (2) usually give rise to Dirichlet's problem in which u is given as a function of x and y on the *whole* boundary or to Neumann's problem in which the derivative along the normal is given, but in the physical problems corresponding to (3) *both* u and $\partial u/\partial t$ are usually given on *part* of the boundary, *e.g.* for $t=0$. In the case of (2) if u and $\partial u/\partial x$ are *both* given as non-analytic functions of x and y for $x=0$, there is no solution. Hadamard and d'Adhemar reiterated this important distinction many times; Hadamard expressed it by saying that for hyperbolic equations the problem was "well-posed" while for elliptic equations it was "not well posed", and at the beginning of [**75**] he quoted Poincaré: "La Physique ne nous donne pas seulement l'occasion de résoudre des problèmes . . . , elle nous fait pressentir la solution." However, this point of view was by no means readily accepted by other mathematicians, and Hadamard wrote [**187**, p. 33, and **261**, p. 40] that "some argued that you may always consider any functions as analytic, as, in the contrary case they could be approximated with any required degree of precision by analytic ones". The fact that Cauchy's problem for elliptic equations could not correspond to any ordinary physical problem was made clear by the following example first given by Hadamard in a lecture in Switzerland in 1917:

$$u(x, y) = n^{-k} \sinh nx \sin ny$$

is a solution of (2) with $f \equiv 0$ such that $u(0, y)$ and $\partial^s u/\partial x^s$, $s = 1, 2, \ldots, k-1$, are arbitrarily small for $x = 0$, while for fixed $x > 0$, u is arbitrarily large for large n. Hence a very small change in the conditions for $x = 0$ can have a very large effect on the solution near $x = 0$. A survey of work on the Cauchy problem was published by Petrovskii in 1946, and later Leray discussed it and explained the need for a return to the study of the singularities of solutions in the general analytic Cauchy problem in connection with certain physical problems.

As regards Hadamard's work on linear hyperbolic equations Temple writes: "In [**187**] Hadamard gives a summary of his researches on the hyperbolic case and resolves the difficulties of Cauchy's problem which he had encountered in [**91**]. His principal contributions to this subject are the determination of the 'elementary solution' and its successful employment in the fundamental Greenian identity by means of the theory of the 'finite part' of improper infinite integrals. Scarcely less important are his discovery of the relation between elemenatry solutions of hyperbolic equations and their related elliptic equations, and his systematic use of the 'method of descent'.

"To appreciate the significance of Hadamard's investigations it is necessary not only to refer to the earlier researches of Volterra, Tedone, Coulon and d'Adhémar, but also to the subsequent applications which Laurent Schwartz has made of the theory of distributions (or generalized functions) to partial differential equations. It will also help to clarify the exposition if we speak first of the elementary problems of the potential equation and the wave equation.

"In the case of the potential equation

$$\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

the elementary solution is

$$G(P, Q) = -(4\pi R)^{-1},$$

where R is the distance from the source point $Q(\xi, \eta, \zeta)$ to the field point $P(x, y, z)$. The fundamental identity is

$$v\Delta u - u\Delta v \equiv \operatorname{div}\{v \operatorname{grad} u - u \operatorname{grad} v\}.$$

Now let u be any solution of the equation $\Delta u = 0$ in a domain τ with boundary σ, and let $v = G(P, Q)$, where Q is in $\tau + \sigma$. Then we integrate the fundamental identity over the domain τ excluding a sphere centre P and radius ϵ, let $\epsilon \to 0$ and find that

$$u(P) = \int (G \operatorname{grad} u - u \operatorname{grad} G)\, \mathbf{d}\sigma. \tag{P}$$

This relation does not express $u(P)$ in terms of independently prescribed values of u and grad u on the boundary σ. Nevertheless the prime object of Hadamard's researches is to discover the appropriate analogy and generalization of this formula for hyperbolic partial differential equations, which are linear but not necessarily with constant coefficients.

"The first difficulty, encountered by earlier investigators, is that the elementary solutions may have singularities which are not concentrated in a single point, such as $P=Q$ for $G(P, Q)$, but are distributed along a line or over a surface. Thus Kirkhoff's elementary solution of the wave equation in three spatial dimensions x, y, z and one time dimension t is of the form $R^{-1}F(R-ct)$, where the singularity at $R \equiv [x^2+y^2+z^2]^{\frac{1}{2}}=0$ is along a line in space-time. Again the elementary solution of the wave equation in x, y and t is

$$(2\pi)^{-1}[x^2+y^2-c^2t^2]^{-\frac{1}{2}},$$

with singularities spread over the Mach cone

$$x^2+y^2=c^2t^2.$$

And in two variables x and t the elementary solution is

$$(2\pi)^{-1}\log(x^2-c^2t^2)^{\frac{1}{2}}$$

with singularities along the lines $x=\pm ct$.

