Philip Hall at the time of his election to the Royal Society

OBITUARY

PHILIP HALL

J. E. ROSEBLADE

Philip Hall, formerly Sadleirian Professor of Pure Mathematics in the University of Cambridge, was a mathematician of great influence and distinction. For many years he was the only algebraist working in England. He was pre-eminent as a group theorist and made many fundamental discoveries; the conspicuous growth of interest in group theory this century owes much to him. He was Honorary Secretary from 1938 to 1941 and 1945 to 1948, and President from 1955 to 1957 of the London Mathematical Society, and its De Morgan Medallist and Larmor Prizeman in 1965. He died on 30 December 1982.

1. *Origins and early life*

Philip Hall, the illegitimate son of George Hall and Mary Laura Sayers (1872–1965), a dressmaker, was born in Hampstead, North London, on 11 April 1904. Making no provision for Philip or his mother, George Hall disappeared from their life soon afterwards. Philip was brought up by his mother; in his grandfather's home in Christ Church Road until he was seven, and then at 8 Well Walk, which remained his home, except for a few years during the Second World War, until 1948.

Laura Sayers and her sister Lois were the younger of two pairs of twin daughters of Joseph Sayers (1837–1917) and his wife Ellen (née Stone) (1839–1901); they lived in Balcombe, West Sussex. Joseph Sayers started out, like his father before him, as a farm labourer. By the time Laura was born he was a Gardener Domestic and in 1915 when his youngest child Ethel left Well Walk to marry, his occupation was described as Market Gardener. At some time between the death of his eldest daughter in 1881 and the marriage of his only son in 1890, Joseph Sayers moved his family from Balcombe to Hampstead (taking three days by horse and cart), so that his girls would not have to go into service; he apprenticed them instead as needlewomen to Peter Robinson. Of the elder twins, one married but Ada remained single and lived the rest of her life with Laura, who became known as Mrs Hall. Lois died in 1936.

In 1910, Ada, Ethel, Laura and Lois bought 8 Well Walk and ran it as a boarding house. They were careful managers, and very little was ever discarded. When rooms remained empty, as happened, for instance, in the Depression, they worried. When Philip went away to school, his room was let; and they persuaded the tenant to find somewhere else during the holidays. As an adult, Philip too showed a prudent regard for the value of money.

Mrs Hall was a devout member of Lyndhurst Road Congregational Church; she remained part of its Prayer Circle, with her surviving sisters, even after she had left Hampstead for good. Philip went to Sunday School there; he was baptised in 1911 and confirmed in 1918; but in later life did not keep his religious beliefs: he found, then, 'quite enough to wonder at in the natural world'. Dr R. F. Horton, the minister

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at Lyndhurst Road, was one of a number of able men who took a friendly interest in Philip Hall's progress.

Philip attended New End Primary School, run by the London County Council (L.C.C.), from 1909 until 1915, when he won a scholarship through the Public Elementary Schools Competition to Christ's Hospital, West Horsham, a school that admitted only those boys whose parents would not have been able to afford boarding school fees.

Hall entered Christ's Hospital in May 1915, when A. W. Upcott was Headmaster, and left in 1922. His Housemaster was Edwin Hyde, who 'could make even mathematics sparkle like a chandelier' (cf. Carter, P. Y. 1953). He was much influenced by C. J. A. Trimble, Head of Mathematics, who was a kind and courteous teacher, something of a recluse, and a considerable mathematician himself, being the author, with T. S. Usherwood, of two textbooks on practical mathematics. Another teacher who had great influence on Hall, as on everyone else of that generation, was the classical scholar W. H. Fyfe, who succeeded Upcott in 1919. Fyfe was the first lay headmaster since 1678, and to the boys appeared benign, liberal and approachable.

When Hall became House Captain in 1921 he ruled, like Fyfe, along the lines of simple reason. He was kind and helpful to the younger boys, who respected and admired him; this was remarkable in those days for he was neither an extroverted leader nor any kind of athlete. He played rugby for his House as a rather clumsy forward, but gave up cricket, being content for a number of years to be official scorer for the school First Eleven. He was in the Officers' Training Corps, but was otherwise unaffected by the Great War. In 1921 he was editor of *The Blue*. His contemporaries, who were well aware of his Olympian intellect, remember him as likeable and cheerful, with a sense of humour, gentle and reserved. Even then Hall had cultured interests that spread widely beyond mathematics. In 1920 he won the Charles Lamb Silver Medal in an obligatory competition for the best English essay. In his last year he was awarded the Thompson Mathematical Gold Medal. He loved the countryside around Horsham and on whole holidays would walk for miles in it.

In December 1921 he won an Open Foundation Scholarship at King's College, Cambridge. Together with an Exhibition from Christ's Hospital and an award from the L.C.C., this gave him for his undergraduate years an annual income of £190, some six times that of his mother.

2. *Cambridge and London 1922–1927*

Hall went up to King's in October 1922 to read for the Mathematical Tripos. He was in the First Class in Part I in 1923 and a Wrangler in Part II in 1925, with special credit in the optional, more advanced, Schedule B. The list of Wranglers that year also contained W. V. D. Hodge and D. E. Littlewood.

In his first letter home after his arrival, Hall wrote

> I am getting on beautifully & enjoying it very much: there are such opportunities of learning that it is about all you can do to make the most of them. I go to four sets of lectures a week, which only means two a day. Two of them are Schedule B lectures, if that means anything to you. I have made the acquaintance of Mr. Littlewood & Mr. Pollard, the two most progressive of the mathematicians here, so I am going on fairly well for a start.

Analysis dominated the pure mathematical scene (E. W. Hobson was Sadleirian professor), but there was a good deal of geometry with H. F. Baker (Lowndean professor), H. W. Richmond and F. P. White. Algebra was quite neglected and it was possible to get a degree without knowing any. Before Hall graduated, however, he had the opportunity of attending advanced courses on group theory given by Baker in 1924 and White in 1925, and of hearing P. A. MacMahon lecture on *Some processes in Combinatory Analysis* in 1925.

He joined many of the religious and semi-religious societies, quite impartially as to denomination, but preferred study to social life; he found the libraries 'more than enough to keep me occupied'. Retiring he may have been but J. T. Sheppard, who was to be Provost of King's from 1933 to 1954, noticed him and took him and a friend to Brittany for a fortnight in June 1923.

Hall owed much to Arthur Berry, who was Assistant Tutor in Mathematics at King's College. While still an undergraduate Hall 'began with Berry's encouragement to study the works of William Burnside, especially his magnificent treatise on the Theory of Groups & some of his later papers'. He offered parts of Burnside's book for the Schedule B examination, and had (amongst other things long since become familiar in the Tripos) to 'shew that there cannot be a simple group whose order is the power of a prime p^r ($r > 1$)'.

Hall graduated B.A. in 1925 (he took his M.A. in 1929); and was elected to an Open Senior Foundation Scholarship, which allowed him to stay on at King's for one more year. That summer during a long holiday in Italy he learnt Italian; and in the following March in London took steps to learn German. At the end of June 1926, undecided as to whether to pursue an academic career, he took part in the Administrative Group Competition of the Civil Service Examination; but was unsuccessful.

Hall stayed in Cambridge, in lodgings, from October 1926 until the end of the Michaelmas Term that year, when he handed in a Fellowship dissertation: 'The Isomorphisms of Abelian Groups'. In many respects this was years before its time, and it must be doubted whether anyone then in Cambridge could properly assess it. It shows signs of hasty preparation; indeed the dissertation finishes abruptly in mid-argument, as if the Fellowship Electors had snatched it away from him. Its contents remained unpublished; some of them will be described later. In March 1927 Hall was elected to a Fellowship at King's College, which he held for the rest of his life.

Hall's first published papers date from 1927. For some months from January that year he worked as research assistant to Karl Pearson (1857–1936) at the Biometric Laboratory, University College, London, where Pearson continued to advocate methods in the theory of correlation that had been superseded by the powerful techniques introduced by R. A. Fisher. As Hall recognised, the results of [**1**], which were derived by classical Pearsonian methods, had already been overtaken by Fisher's; and whereas there was in [**2**] an elegant geometric argument, the essentials of which are still quoted in M. G. Kendall and A. Stuart's *The advanced theory of statistics*, it can have surprised no-one that Hall was not inspired to become a statistician. Indeed his principal task whilst with Pearson was to work out the tables of the Incomplete Beta Function (cf. Karl Pearson 1934), and as there were to be finally about a quarter of a million entries, he took the view that, as far as he was concerned, it could stay incomplete.

