PERCY JOHN HEAWOOD

G. A. Dirac

Percy John Heawood died in Durham on 24 January, 1955, at the age of 93. He played a decisive part in re-establishing in their proper places the four colour problem and Durham Castle, which may be regarded as the two great and fitting monuments of his noteworthy activities as a mathematician and as an administrator.

P. J. Heawood was born in September 1861 at Newport, Shropshire, the eldest of four sons of the Reverend J. R. Heawood, who was rector of a church near Ipswich. He was educated at Queen Elizabeth's Grammar School, in Ipswich. In 1880 he went up to Oxford with an Open Scholarship from Exeter College. He stayed in Oxford until 1887, when he became lecturer in mathematics at the Durham Colleges.

In Heawood's time the most distinguished mathematicians at Oxford were Henry Smith, Savilian Professor of Geometry, who died in 1883, and his successor, J. J. Sylvester. Mathematical teaching was by lectures and tutorials. The lectures were given by the professors and lecturers, some of the latter taught on a college basis, others on an inter-collegiate basis. Each student was assigned to a tutor (*i.e.* supervisor of studies or coach), with whom he had a private hour at least once a week. The function of tutor was performed by mathematics professors and lecturers, for example, Henry Smith was Tutor of Balliol College as well as Savilian Professor.

The following account of the attitude on teaching, which prevailed among the mathematicians at Oxford at the time when Heawood studied there, is taken from Professor E. B. Elliott's address delivered at the 200th meeting of the Oxford Mathematical and Physical Society, on 16 May, 1925 :—

" We thought, too exclusively probably, but I still hold soundly, that our great business as teachers in a University was to educate, to assist young men, many of whom had no too strong a sense of responsibility, or care about the application of what mind they possessed, to learn how to make the best use of their powers. Those of mediocre capacity were as important to us, or we thought they ought to be, as the clever ones who gave us the greatest intellectual satisfaction. To encourage effort, and secure concentration of powers, we thought all-important. We should have repudiated indignantly the notion, now not unpopular, that high honours in the Mathematical Schools should be reserved for those who prove inclination and the right sort of capacity to extend mathematical knowledge, and were in no hurry as teachers to encourage specialisation. What we said was, ' Slur nothing. Be precise. Be thorough. Face little

[Journal London Math. Soc. 38 (1963), 263–277]

difficulties, and rest not till you conquer them.' The man who can turn from one problem to another, from prevailing once to grappling again, is qualifying for success in the battle of life. . . .

"Accordingly we laid great stress on elements, tried to secure adequate attention to a sufficient variety of subjects in the common ground of mathematicians, to Algebra, Geometry, the older analysis—we had not heard of the newer—the Mechanics of Solids and Fluids, Optics and Astronomy, fixing the range examined on in a subject (except perhaps in Geometry) rather low, but demanding a thoroughness of knowledge which was decidedly high. . . .

"But how about original work . . .? It had not yet occurred to people that systematic training for it was possible. Even the Senior Mathematical Scholarship was not given for an original or learned dissertation. . . . When we had graduated, Henry Smith and others saw that we were elected into the London Mathematical Society, and our ambition was stimulated, but it was ambition for the future, and we needed more encouragement in the present. The little things which were beginning to accumulate in our note-books could not compare with the massive works in the *Proceedings*, and if we ever attended a meeting we were rather dismayed than otherwise by the unintelligibility of papers as read."

There were two honours examinations in mathematics: "Moderations", taken at the end of a student's first year, and "Greats" or "Finals", taken at the end of his third year, and qualifying him for the B.A. degree. There was no public order of merit for these examinations. The "blue ribbon" was conferred by the Junior Mathematical Scholarship, awarded to the winner in a written examination held in January which the students could attempt in their first and in their second year. Students in their fourth year could compete for the Senior Mathematical Scholarship in a written examination held in January. There was also a periodic graduate award on an astronomical subject, called the Lady Herschell's Prize.

Heawood's mathematical career at Oxford was extremely distinguished. He obtained a First Class in Mathematical Moderations in 1881 and a First Class in Mathematical Finals in 1883. He was awarded the Junior Mathematical Scholarship of the University in 1882 and the Senior Mathematical Scholarship of the University and the Lady Herschell's Prize in 1886. In addition, he obtained a Second Class in Classics in 1885. He became a B.A. in 1883 and an M.A. in 1887. He once told his colleague Professor J. R. Burchnall that he belonged to the school of Henry Smith at its height in his undergraduate days. Henry Smith was the leading mathematician at Oxford until his death in February 1883. His lectures on Modern Geometry were widely attended. He and Esson, C. J. Faulkner and C. J. C. Price, were the first to coordinate their lectures on an inter-collegiate basis; their association was called the Mathematical Combination, and Heawood's remark to Professor Burchnall suggests that he was probably enrolled with the Combination.

In 1887 Heawood became Lecturer in Mathematics at the Durham Colleges—later Durham University—where at that time R. A. Sampson (afterwards Astronomer Royal for Scotland) was Professor. He remained in Durham for the rest of his life. The University was until 1937 controlled by the Dean and Chapter of Durham Cathedral. This status was probably congenial to Heawood from the start: he was a devout Anglican layman, an excellent Hebrew scholar, and well versed in classics and theology.

In 1890 he published his celebrated paper "Map colour theorems", which is undoubtedly the greatest contribution so far made to the mathematical theory of the colouring of maps. How he was led to the subject is not known.