"This difficulty was evaded by Volterra and Tedone by operating, not with the prescribed dependent variable $u(x_1, x_2, \ldots, x_m, t)$, but with integrals of the form

$$\int_0^t (t-\tau)^{m-2}u(x_1, x_2, \ldots, x_m, \tau)\,d\tau$$

for the wave function in m spatial dimensions.

"In the case of hyperbolic equations with variable coefficients a second difficulty presented itself to Coulon and d'Adhémar, viz., the absence of any guide to the *functional form* of the elementary solution.

"Hadamard's great contribution was to carry out the complete and direct generalization of the familiar formula (P) for the potential function. This achievement appears all the more remarkable when we remember that Hadamard did not have available the powerful methods of the theory of distributions (see Schwartz, II, 64).

"In that theory a partial differential equation with constant coefficients is expressed in the form

$$A * T = 0, \tag{A}$$

where T is a distribution replacing the wave function u, A is a distribution replacing the differential operator, and the star represents a convolution product or product of compositions. Thus if $\delta(x, t)$ is the delta function in x and t, and $A \equiv \delta_{xx} - c^2 \delta_{tt}$, then

$$A * u = \int \{\delta_{\xi\xi}(\xi, \tau) - c^2 \delta_{\tau\tau}(\xi, \tau)\} u(x-\xi, t-\tau)\, d\xi\, dt$$
$$= u_{xx} - c^2 u_{tt}.$$

"An elementary solution of the equation (A) is defined to be a distribution E such that

$$A * E = \delta.$$

To discuss the boundary value problem for a domain τ, and a distribution T such that (A) holds, we operate with the restriction T^* of T to the domain τ and calculate the product $A * T^* * E$, which gives the appropriate generalization of the simple classical formula (P).

"In Hadamard's researches the elementary solution in m-dimensional space is constructed explicitly, in terms of the characteristic conoid $T=0$ of the given hyperbolic equation, in the form

$$U \,.\, T^{-\frac{1}{2}m+1}, \text{ if } m \text{ is odd.}$$

The case of m even is more difficult, and in Hadamard's researches the function

$$U \,.\, T^{-\frac{1}{2}m+1} - V \log T, \quad m \text{ even,}$$

where U and V are holomorphic near the vertex of the conoid, played the part of the elementary solution.

"There remains the problem of carrying out the integrations which generalise the classical formula (P). To solve this problem Hadamard introduced the concept of the finite part of an infinite integral, which we should now describe in terms of the distribution which is associated with the elementary solution. In Hadamard's work this concept is obtained by removing from the domain of integration the region in which $|T| < \epsilon$. It is then found that the integral consists of terms of the form

$$(B_0 + B_1 \epsilon + \ldots + B_{p-1} \epsilon^{p-1}) \,.\, \epsilon^{-p+\frac{1}{2}},$$

together with another term $I(\epsilon)$ which converges to a limit as $\epsilon \to 0$. It is the limit of $I(\epsilon)$ as $\epsilon \to 0$ which Hadamard calls the 'finite part' of the integral.

"These researches were completed by the 'method of descent' in which the solution of a partial differential equation in the space variables $x_1, x_2, \ldots, x_m$ and a time variable t is used to obtain the solution in the space variables $x_1, x_2, \ldots, x_{m-1}$, and a time variable t.

"Finally it must be noted that Hadamard's researches are not restricted to the analytic case but that the effect of nonanalytic coefficients is carefully studied."

A definition of the elementary solution of (A) is given in Schwartz. The application of topological methods to partial differential equations developed by Leray and Schauder, although it has little direct connection with Hadamard's work, was no doubt stimulated by it.