3. *Cambridge and Bletchley 1927–1945*

Hall returned to Cambridge in September 1927 to take up residence as a Fellow of King's. Almost at once he made a discovery as important for the theory of finite soluble groups as Sylow's Theorem of 1872 is for finite groups in general. In 'A note on soluble groups' Hall proved that if G is a soluble group of order mn, where m and n are coprime, then every subgroup of G whose order divides m is contained in some subgroup of order m, and these subgroups of order m are all conjugate in G. A subgroup whose order and index in a finite group are coprime is now known as a Hall subgroup. Ten years later, in [9], Hall characterised soluble groups by such arithmetic properties, and went on to develop a general theory of finite soluble groups. In the late fifties, the Hall–Higman paper [21] and 'Theorems like Sylow's' [22] had a profound influence on J. G. Thompson, and were indispensable for the great achievements of the sixties.

In 1932 Hall wrote his famous 'A contribution to the theory of groups of prime-power order' in which he tried 'to discover some of the more obvious features which underlie the structure of the most general p-group'. This is one of the fundamental sources of modern group theory, which can be read now by any graduate student just for the enjoyment of the beauty of the presentation. Hall regarded as his most interesting contribution the theory of regular p-groups; but he also laid down the basic laws of the commutator calculus, introduced the Commutator Collecting Process and discovered one of the links connecting the study of groups with that of Lie rings. He replied to a letter of congratulation on the fiftieth anniversary of its submission that

> There is a story behind that paper. My Fellowship at King's had been renewed in 1930 but, —sometime in 31 I think, —it was intimated to me that a second renewal would be unlikely, unless I showed signs of mathematical life; before then I had only produced one short note in 1928, so there was some justification for their warning & I obviously had to make a bit of an effort.

A. E. Ingham, passing on the high praise of the referee, urged Hall to go to the International Congress in Zürich that September and tell the world about groups of prime-power order; Hall took only the first part of the advice.

Between 1930 and 1933 Hall gave courses on *Finite Groups*; amongst those to hear him were H. S. M. Coxeter, who often walked over to his rooms from Trinity for discussions, and M. Hall. Alfred Young was also lecturing on *Group Characteristics*. Hall was appointed Lecturer in 1933 and with M. H. A. Newman began to give Algebra its footing in Part II. Of all the advanced courses he gave before the War, perhaps *Representation Theory* had the widest appeal. In the Lent Term of 1939, to satisfy all 15 who came to the meeting to arrange times, Hall offered 8 a.m. Nobody could plead that he already had a lecture, so he gave 24 lectures at 8 in the morning; and the number attending remained constant. This was remarkable considering Hall normally worked into the early hours and rose late.

Hall's first research student, though officially supervised by R. H. Fowler, was Garrett Birkhoff who spent 1932–33 in Cambridge and whose first paper on lattice theory (1933) testifies to the help and encouragement he received from Hall. Three others worked as Hall's research students before the war: B. H. Neumann on identical relations in groups, K. A. Hirsch on infinite soluble groups and T. E. Easterfield on groups of order p^6. Olga Taussky (now Taussky-Todd), like Hirsch and Neumann a refugee from the Nazi terror, had many discussions with Hall on p-groups; and D. Derry during his year in Cambridge discussed with him torsion-free abelian groups.

Hall had also to teach undergraduates reading mathematics or natural sciences at King's for up to eight hours a week. R. E. Macpherson has written 'Whatever the question he would bring that part of mathematics to life & nearly always extend it into neighbouring fields. It was assumed pre-war that a supervisor would cover the whole field, pure and applied ... A supervision with Philip often ran beyond the standard hour to two hours plus, but I have never heard this resented. King's undergraduates whether first class or third class regarded themselves as singularly lucky in the thirties'. Some of his pupils from that time had distinguished careers in fields other than mathematics, and some became life-long friends. Among those who remained mathematicians were D. G. Champernowne, R. O. Gandy, J. M. C. Scott and A. M. Turing. When Taussky accused Hall of being the worst recluse in Cambridge, he replied 'No, Turing is worse'.

In 1935 J. K. Senior wrote to Hall suggesting they should combine their studies of groups of order 64. The collaboration which then began was vigorous until 1939; it continued after the war until 1959, when Hall withdrew. Marshall Hall took over from him and with Senior published *The groups of order* 2^n ($n \leqslant 6$) in 1964. The introduction makes clear what debt the work owes to Philip Hall.

Hall went to the International Congress in Oslo in 1936, but did not give a talk. He did speak however in Section A* of the British Association Meeting held in Cambridge in August 1938 on *The verbal classification of groups*, which discussed in general terms some of his ideas in [**15**]; the programme that year included talks by G. D. Birkhoff, S. Lefschetz, W. V. D. Hodge, A. Speiser, B. H. Neumann, Olga Taussky, Garrett Birkhoff, J. H. C. Whitehead and S. Eilenberg.

Hall was invited in 1939, not for the first time nor the last, by H. Hasse, whom he had met in Cambridge in March 1935, to give a series of lectures on his work at the Mathematical Institute in Göttingen. A minor congress took place in June of that year. A number of workers from German universities contributed lectures: W. Magnus, E. Witt, B. van der Waerden, W. Krull, H. Wielandt, A. Scholz, W. Specht and H. Zassenhaus; O. Grün was also there; but the only other outsider besides Hall to attend was A. Speiser from Zürich. The lectures, including the four ([**14**, **15**, **16**, **17**]) given by Hall, were subsequently published in Crelle's *Journal*. Hall sent in his manuscript through neutrals, and received his offprints via the Royal Navy after the war. There was some criticism of his going, but Hall 'felt that the German mathematicians ... were as little responsible for the present situation (& probably enjoy it as little) as you or I do'. In 1946 Hall wrote letters to the Allied Authorities on Hasse's behalf and sent encouraging and sympathetic ones to Hasse himself. He helped the resurgent Mathematics Department of Hamburg by sending books to Zassenhaus, in exchange for copies of the *Abhandlungen*. During the Berlin Blockade, Hall arranged for Hasse to be sent *The Times*, and received Hasse's books on number theory in return.

He was invited to speak at the International Congress in June 1940, but war prevented him from going; indeed, then, he was 'having great fun with the Local Defence Volunteers, expecting later on to be called up'. (He had offered his services to the Air Ministry.) In January 1941 the British Council, who were looking for 'someone who could do credit to our national name' asked Hall if he would consider accepting the Chair of Mathematics at the University of Istanbul, as a form of war service; he declined this offer. Instead, in September 1941, he obtained leave of absence from Cambridge University for the duration and went to work at the Government Code and Cypher School at Bletchley Park. He worked first on Italian cyphers, and after they had become impossible or of limited value he worked on the Japanese diplomatic cyphers, in particular those used by the Japanese military and naval attachés in Europe. The success that was had against these was of great importance after the autumn of 1943. A knowledge of Japanese was not essential, but Hall taught himself some 1,500 characters. He lived with his mother and Ada, who had moved to Little Gaddesden in 1939; and travelled the 20 miles to Bletchley by motorcycle and train. Ethel rejoined her sisters in 1944 when her husband died.

By 1940 Hall's work commanded great respect and admiration; he was recognised as knowing more about groups than anybody else and was thought of as the very successor of Burnside. In 1942 he was elected to the Royal Society. Hall wrote at the time

> The aim of my researches has been to a very considerable extent that of extending and completing in certain directions the work of Burnside. I asked Burnside's advice on topics of group-theory which would be worth investigation & received a post-card in reply containing valuable suggestions as to worth-while problems. This was in 1927 and shortly afterwards Burnside died. I never met him, but he has been the greatest influence on my ways of thinking.

Burnside had more than group theory in common with Hall; he also was educated at Christ's Hospital and left as an Exhibitioner after winning the Thompson Gold Medal; his Obituary Notice appears in the same volume as Hall's short note of 1928.

4. *Cambridge 1945–1982*

Hall was released from Government Service at the end of July 1945 and returned to King's College. His mother returned to Hampstead, and he tended to go up to London for week-ends to see her; which is how in 1946 he could give a helping hand to A. W. Goldie, once a month or so on Sundays, at 8 Well Walk. During the next five years he gave lectures in which he explored not groups so much as aspects of all the other algebras. He was full of vigour and brimming with ideas. When he was not presenting his own results, he illuminated others' with his own interpretation, always going further than they had gone. Students felt they were hearing today what the rest of the world would only hear tomorrow. P. M. Cohn in the preface to his *Universal Algebra* refers to information passed on by oral tradition whilst acknowledging how much the book owes to Hall's 'most lucid and stimulating' lectures. In 1949 Hall talked on Universal Algebra at the first British Mathematical Colloquium held in Manchester; and was invited to talk at the International Congress in Cambridge, U.S.A. in September 1950, but could not go 'for family reasons'.

Although after his promotion to Reader in Algebra in 1949 Hall's lectures were aimed at Part III students and graduates, he continued to teach undergraduates in King's until 1953 when he succeeded L. J. Mordell to the Sadleirian professorship. After 1949 when D. R. Taunt, D. G. Northcott and D. Rees were all appointed to junior positions, there were always other algebraists on the staff besides Hall. He was Chairman of the Faculty Board for two years from 1956, and conducted its business with courtesy and authority; he did not appear at all interested in, or good at, getting his own way. What he thought was always made entirely clear; the rest of the Board could agree or disagree.

In 1955 Hall gave lectures in St Andrews at a Colloquium of the Edinburgh Mathematical Society. In accepting the invitation, he wrote to W. L. Edge 'The subject I have in mind is symmetric functions, in relation to various branches of the theory of groups. I think I can find something to say on that which will not be too

Philip Hall at Oberwolfach 1960

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trite.' The lectures, far from trite, will be described later; they included an account of what are now known as the Hall algebra and the Hall–Littlewood polynomials. His ideas are of great importance, for example, in the representation theory of the symmetric group. The shorter account of some of his results given at the 4th Canadian Congress in Banff in September 1957 is all that Hall published on the subject. The lectures on *Nilpotent groups* that he gave the month before in Edmonton, Alberta, were published; they have had great influence ever since.

Hall's most important work on infinite groups, begun in 1952 and continued in 1959 and 1961, took the form of a systematic investigation of certain finiteness properties, including residual finiteness, which given finitely generated soluble groups might or might not possess. The papers [**18**, **27**, **30**] were seminal. In them he initiated, for example, the representation theory of polycyclic groups, which has developed strikingly over the past decade.

Although Hall was excited by the spectacular progress made in the theory of finite simple groups and followed it with interest, he never took any public part in its development, but influenced it through correspondence with J. G. Thompson, who has described (Thompson 1984) how encouraging it was at the end of 1958 to read a letter which for the first time put him in touch with a 'resonating mind equally aware of the technical nuances which gripped me'. Hall never discovered a finite simple group, but he did construct a profusion of infinite ones. His universal countable locally finite simple group appeared in 1959; in 1963 he constructed for the first time a non-strictly simple group and in his last paper, begun in 1968 but not finished until 1972, he proved very general theorems about embedding groups in simple groups.

In 1948 his mother and aunts moved to Histon, a village some three miles to the north of Cambridge within what for him was easy walking distance from King's. Hall kept contact with his 'Dear Old Hampstead' through his cousin Dorothy Tribe until her death in 1974 (and found pleasure in renewing it a few years later through the daughter of an erstwhile pupil). His Aunt Ada died in 1950 aged 84. From 1958, when his surviving aunt died, he lived in Histon finding it difficult to be away except briefly during the day; absence for a longer period required Dorothy to come and look after his mother; and if Dorothy was ill or otherwise unavailable, he was stuck. Thus it was that he could hardly ever accept any of the invitations that came to him from all over the world.

In 1965 Hall took a year's sabbatical leave, hoping to concentrate on writing as he had done during 1958, but he was unable to do so; his mother could not very well be left at all, and in September that year she died. Two years later he retired.

He had a special affection for Germany and in 1960 enjoyed a visit to Oberwolfach, even though 'for family reasons' he managed only two days there; he gave a talk in Tübingen afterwards at the invitation of Wielandt, for whom he had a high regard. Some of Hall's pupils and their pupils went to work with Wielandt for their post-doctoral years, including N. Blackburn, R. W. Carter, J. S. Rose and J. E. Roseblade; so it was particularly appropriate that he should become *Dr. rer. nat. honoris causa* in November 1963 as part of the celebrations of the centenary of the Faculty of Science and Mathematics of Tübingen University. He went to Oberwolfach once more, in 1973.

In 1969 Hall found a subject 'as inexhaustible as mathematics and much more restful'. The fact of his mother living to be 93 gave him an initial impetus to study longevity; and another stimulus came from a paper (circulated but not published) by a contemporary of Hall at King's, I. M. Stephens. Hall mostly compiled his collection of nonagenarians, or *nons* as he called them, in 1970–72; and by March 1975 his register contained nearly 3,900 items. He measured 'the genealogical *distance* of two people rather crudely but in a way that is very easy to apply viz. if one is the *m*th-cousin of the other *n* times removed' he took 'the distance to be $2m+n+1$, with the addition of $\frac{1}{2}$ if the cousinship is of the half-blood; and in the usual event of multiple cousinship,' he took 'the smallest of these distances'. He analysed the nons in connected classes, an N_i being a set of nons such that one may pass from one to any other in steps of distance at most i. His largest N_3 had somewhere between 250 and 300 members; but he had shrunk (in 1975) from working out thoroughly an enormous 'British Gentry' N_4. He enjoyed making up genealogical tables, and his letters seldom failed to give some amusing account of his latest finds. He knew quite as much about the stems and branches of his nons as ever he did about his 2-groups.

During his long retirement he lived alone at Histon, as always caring nothing for hot water or central heating, which he thought unhealthy. He remained as studious as ever, but apart from intermittent work on his last paper, seldom thought about groups. He gave up radio and television, but remained well informed. His attitude to authority was irreverent and he was sceptical of the value those in positions of power or influence put on themselves; one of his regrets was not having come across *Private Eye* sooner. He saw little of King's after relinquishing his rooms in 1970, and did not care for the changes taking place there; but he delighted his surviving friends by going to a dinner party to mark his and G. Rylands' completion of 50 years as Fellows. Fortunately some of his younger friends remained in Cambridge and with their children formed a sort of family for him. He also renewed with evident pleasure some of the friendships that had lapsed over the years.

He died from pneumonia in Addenbrooke's hospital, following a stroke, at the end of 1982.

Hall had an unusually wide range of interests and his knowledge was encyclopedic. At King's, this made him for many years a redoubtable Fellowship Elector; he was qualified to express an opinion on almost any subject, in the sciences or in the humanities. But it was not only this: his personality established him in a leading position. He rarely spoke; when he did it was decisive. He once, reluctantly but as always truthfully, spoke against the claims of a mathematician saying that his axioms were 'somewhat too reasonable'; he gave a positive welcome to adventurousness, and secured the election of one candidate by referring to his 'almost repellent originality'.

Hall was liberal in his hospitality to visitors, whom he would put up either in The Bull just next to King's or, when that had gone, in The Garden House overlooking the river, and whose bills he would pay before they could object. Whilst he lived in college he frequently invited young friends to dinner in his rooms or at an hotel; he also enjoyed evenings with them at the theatre or cinema. Supervisions with research students would more often than not involve lunch at a restaurant such as The Bath. He made generous individual gifts; and contributed, handsomely and anonymously, to hardship funds in both King's and Jesus Colleges. Particular charities, like the National Children's Home and the Mansfield House Settlement, also benefitted from his impulse to give, but contributing to some of the others, like funds for saving certain animal species which he believed would inevitably become extinct, he thought a waste of money. Hall was pessimistic too about the prospect for the human race; he foresaw only slow decline.

He was most generous with his ideas and very modest in claiming any credit for them. N. Blackburn has written 'Every time I see a lecturer solemnly writing his own name on the blackboard as a discoverer of something, I think of Philip Hall, whose lectures consisted very largely of his own discoveries but who never once mentioned that fact. He stuck to the mathematics and made that as interesting as he could.' And yet he had a proper regard for his own standing in Algebra and was prepared, at least in private, to refer to certain other mathematicians whose reputations he thought unjustified as 'pretty dim'. His students loved him and he them. Writing so lucidly and elegantly himself, he must have found painful much of what they first wrote; but whenever he had strong criticism to make of their work, he always found a way to soften the blow and never failed to suggest effective improvements. Nor did he abandon them when they had completed their dissertations; he wrote them helpful and stimulating letters, often very long and always by hand. (He was not always so helpful to others with whom he was in serious disagreement.) Not all of his students did group theory, for he was comfortable in a much broader band of mathematics than his publications might suggest, and some of them migrated to other branches of algebra. Hall never wrote a book himself, but many of his ideas have been propagated throughout the world of mathematics by those of his friends and pupils.