Hadamard later recorded [**207**] his own approach to this part of his researches as follows : " Frappé, comme tous les géometres, de la beauté du résultat de M. Volterra, je me proposai de les étendre aux équations linéaires aux dérivées partielles à coefficients quelconques, donc en général variables, c'est a dire à l'étude de propagations en millieux hétérogènes. Dans ce temps la c'était essentiellement une idée de mathématicien; les physiciens savaient, si l'on veut, qu'on peut avoir à s'occuper de milieux hétérogènes; mais on ne peut pas dire qu'ils eussent appris à s'y intéresser.... Pour autant que je puis remonter à ma propre psychologie d'il y a une vingtaine d'années,—le fait que la méthode était limitée au cas des coefficients constants me semblait dénoter qu'elle n'avait pas sa véritable forme, qu'il lui manquait quelque chose."

The general elementary solution was obtained by the consideration of the geodesics of a properly chosen linear element and it may have been his mastery of geodesics developed through his earlier dynamical work that enabled him to see the " véritable forme " through all the subsidiary complications. He also recorded [**295**, p. 110] how he was led to the idea of the " finite part " of an integral step by step [see **100**, p. 121] : " I could not avoid it any more than the prisoner in Poe's tale, the Pit and the Pendulum, could avoid the hole at the centre of his cell." On the other hand, Hadamard wrote that he could not undertsand [**295**, p. 53] how he failed to notice a method given in Hilbert-Courant based on the work of John and Asgueierson which enlightened the whole of another problem in this field.

Hadamard's discussion of Huygen's principle in [**187**, p. 53] and [**204**, p. 610] is justly famous; he pointed out that when n is odd and not less than 3 the ordinary wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x_1{}^2} - \dots - \frac{\partial^2 u}{\partial x_n{}^2} = 0 \tag{B}$$

has a remarkable property which he called " Huygen's minor premise ". It is that, if a solution is zero outside a sphere $r = (x_1{}^2 + x_2{}^2 + \dots + x_n{}^2)^{\frac{1}{2}} = \epsilon$ when $t = 0$, and not zero inside, then for $t > 2\epsilon$ the nonzero part of the solution lies in the spherical shell $t - \epsilon < r < t + \epsilon$. Hadamard showed that a general hyperbolic equation of normal hyperbolic type has this property if, and only if, n is odd and $n \geqslant 3$, and his elementary solution has no logarithmic part. (In the contrary case Hadamard said that there is " diffusion of waves "). He then raised the question of characterising the equations for which the " minor premise " holds. The so-called " Hadamard conjecture " states that the minor premise holds only for general hyperbolic equations equivalent to (B) for which n is odd and not less than 3, but that conjecture was not in fact categorically stated by Hadamard (see Courant-Hilbert, p. 765). A good deal of work has been done on it; although it has been proved correct for some general classes of equations, examples have been constructed to show that it is false for others, and in particular in the physically important case $n=3$, and the general problem has not been completely solved. [See Günther, where further references are given.]

In spite of Hadamard's other great achievements, including the proof of the prime number theorem, and in spite of the lapse mentioned above, it is not surprising that one admirer (see P. Lévy) said : " C'est dans la théorie des équations aux dérivées partielles qu'il a donné les preuves les plus éclatant de son génie."

Addendum to the Notice about J. Hadamard

In *A mathematician's miscellany* (pp. 82–83) J. E. Littlewood describes his recollections of the discovery of the proof of the Abel-Tauber theorem and of the derivates theorem which was essential to it and to many other Tauberian theorems. (The derivates theorem states that if $f(x)=o(1)$ and $f''(x)=O(1)$, then $f'(x\ =o(1)$ as $x\to\infty$, and can be extended at once to derivates of higher order.) Littlewood goes on to say " The derivates theorem was actually known, but buried in a paper by Hadamard on waves ". In fact it occurs in [**30**, p. 334] in connection with dynamical trajectories (not waves) and probably formed part of the material submitted for the Prix Bordin in 1896. Hadamard pointed out that it was also proved by A. Kneser, *Journel f. reine u. angewandte Math.*, 118 (1897), 199, again in connection with dynamical problems (see G. H. Hardy and J. E. Littlewood, *Mess. of Maths.*, 43 (1914), p. 147 f.n.).

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A complete bibliography of Hadamard's writings up to 1935 is printed at the end of the volume of *Selecta* [**284**] and of his mathematical works in Mandelbrojt and Schwartz. The present bibliography is intended to complete these. In addition, the works referred to in the text are listed below with the same numbers as in *Selecta* bibliography which ended with 280. I have assigned numbers [**222a**], [**292a**], etc., to the works which do not appear in the other bibliographies, as if they came at the end of the year in which they were published, having assigned numbers after [**280**] consecutively, according to the list in Mandelbrojt and Schwartz.

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