Hall was not gregarious, and cared little for large gatherings or formal occasions; he was reticent rather than shy. When he was with friends he was the best company in the world. Some will remember him best for his extensive, but discriminating, love of poetry, which he spoke beautifully, not only in English; Dante was a love of his for over 50 years; his learning Japanese might just as well have been to read poems in it as for any reason to do with cyphers; there was a time in the seventies when he learnt a sonnet a day, admitting it to be easier if you lived alone. Others will prefer to recall his enjoyment of music and art, or his love of flowers and country walks. He could meet anyone on their own ground and find something to interest them. If help was needed, he gave it. He was a wonderful person: gentle, amused, kind; and the soul of integrity.

5. *Post-war research students*

The following were research students of Hall after the war: D. R. Taunt, H. A. Thurston, W. V. D. Hale, P. M. Cohn, J. A. Green, N. J. S. Hughes, A. J. Weir, G. D. Findlay, W. A. Coppel, K. W. Gruenberg, D. J. Harris, D. H. McLain, N. Blackburn, J. F. Rigby, J. Anson, C. R. Kulatilaka, A. M. Duguid, R. Armstrong, A. Learner, J. E. Roseblade, K. E. Pledger, D. J. S. Robinson, B. Hartley, A. Rae, A. H. Rhemtulla, P. W. Stroud.

6. *Honours, etc.*

Hall was elected to the Royal Society in 1942 and awarded the Sylvester Medal of the Royal Society (1961). He was President of the London Mathematical Society (1955–57) and received its De Morgan Medal and Larmor Prize (1965) and Senior Berwick Prize (1958). He received honorary doctorates from the Universities of Tübingen (1963) and Warwick (1977). He was elected to an Honorary Fellowship at Jesus College, Cambridge in 1976.

Hall was a joint editor of the *Cambridge Tracts in Mathematics and Mathematical Physics* (1953–61), and an editor of the *Journal of Algebra* (1964–67).

7. *Sources*

Most of the facts and quotations come from Hall's personal papers, which are in my possession. I have also made free use of the very many letters I have received from Hall's contemporaries at school, from Fellows and ex-Fellows of King's College and from Hall's other friends and colleagues. I am most grateful to them all. I acknowledge too the valuable help with Hall's work on Statistics given me by R. L. Plackett and B. W. Silverman. The information about the family move from Balcombe was kindly given me by Mrs D. Sayers. Obituary Notices have appeared in *The Times* of 5 January, 1983, the *Annual Report of King's College, Cambridge* 1983 and the *Annual Report of Jesus College, Cambridge* 1983. There is a mathematical profile (written by Hall himself) in the *McGraw–Hill Modern Scientists and Engineers, Volume* 2 (1980).

THE MATHEMATICAL WORK OF PHILIP HALL

This will be summarised in three sections: *Finite groups* by J. G. Thompson, *The algebra of partitions* by J. A. Green and *Infinite groups* by J. E. Roseblade. These brief accounts may be supplemented by reference to *Group theory: essays for Philip Hall*, which contains eight survey articles commissioned through the London Mathematical Society for a volume to have been published in Hall's honour on his eightieth birthday. In these essays, which describe the present state of knowledge in areas where Hall made significant contributions, Hall's influence on algebra is allowed to speak for itself. Hall left a large collection of manuscript papers which still have to be properly sorted and assessed; except in regard to symmetric functions no account has been taken of these.

FINITE GROUPS

J. G. Thompson

Following an initial scepticism, Burnside was won over to the theory of characters developed by Frobenius. He became an advocate of representation theory and its adjunct, character theory, not least through his proof that all groups of order $p^a q^b$ are soluble. By the early years of this century, representation theory had recorded several triumphs and was well-placed to claim the undivided attention of aspiring group theorists. That Hall was aware of these developments and of their potential is amply demonstrated in both his published and unpublished work. The unique imprint which he has left on finite group theory can be discerned most easily by tracing the influence of Burnside on Hall, although this involves taking certain liberties with the chronology.

Burnside proved that if G is a finite group, S is a normal subgroup of the Sylow p-subgroup P of G, and Q is a Sylow p-subgroup of G that contains S as a non normal subgroup, then for some element x of G whose order is prime to p, there is a subgroup T with $S \subseteq T \subseteq P$ such that x normalizes T but not S. This result leads among other places to the notion of p-normality and to the transfer theorems of Grün and Wielandt, culminating in the Hall transfer theorem, the best result yet proved about the transfer map.

Transfer theorems attempt to describe $P \cap G_p$, where P is a Sylow p-subgroup of G and $G_p = [G, G] \cdot G^p$. Frobenius proved and exploited the result that this subgroup of P is generated by P^p, together with the ratios xy^{-1} of elements of P that are conjugate in G. Hall showed that the subgroup in question is generated by $P \cap N_p$, where N is the normalizer of P in G, together with certain elements of P of the form $[u, z, \ldots, z]^g$. Here $u, z \in P, g \in G$, and the commutator has weight p. This result was never published by Hall himself. A proof can be found in M. Hall 1959. The original result of Burnside has been carried forward, in the same spirit, to a much more useful position.

Burnside recognized the importance of the transfer map, and he showed that if a Sylow p-subgroup of G is contained in the centre of its normalizer, then G has a normal p-complement. Burnside also noticed, as did Frobenius, that if p is the smallest prime that divides the order of G, and if the Sylow p-subgroups of G are cyclic, then the preceding result may be applied. Results of this sort are provocative and not without interest. They cannot, however, be said to provide a theory. Hall's characterization of finite soluble groups as precisely those finite groups that contain Sylow p-complements for every prime p succeeds in crossing the line separating isolated results from a general framework. The theory implicit in this characterization has led to much research and has not yet run its course.

Burnside's influence on Hall is apparent, too, in the theory of p-groups. Burnside showed that a group of order p^n has a normal Abelian subgroup of order p^a, provided $\frac{1}{2}a(a-1) < n$. He also determined, for example, all the p-groups that have a cyclic subgroup of index p. In Hall's paper [**4**], these and similar results became single facets of a general theory of p-groups, already mentioned, and general principles relating power maps and commutation to the lattice of subgroups were laid down. The ability to exploit general properties of the commutator calculus served Hall well, and was needed, to mention one case only, in the proof of his transfer theorem. Still later, Hall discovered an exact analogue for groups of the Jacobi identity for Lie algebras, namely, in Hall's notation which has become widely accepted,

$$[x, y^{-1}, z]^y [y, z^{-1}, x]^z [z, x^{-1}, y]^x = 1\,,$$

a relation valid in all groups.

The well known Burnside problem also influenced Hall. The commutator collection formula of Hall provided a glimmer of hope that an attack could be made on the Burnside problem for groups of prime exponent, and in his Presidential Address to the London Mathematical Society in 1957, he closed with an allusion to the Burnside problem and Engel groups: 'In spite of, or perhaps because of, their relatively concrete and particular character, they appear, to me at least, to offer an amiable alternative to the ever popular pursuit of abstraction.'

Aware that the Burnside problem might have a negative solution, as indeed it does (P. S. Novikov, S. I. Adyan 1968), Hall and Higman [**21**] took care to prove a number of results which are related to, but independent of, the Burnside problem. They remarked in passing that 'the order of the greatest two-generator group of exponent 15 with metabelian Sylow subgroups is $3^{9{,}93418{,}37570{,}31251} \cdot 5^{5{,}68225}$'. They also proved that the restricted Burnside problem for groups of exponent 6 has a positive solution, and that the largest k-generator finite group of exponent 6 has order

$$2^a \cdot 3^{b+\binom{b}{2}+\binom{b}{3}}\,,$$

where $a = 1+(k-1)\cdot 3^{k+\binom{k}{2}+\binom{k}{3}}$, $b = 1+(k-1)\cdot 2^k$. This result was helpful to M. Hall in his proof that the Burnside problem for groups of exponent 6 has a positive solution. The main reason Hall and Higman were able to obtain so casually such dauntingly precise results stems from the manner with which the earlier results in this paper were organized. The study of p-soluble groups leads to representation theory. In the spirit of MacMahon, they show that in the critical case to which they have been led, the Jordan canonical form of certain p-elements is explicitly determined. A proof using the Green correspondence can be given, but the sure-footedness of their argument is evident. The path leads to groups of the form $G = QP$, where Q is normal and an extraspecial q-group, while P is a p-group acting trivially on the centre of Q, faithfully and irreducibly on the central quotient of Q. In this situation, one needs to study the absolutely irreducible kG-modules on which G acts faithfully, where k is a field of characteristic p. That the essence of the difficulty lies here is in itself a major contribution to the theory of groups. That the difficulty was completely surmounted established this paper as a landmark.

Hall's first contact with extraspecial groups was already made by the time he submitted his 1926 essay 'The Isomorphisms of Abelian Groups'. The word 'Abelian' of the title refers to subgroups of $\mathrm{GL}(n, \mathbb{C})$ whose commutator subgroup consists of scalar matrices, that is, it refers to Abelian subgroups of $\mathrm{PGL}(n, \mathbb{C})$. In this essay, Hall expresses an indebtedness to the paper of Study 1912. The essay also contains references to Speiser's book, to the paper of Scorza 1909, to the one of Schottky 1903, and to Burnside. Although the essay is incomplete, it contains a proof that a p-group of order exceeding p in which every characteristic Abelian subgroup is of order at most p is a central product of non Abelian groups of order p^3 (such a group was called azygetical by Study). Moreover, Hall showed that when $p = 2$ the isomorphism type of an extraspecial group of a given order depends only on the parity of the number of factors that are quaternion groups. As already mentioned, the essay shows signs of having been written in haste, and suffers from unwise use of the word 'obvious', a trait common to the young, but not always confined there. It is a trait which Hall did not retain.

Hall's paper [13] begins with these words:

> Let p be a given prime. The object of this note is to prove the following rather curious result.
>
> *The sum of the reciprocals of the orders of all the Abelian groups of order a power of p is equal to the sum of the reciprocals of the orders of their groups of automorphisms.*

There is nothing of Burnside here, and while the paper refers to MacMahon, one is scarcely entitled to attribute any particular influence of that remarkable combinatorialist on the genesis of this paper. To use Hall's own words, the result is 'curious' and has an 'almost repellent originality'. The proof uses the well known combinatorial identities of Euler,

$$\frac{1}{\prod_{n=1}^{\infty}(1-x^n)} = \sum_{n=0}^{\infty} \pi_n x^n = \sum_{n=0}^{\infty} \frac{x^n}{f_n(x)},$$

where π_n is the number of partitions of n and $f_n(x) = (1-x)(1-x^2)\dots(1-x^n)$. Since there are π_n Abelian p-groups of order p^n, the sum of the reciprocals of their common order is $\pi_n \rho^n$, where $\rho = 1/p$. The proof of the theorem is completed by showing that the sum of the reciprocals of the orders of the groups of automorphisms of the π_n groups of order p^n is $\rho^n/f_n(\rho)$. Thus, Euler's identities are given a group-theoretic interpretation. A connection between this result and the Hall algebra is mentioned in the essay by Macdonald 1984.

Having discovered regular p-groups and having proved a number of theorems about them, Hall found an opportunity to use them in extending a theorem of Frobenius concerning solutions of equations in finite groups. Hall writes, 'The crucial result of the present paper is an improved criterion of regularity' [5, p. 472]. The remarkable thing about this assertion is the hidden way in which regularity enters the arguments. Neither Frobenius's original theorem nor the improvement of it which Hall found requires any reference to regularity, and it may be guessed that only the serious student of p-group theory can savour the arguments in this tightly reasoned paper. In any case, the 'improved criterion of regularity' is the following: '*G is regular whenever the order of* $G/\mho_1$ *is less than* p^p'. Here G is a p-group and $\mho_1 (= G^p)$ is the subgroup generated by the p^{th} powers of all elements of G. From this result, Hall deduces detailed information about the number of elements X in an arbitrary group G for which $f = g = \ldots = 1$, where $f, g, \ldots$ are words in $X, A_1, A_2, \ldots, X$ is an unknown and $A_1, A_2, \ldots$ are elements of G. The theorem of Frobenius which this work generalizes is the one about the number of solutions of the equation $X^n = A$ in a finite group G, and which Frobenius, not noted for over-stating the case, called a 'Fundamentalsatz der Gruppentheorie'. The ideas in these papers of Frobenius and Hall are still not fully explored, nor is their relevance to other questions understood.

Just as Hall used regular p-groups as a tool in [5], so he also used an enumeration principle in [8]. It is to be noted that regular p-groups and the enumeration principle first appeared in [4]. In [8] it was necessary to extend the enumeration principle from p-groups to arbitrary finite groups, and the Eulerian functions ϕ_n, which Hall tackled head-on, became, in certain cases, amenable to calculation. The definition of $\phi_n(G)$ is straightforward: $\phi_n(G)$ is the number of ordered n-tuples $(X_1, \ldots, X_n)$ of elements of G such that G is generated by $\{X_1, \ldots, X_n\}$. The calculation of $\phi_n(G)$ is not straightforward, and has been carried out for very few groups. The difficulty seems to lie in our inability to compute the generalized Möbius function μ_G, defined inductively on the lattice of all subgroups of G by the requirements that

$$\mu_G(G) = 1\,, \qquad \sum_{K \supseteq H} \mu_G(K) = 0 \quad \text{for all proper subgroups } H \text{ of } G\,.$$

Hall showed that $\mu_G(H) = 0$ unless H is the intersection of maximal subgroups of G, G itself being viewed as the empty intersection. If H is the intersection of maximal subgroups of G, then

$$\mu_G(H) = \sum_k (-1)^k \nu_k(H)\,,$$

where $\nu_k(H)$ is the number of sets of k maximal subgroups of G whose intersection is H. From this description of μ_G, it is not surprising that its explicit determination in any given case should be difficult. But, as Hall says, 'The essential step is always the calculation of the Möbius function $\mu_G \ldots$'. One of the charms of this paper is the way in which μ_G is determined when $G = L_2(p)$, and p is any prime. The value of the paper lies, however, not so much in the particular calculations carried out so expeditiously, but rather in having set up the framework in terms of which ϕ_n must be studied, if it is to be studied at all. The functions ϕ_n and μ_G are inextricably intertwined, and so properties of ϕ_n are not easy to obtain.

The result obtained by Hall in [**6**] is now part of the folklore, known as the marriage theorem. It is of a purely combinatorial nature, although it was motivated by group-theoretic considerations. Apart from the intrinsic interest of this paper, its lucidity and easy comprehensibility make it a model of mathematical clarity.

Hall's last published paper about finite groups is [**22**]. Although the paper deals exclusively with Hall subgroups of a finite group, Hall did not use this term. What is generally called a Hall π-subgroup of G was called a S_π-subgroup of G by Hall. There is a torrent of theorems in this paper, which are proved, for the most part, by appealing to a central theme. As Hall puts it, 'The principal theorem of the present paper is

THEOREM D 5. *If K is a normal subgroup of G such that K satisfies E_π^n and G satisfies D_π^s, then G satisfies D_π^s.*'

The terminology here is compact. A group G satisfies E_π^n if G contains a nilpotent Hall π-subgroup, and satisfies D_π^s if it contains a soluble Hall π-subgroup H with the property that every π-subgroup of G is contained in a conjugate of H. The results in this paper present in concise form the most detailed information about Hall subgroups of finite groups that can be proved without a direct assault on the structure of simple groups. Even so, Hall obtains all the soluble Hall subgroups of the symmetric groups: 'Let Σ_n be the symmetric group of order $n!$ and let $p < q \leqslant n$, where p and q are primes. Then Σ_n satisfies $E_{p,q}$ only when $p = 2$, $q = 3$, and $n = 3, 4, 5, 7$, or 8. For $n = 5, 7$, and 8, Σ_n satisfies $C_{2,3}^s$ but not $D_{2,3}$.'

A predecessor of this paper is Wielandt's theorem that E_π^n implies D_π. Hall appeals to Burnside and notes that it is not possible to extend Wielandt's theorem by relaxing the nilpotency hypothesis to supersolubility; the groups $L_2(p)$ provide counterexamples.

THE ALGEBRA OF PARTITIONS

J. A. Green

Apart from the article [**24**] (and some notebooks and loose papers which remain to be studied), the only account which Philip Hall left of his work on the combinatorial questions which arise from the appearance of partitions in different parts of group theory, are the following.

A. Handwritten notes for a series of five lectures entitled* 'Symmetric Functions in the Theory of Groups', given at the St Andrews Colloquium, 13–23 July 1955.

B. An undated manuscript of 21 pages headed 'Abelian p-groups and Similar Modules, by P. Hall. 1. Results'.

*This title is not in Hall's manuscript; I have taken it from W. Ledermann's notes of the lectures as they were delivered. I am much indebted to Professor Ledermann for lending me his notes.

Of these, A is the more extensive and is the only document that contains substantial proofs. B is apparently the introduction to a paper that was never completed. Article [**24**] is a brief account of things discussed in more detail in A.

The part of this work that has been most developed and used by others is that expressed by Hall's elegant 'algebra of Abelian p-groups' or 'algebra of partitions'. This, with a full account of its applications to the representation theory of the general linear groups GL_n, both over finite fields and over local fields, is given by I. G. Macdonald in his book *Symmetric functions and Hall polynomials*, and summarized in his 1984 essay. So in the brief description of the contents of A and B that follows I refer to these (as [**M**] and [**M**′] respectively) wherever convenient; to make this reference easier I have used Macdonald's notation for symmetric functions rather than Hall's.

Hall gives as the object of the St Andrews lectures

> ... to point to problems mostly unsolved in the theory of symmetric functions which arise from certain considerations in the theory of groups.

He excludes from his concern, except incidentally, the 'widely known' relation of symmetric functions with the representations of the symmetric groups and the classical general linear groups. A characteristic passage (it occurs in Lecture 4) describes something of Hall's special insight into these problems,

> ... wherever in mathematics you meet with partitions, you have only to turn over the stone or lift up the bark and you will, almost infallibly, find symmetric functions underneath. More precisely: if we have a class of mathematical objects which in a natural and significant way can be placed in one-to-one correspondence with the partitions, we must expect the internal structure of these objects and their relations to one another to involve sooner or later ... the algebra of symmetric functions.

Lecture 1 describes 'the formal relations of symmetric function theory'. Let $\Lambda = \oplus \Lambda_n$ be the ring (or $\mathbb{Z}$-algebra) of all symmetric functions with rational integral coefficients (Hall takes symmetric functions with rational coefficients) in a denumerable set of independent variables $x_1, x_2, \ldots$; Λ is graded by degree (see [**M**, I, 2] or [**M**′, §2]). The elementary symmetric functions $e_1, e_2, \ldots$ freely generate Λ as commutative ring; the complete (homogeneous) symmetric functions $h_1, h_2, \ldots$ form another such free generating set, a fact which follows from the *fundamental recurrence relations*

$$\sum_{r=0}^{n} (-1)^r e_r h_{n-r} = 0 \qquad \text{for} \quad n \geqslant 1 .$$

The module Λ_n of symmetric functions that are homogeneous of given degree n, has important $\mathbb{Z}$-bases $\{m_\lambda\}, \{e_\lambda\}, \{h_\lambda\}$, each parametrized by the set of all partitions $\lambda = \{\lambda_1, \lambda_2, \ldots\}$ of n. Here m_λ is the monomial symmetric function $\sum x_1^{\lambda_1} x_2^{\lambda_2} \ldots$, and $e_\lambda = e_{\lambda_1} e_{\lambda_2} \ldots$, $h_\lambda = h_{\lambda_1} h_{\lambda_2} \ldots$. There is a symmetric, bilinear function on Λ_n defined by $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$; this is positive definite relative to the basis $\{m_\lambda\}$. Hall suggests as the *central theorem about symmetric functions*, that there exists a $\mathbb{Z}$-basis $\{s_\lambda\}$ of Λ_n which is *orthonormal*, $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$. Such a basis is provided by the Schur functions (or S-functions, [**M**, I, 3]). Hall gives two alternative (and well known) descriptions of the s_λ; the first defines s_λ by Jacobi's bialternant, and the second as the generating function ('characteristic') of an irreducible character of the symmetric group S_n. (Perhaps surprisingly, Hall does not mention Schur's description of s_λ as an irreducible rational character of $GL_n(\mathbb{C})$.)

Lecture 2 begins by introducing the 'Gaussian polynomials'

$$g_{r,s}(u) = \frac{(1-u^{r+1})\dots(1-u^{r+s})}{(1-u)\dots(1-u^s)} = \frac{f_{r+s}(u)}{f_r(u)f_s(u)}$$

in an indeterminate u; one defines $f_n(u) = (1-u)(1-u^2)\dots(1-u^n)$ for $n \geqslant 1$, and $f_0(u) = 1$. These appear when one specializes suitably the variables $x_1, x_2, \dots$ in e_n and h_n (see [M, p. 18]). Hall mentions three contexts in which the $g_{r,s}$ arise naturally:

(i) The coefficient of u^k in $g_{r,s}(u)$ is the number of partitions of k which are 'subordinate' to the partition s^r (a partition λ is subordinate to a partition μ if $\lambda_i \leqslant \mu_i$, $i = 1, 2, \dots$). This shows, in particular, that the coefficients of $g_{r,s}(u)$ are non-negative integers.

(ii) Put $u = q$, the number of elements of a finite field $k = GF(q)$. Then $g_{r,s}(q)$ is the number of subspaces of dimension r, in a k-space $L_n(q)$ of dimension $n = r+s$. Equivalently, it is the number of points (Hall uses the nice term 'population') of the Grassmannian $G_{r,s}(q)$ over k.

(iii) Ehresmann showed in 1934 that the Poincaré polynomial of the compact manifold $G_{r,s}(\mathbb{C})$ is $g_{r,s}(t^2)$; that is, that the m^{th} Betti number b_m of this manifold is zero if m is odd, while b_{2k} is precisely the number of partitions of k described in (i).

Hall remarks that there are other cases where the correspondence between the 'finite' geometry and the geometry over the complex field is exactly as in the case of the Grassmannians. (These are all special cases of the 'Weil conjectures', which were published in 1949 (see Hartshorne 1977). It is not clear whether Hall knew of these conjectures in 1955.) As generalization of (i), (ii) above, Hall proves the following result: *for any partition λ, the population $h_\lambda(q)$ of the Schubert variety* Sh_λ *over the finite field $k = GF(q)$ is a polynomial in q, for which the coefficient of q^k is the number of partitions of k which are subordinate to λ.*

Lecture 2 ends with two proofs of the fact that the number of nilpotent $n \times n$ matrices over $GF(q)$ is $q^{n(n-1)}$. The second of these proofs uses the beautiful identity

$$\frac{u^n}{f_n(u)} = \sum_\beta \frac{u^{\sum \beta_j^2}}{f_{\beta_1}(u)} \prod_j g_{\beta_{j-1},\beta_j}(u),$$

where the main sum is over all partitions of n.

Lecture 3 deals with some numerical invariants of a finite group G, particularly the number $w_d(G)$ of elements x in G that satisfy $x^d = 1$. Here d is a given positive integer, which we may assume is a divisor of the order g of G. A famous theorem of Frobenius says that then d divides $w_d(G)$. Hall gives some detailed divisibility properties of $w_d(G)$ in the case $G = S_n$, the symmetric group on n symbols. These rest on the following generating function for $w_d(S_n)$, which comes from the algebra of symmetric functions by a suitable specialization of the variables,

$$\exp \sum_{r|d} \frac{1}{r} t^r = \sum_{n=0}^{\infty} \frac{w_d(S_n)}{n!} t^n.$$

Hall conjectures *en passant* that if a finite group G has a subgroup H all of whose elements are d^{th} roots of unity, then $w_d(G)$ should be divisible 'by something approaching' the order of H, and states without proof a theorem of this sort: *Let G be a finite group, d a positive integer and p a prime such that p^k divides d. If G has an elementary Abelian subgroup H of order p^n, then $w_d(G)$ is divisible by p^N, where* $N = n-\left[\dfrac{n}{p^k}\right]$.

The lecture ends with some remarks on the numbers $\bar{w}_p(G)$, $\bar{w}_{p'}(G)$ of elements of G whose order is a power of p, or is prime to p, respectively (here p is a given prime). Hall observes that the 'square law' $\bar{w}_p(G) = |G_p|^2$ (G_p being a Sylow p-subgroup of G), holds for many finite simple groups G which are associated to the classical linear groups. (To put this in historical perspective, one may note that Chevalley's paper 'Sur certains groupes simples', which provides a unified treatment of all finite simple groups of 'Lie type', appeared in 1955. Steinberg 1968 proved that the square law holds for all such groups.)

In Lectures 4 and 5 Hall describes the algebra of finite Abelian p-groups. Because this work and the subsequent researches that it inspired are described in detail in Macdonald's book, and summarized in his essay, I shall give here only a bare outline.

If p is a prime number and $\rho = (\rho_1, \rho_2, \ldots)$ is a partition, we say that a finite Abelian p-group has *type* ρ if it is isomorphic to $\bigoplus_i \mathbb{Z}/p^{\rho_i}\mathbb{Z}$. If λ, μ, ρ are partitions, denote by $g^{\rho}_{\lambda\mu} = g^{\rho}_{\lambda\mu}(p)$ the number of subgroups X of type λ in an Abelian p-group A of type ρ, such that A/X has type μ. Hall defines a ring $H = H(p)$ as follows: H is a free $\mathbb{Z}$-module with free basis $\{u_\lambda = u_\lambda(p)\}$ indexed by the set of all partitions λ. Multiplication in H is defined by the rule

$$u_\lambda(p)u_\mu(p) = \sum_\rho g^{\rho}_{\lambda\mu}(p)u_\rho(p).$$

It is not hard to prove that H is associative, commutative and with identity element $u_0(p)$. Moreover $H = \bigoplus_n H_n$ is graded, taking H_n to be the $\mathbb{Z}$-submodule having $\{u_\lambda \mid \lambda \text{ a partition of } n\}$ as basis. As commutative ring (or $\mathbb{Z}$-algebra) H has free generating seet $u_{\{1\}}, u_{\{1^2\}}, u_{\{1^3\}}, \ldots$. It would therefore be possible to define an isomorphism between H and the ring Λ of symmetric functions, by mapping each $u_{\{1^r\}}$ to the elementary symmetric function e_r. But it turns out better if we map $p^{\frac{1}{2}r(r-1)}u_{\{1^r\}}$ to e_r. This gives an isomorphism between the $\mathbb{Q}$-algebras $H(p) \otimes \mathbb{Q}$ and $\Lambda \otimes \mathbb{Q}$, which Hall treats as an identification. (An explicit formula for the symmetric function which corresponds in this identification to $u_\lambda(p)$, was given by D. E. Littlewood 1961. See [M, III].) Hall's main theorem says that *for all partitions λ, μ, ρ the function $g^{\rho}_{\lambda\mu}(p)$ is (given by) a polynomial in p with integral coefficients, whose leading coefficient is the coefficient $c^{\rho}_{\lambda\mu}$ which appears in the product of Schur functions*

$$s_\lambda s_\mu = \sum_\rho c^{\rho}_{\lambda\mu} s_\rho.$$

In particular if $c^{\rho}_{\lambda\mu} = 0$, then $g^{\rho}_{\lambda\mu}(p)$ is zero for all p. If $c^{\rho}_{\lambda\mu} \neq 0$, then the polynomial $g^{\rho}_{\lambda\mu}(p)$ has degree $n(\rho)-n(\lambda)-n(\mu)$, where for any partition ν, $n(\nu) = \sum_i (i-1)\nu_i$. A fairly detailed sketch of the proof of this theorem is given in Lecture 5.

To end this section, I should say something about the document B mentioned at the beginning. This describes the 'Hall algebra' H in a rather more general setting, replacing finite Abelian p-groups by R-modules of finite length, where R is a complete discrete valuation ring having finite residue-class field of order q. Then the same 'Hall polynomials' $g^{\rho}_{\lambda\mu}(q)$ appear as before, evaluated at q instead of p. Hall analyses $g^{\rho}_{\lambda\mu}(q)$ as follows. Let λ, μ, ρ be partitions, and let $G = G_{\rho}(R)$ be an R-module of type ρ. Consider the chains of submodules

$$(1) \qquad 0 \leqslant H \leqslant G = G_{\rho}(R)$$

such that H has type λ, and G/H has type μ. These are permuted among themselves by the automorphism group $A_{\rho}(R)$ of G, and of course $g^{\rho}_{\lambda\mu}(q)$ is the sum of the lengths of the $A_{\rho}(R)$-orbits so obtained. Unfortunately the number $t^{\rho}_{\lambda\mu}(R)$ of these orbits may depend on R, as well as on λ, μ, ρ. Hall shows that the endomorphism ring of any chain (1) has R-composition length $\geqslant \frac{1}{2}\sum_i (\rho_i'^2 + \lambda_i'^2 + \mu_i'^2)$, where λ', μ', ρ' are the partitions conjugate to λ, μ, ρ; he calls the orbit of (1) *dominant* if equality holds here. The number of chains in any orbit is given by a monic polynomial in q whose degree is that of $g^{\rho}_{\lambda\mu}(q)$ (namely $n(\rho) - n(\lambda) - n(\mu)$) if and only if the orbit is dominant; for all other orbits the degree is less. Hall conjectures that there exist precisely $c^{\rho}_{\lambda\mu}$ dominant orbits. (This was proved by T. Klein 1969. Macdonald [**M**, II, 4] gives another proof.)

INFINITE GROUPS

J. E. Roseblade

Finitely generated soluble groups. Hall called the S-groups studied by K. A. Hirsch polycyclic because they have a series $1 = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$ in which each factor G_{i+1}/G_i is cyclic. All finitely generated nilpotent groups are polycyclic. There are few theorems or arguments in the representation theory of polycyclic groups which do not have their origin in Hall's three papers [**18**, **27**, **30**] on finitely generated soluble groups and finiteness conditions. In the first, Hall characterised polycyclic groups as the soluble groups whose integral group rings satisfy the maximum condition on right ideals. In part, this reflects the fact that if G is expressible as $K\langle x\rangle$ with K normal in G, then $\mathbb{Z}G$ is akin to a polynomial ring over $\mathbb{Z}K$. One corollary is that a finitely generated group Γ with an Abelian normal subgroup M and polycyclic quotient group $G = \Gamma/M$ has Max-n, the maximum condition for normal subgroups; this follows because, as Hall proved, polycyclic groups are finitely related so that M is finitely generated as a $\mathbb{Z}G$-module. He raised the question of which soluble groups are finitely related, by asking whether every finitely related soluble group has Max-n. H. Abels 1979 showed that this is not so; but the general question is still being vigorously investigated. Significant results have been obtained by R. Bieri and R. Strebel, which are discussed with many others in the essay by Strebel 1984.

Another corollary of the main result of [**18**] is that there is only a countable number of finitely generated groups that have some term of their lower central series Abelian. This is in marked contrast with the 2-generator groups with a metabelian group of inner automorphisms. Hall showed that there were uncountably many such groups H with centre any given non-trivial countable Abelian group C. He related this to a particular dichotomy of the set $\mathscr{C}$ of all commutator words. The same dichotomy is exhibited in relation to residual finiteness: in [**27**] Hall shows that every finitely generated Abelian by nilpotent group is residually finite. On the other hand H cannot be residually finite if C is divisible.

The arguments in [**27**] are much subtler than those in [**18**] and rest on a careful analysis of finitely generated modules M over group rings KG, where K is a principal ideal domain and G is polycyclic. Hall proved that a free K-submodule M_0 exists such that M/M_0 is a π-torsion module, with π some finite set of primes of K (that is, he proved what is now called generic flatness). From this, but only if G was nilpotent, he proved that if M is irreducible then it is of finite dimension over K if K is a locally finite field, and finite if K is the ring of integers. It seemed plausible to Hall that this result on irreducible modules should hold without the restriction; and that residual finiteness should be enjoyed by all finitely generated Abelian by polycyclic groups. The finiteness of the irreducible modules was proved by J. E. Roseblade 1973, many of whose arguments directly generalize Hall's; and the step to residual finiteness was made by A. V. Jategaonkar 1974. Hall showed also that the irreducible modules need not be finite when K is a field with an element of infinite multiplicative order, unless G has an Abelian subgroup of finite index. Much work has recently been done on groups all of whose irreducible modules over a given field are finite dimensional, especially by B. A. F. Wehrfritz 1981, 82.

The Frattini subgroup Φ of a finitely generated soluble group Γ was proved to be nilpotent by N. Ito 1953 and K. A. Hirsch 1954 if Γ is polycyclic, and by Hall himself in [**27**] if Γ is metabelian. In [**30**] Hall gave the first example of such a Γ where Φ is not nilpotent, proved more extensive theorems and discussed the corresponding dichotomies of $\mathscr{C}$. The most important result was that Φ is still nilpotent when Γ has some member, M say, of its lower central series nilpotent; and the crucial case was when M was Abelian. Hall first noted (in this case) that the residual finiteness of all the quotient groups of Γ together with the original result of Frattini 1855 showed that Φ centralized every chief factor of Γ; and then considered the elements of $G = \Gamma/M$ that acted trivially on every irreducible quotient module M_1 of M. The result on which all else hinges is that if z is a central element of $\mathbb{Z}G$ that annihilates every such M_1 then some power of z annihilates M. Just as the finiteness of the irreducible modules in [**27**] is a 'weak' Nullstellensatz, so this result is a 'strong' one. Hall remarked that the classical proof of the Nullstellensatz by J. L. Rabinowicz 1929 can be taken over without modifying the essential idea. J. E. Roseblade 1973 proved that Φ is nilpotent if Γ is Abelian by polycyclic. An account of some of the ring theory that developed from Hall's work can be found in D. S. Passman's book *The algebraic structure of groups rings*, and some of the most recent in his 1984 essay.

In another direction, A. I. Mal'cev showed that every soluble subgroup of $GL_n(\mathbb{Z})$ is polycyclic. This appeared first in Russian in 1951 and in translation five years later. In the Edmonton Notes Hall began the task of proving the converse: he showed that every finitely generated nilpotent group can be embedded in $GL_n(\mathbb{Z})$ for some n; and L. Auslander 1967 and R. G. Swan 1967 completed it. It is significant that the impetus for proving the converse came from Hall's Canadian lectures.

Stability groups. L. Kaloujnine 1950 introduced the stability group of a chain $1 = G_0 \leqslant G_1 \leqslant \ldots \leqslant G_n = G$ and proved it nilpotent if the G_i are all normal in G; Hall proved it nilpotent in general, first in the Edmonton Notes and then in [**25**]. A subgroup H of $\mathrm{Aut}(G)$ is contained in the stability group of some finite chain if and only if H is a subnormal subgroup of the split extension of G by H; and this connexion extends to one between ascendant and descendant subgroups and stability groups of ascending and descending series. The rather basic area of group theory that embraces these topics was extensively studied by Hall and his students after 1959, but all that Hall published was in his joint paper [35] with B. Hartley. They proved that a stability group of an ascending series has a descending hypercentral series; and that a finitely generated group can faithfully stabilize an ascending series if and only if it is residually nilpotent. They applied a *collecting process* to free products to prove that any group K is descendant in $K * K$, and hence that any group can faithfully stabilize a descending series of some group. Some progress was made with the difficult question of what groups can faithfully stabilize a descending invariant series, but it was left unresolved. This is connected with the class, $\mathfrak{X}$ say, of groups G which are descendant in the standard wreath product $A \wr G$ for every Abelian group A. Hall and Hartley proved that a group which has a descending series with infinite cyclic factors belongs to $\mathfrak{X}$. The study of $\mathfrak{X}$-groups is closely connected with that of powers of augmentation ideals and of when some (transfinite) power is zero. This has received much attention since 1966, notably from Hartley and his followers. Many of the methods ultimately depend on the construction given by Hall in Chapter 7 of the Edmonton Notes and used by him to prove the theorem of S. A. Jennings 1955 on dimension subgroups. In this connection, too, the discussion given by Hall in his Presidential Address to the London Mathematical Society [26] of basic sequences in Lie rings and collecting processes in groups has proved valuable. (Hall did much of the early work on basic commutators in the thirties, but did not publish it.) The essay by Hartley 1984 contains a full account and very many references.

Simple groups. Hall made three constructive contributions to the study of infinite simple groups. First there is his universal countable locally finite group C in [29], which is defined to within isomorphism by the properties that every finite group can be embedded in C and any two isomorphic finite subgroups are conjugate in C. It is universal in the sense that it contains $2^{\aleph_0}$ copies of every non-trivial countable locally finite group. The easiest description of C is as a union of an ascending tower $1 = L_0 < L_1 < \ldots < L_n < L_{n+1} < \ldots$ of subgroups, where L_1 is any finite group of order greater than 2 and for $n \geqslant 1$, L_{n+1} is the symmetric group of all the permutations of L_n and L_n is embedded in L_{n+1} via its regular representation. Hall adapted the construction for other purposes: if L_1 is cyclic of prime order p and $L_{n+1} = L_n | \wr | L_1$ with L_n embedded as a direct factor of the base group of L_{n+1} if n is even, and as the diagonal subgroup if n is odd, then he proved the union to be *verbally complete*. With choices of L_n and embeddings only slightly more elaborate, Hall constructed countable locally finite verbally complete p-groups having centre any preassigned countable Abelian p-group.

Because forming a wreath product with infinitely many factors as a union of finite wreath products with a suitable embedding rule is cumbersome if the number of factors is large or their ordering elaborate, Hall defined in [31] the general wreath product $W = \operatorname{Wr}_{\lambda \in \Lambda} H_\lambda$ directly for any ordered set Λ and arbitrary permutation groups H_λ. The definition generalized that given for finite wreath products by L. Kaloujnine and M. Krasner 1950; and Hall proved its essential property: the associative law. When all the H_λ are permutationally isomorphic with a given H then W is what he called a wreath power $\operatorname{Wr} H^\Lambda$. On the assumption that no H_λ is trivial, Hall proved that if Λ has no greatest element then $W' = W''$ and every normal subgroup of W' is normal in W. When W is a wreath power and the group A of order preserving automorphisms of Λ acts irreducibly on Λ (as happens if Λ is the set of integers in their natural order), he proved that W' is a minimal normal subgroup of the split extension of W by A, and consequently characteristically simple. With H a non-trivial p-group, this gives a wide range of characteristically simple p-groups, differing from those first discovered by D. H. McLain 1954. All the McLain groups are generated by Abelian normal subgroups; Hall proved that if Λ has no least element then W even has no finitely generated subnormal subgroup.

The simple ordered group of S. Chehata 1952 showed that it is possible for infinite simple groups to have non-trivial series, though whether these could be ascending was a question raised by B. I. Plotkin 1958. To answer this, Hall gave in [**32**] a general discussion of series in simple groups. Using a combination of wreath power and tower-building arguments, Hall constructed for any non-trivial group H and limit ordinal λ a simple group $g_\lambda(H)$ with an ascending series of ordinal type λ. By varying H, simple groups of different complexions were exhibited. For example, if I is an infinite cyclic group, then $g_\omega(I)$ has I as a serial subgroup and lies in $(\mathrm{LE})^{\omega+1}\mathfrak{A}$.

Much of Hall's later work, both in lectures and papers, was marked by the systematic use of closure operators such as L and E acting on classes of groups, which may have seemed sometimes to serve only to express results more succinctly or elegantly than could be done otherwise, but frequently revealed fresh ways of viewing old problems and often suggested new ones. Thus Hall in [**32**] could ask if $(\mathrm{LE})^{\omega}\mathfrak{A}$ contained a non-cyclic simple group. B. Hartley and S. E. Stonehewer 1968 showed that it did not. Closure operations were ideal for bringing order to the chaos of classes which had been introduced in the A. G. Kuroš and S. N. Černikov survey of 1947. Hall made a number of contributions settling differences between various classes of generalized soluble groups in [**31**, **32** and **33**].

Hall's third contribution to simple groups is in his very long and intricate last paper. In 1949 G. Higman, B. H. Neumann and H. Neumann proved that any countable group can be embedded in a group generated by 2 elements. In 1968 this was improved by R. S. Dark who showed that the embedding could be made subnormally, and by F. Levin who showed that the 2 generators could be taken to have preassigned orders $m > 1, n > 2$. Hall combined these improvements into an all-embracing one: given three groups H, K and L such that $|L| \leqslant |H * K|, |H| > 2$ and $|K| > 1$ then there is a group J of the form $\langle H, K \rangle$ that contains L as a 2-step subnormal subgroup. At the other extreme, Hall asked when L could be embedded in a simple group J of the same form. P. E. Schupp 1976 showed that this could always be done; but he used small cancellation theory. None of Hall's various embedding methods seemed capable of proving such a result. For a non-trivial group K he defined $l(K)$ to be the least cardinal such that whenever $|L| \leqslant |K * K|$ then L can be embedded in some simple group generated by $l(K)$ copies of K, and showed that $l(K)$ was finite. This was the decisive step in proving that if K_1, K_2 and K_3 are non-trivial and $|L| \leqslant |K_1 * K_2 * K_3|$, then L can be embedded in a simple group generated by a copy of K_1, K_2 and K_3; except when the K_i are all of order 2 or two of them are of order 2 and the third uncountable. It followed that $l(K) \leqslant 4$ always and $l(K) \leqslant 3$ unless $|K| = 2$. Hall thought it likely that these bounds were just 1 too high. Schupp's theorem shows that he was right.

The present state of affairs in relation to universal locally finite groups and embedding theorems is discussed by D. J. S. Robinson 1984.

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