In the same year he married Christiana, daughter of Canon H. B. Tristram, of Durham, a distinguished naturalist and traveller. She was a lady of great character and evangelical piety. The marriage was a long and happy one.

Heawood was appointed to the Chair of Mathematics at Durham in 1911; he retired in 1939 at the age of 78. By the time of his appointment he had already shown great capacity and liking for administrative and committee work, and he had an established reputation as a mathematician.

His later career at Durham University was spectacularly prominent. He occupied in turn a number of responsible administrative posts, including those of Censor and of Proctor, and finally of Vice-Chancellor from 1926 till 1928. He also became the first chairman of the University of Durham Schools Examination Board, and remained for many years. He maintained a lively interest in the cultivation of good relations between schools and the University, often taking long journeys to attend Governors' meetings.

Outside the University his strongest loyalty was for Durham, and here Durham Castle is his great memorial. The historically important castle dates back at least as far as the 10th century, from the 11th century until 1837 it was one of the palaces of the Bishops of Durham, and today it is one of the great historic monuments of England. In 1928 it was discovered that the foundations of the castle were insecure and the castle was gradually sliding down the cliff. The University exhausted all its available means, and despaired of raising the vastly greater sums needed to save the castle from destruction. Heawood alone would not give up. He assumed the secretaryship of the Durham Castle Restoration Fund, and attempted to collect the enormous amounts needed, toiling on year after year practically single-handed. His heroic efforts were finally crowned with success, the newly founded Pilgrim Trust came to the rescue with a very large grant, and after prolonged works the castle was permanently rescued. Heawood's devotion was rewarded by an Honorary D.C.L. conferred by Durham University in 1931, and his success was recognised by the award of the O.B.E. in 1939.

Heawood's religious devotion and Greek and Hebrew scholarship found their expression in numerous articles contributed to a number of theological journals. The last one appeared in 1951, when he was 90, in an American journal: its subject was the date of the Last Supper. Like his wife, he was interested in the work of the Church Missionary Society; he acted as Diocesan Treasurer almost until his death, and for many years he represented the diocese as a layman on the Church of England Assembly.

In his appearance, manners and habits of thought, Heawood was an extravagantly unusual man. He had an immense moustache and a meagre, slightly stooping figure. He usually wore an Inverness cape of strange pattern and manifest antiquity, and carried an ancient handbag. His walk was delicate and hasty, and he was often accompanied by a dog, which was admitted to his lectures. He had a very loud voice singularly lacking in modulation; he detested and was impossible on the telephone. But strangers who presumed that the inner man corresponded to outward appearances were often rudely shocked. For example, during his early days at Durham while he was teaching a class of divinity students mathematics, one of them asked "You have taught us to cast out the nines, can you cast out devils?" "Yes, I can," replied Heawood, "get out at once!" He set his watch once a year, on Christmas Day. "No, it's not two hours fast, it's ten hours slow." All his life was organised on equally logical but fantastic lines. He was a precisionist even in small matters. Soon after he appeared in Durham legends began to collect and cluster thick around him. He did not cultivate his reputation as a character, he just went his way and did not care whether people laughed. His transparent sincerity, piety and goodness of heart, and his eccentricity and extraordinary blend of naivité and shrewdness secured for him not only the fascinated interest, but also the regard and respect of his colleagues. He was acute and competent in the transaction of business, and conservative in his views and politics. He had some prejudices and limitations of sympathy where he was not open to conviction, but he was fair-minded in his judgment and tolerant of others.

There was one professor of and two lecturers in mathematics at Durham up to 1939. The professor was *primus inter pares*, and Heawood regarded himself as such. His lectures were considered good, the chief criticism being that he paid much more attention to what was on the blackboard than to his audience. Right up to his retirement from his professorship in 1939 at the age of 78 he took his full share in the teaching.

The University was founded and, until 1937, controlled by the Dean and Chapter of Durham Cathedral, and the main tension in the University during the period 1900–1937 was between those who wished to maintain the ecclesiastical ethos and those who wished to make the University more modern. Heawood, with his strong religious connections, was generally numbered with the former. He was, for instance, reluctant to make the initial application for a Government grant in 1921, and was unsympathetic to the introduction of science teaching a little later. He did, of course, loyally accept the changes. It seems that while on everyday matters of administration and in the resolving of difficulties his judgment and advice eminently sound and wise, and his energy and patience very great, his attitude to major questions of policy was backward rather than forward looking.

Towards the end of his long life he was indeed a Nestor in Durham. He had been teaching there for some years before the oldest member of the contemporary University Senate was born. He wrote a noteworthy mathematical paper when almost 90, an astounding achievement. He conducted correspondence with mathematicians up to 1954, and his fine intellect remained to the end.

His wife died a few months before him. Their two children had distinguished careers as headmaster and headmistress respectively. His son Geoffrey survives him.

Heawood published papers on the colouring of maps, continued fractions, the theory of approximations, quadratic residues and reciprocity, the theory of functions and geometry, and he made numerous contributions to the *Mathematical Gazette*. His most prominent contributions to mathematics were those concerned with the colouring of maps; he was the chief architect of this branch, the central subject of which is the Four Colour Problem.

In 1852 De Morgan, in a latter to Hamilton, wrote that a student had noticed, while colouring a political map of England, that using a stock of four colours the counties of England could be coloured in such a way that each county had one of the four colours, and no two neighbouring counties—that is to say counties with a common line of frontier—were coloured alike; he (De Morgan) was asked whether all maps could be coloured with four colours in this way, and he thought that they could be but he was unable to discover a proof. Hamilton replied that this was a quaternion which he did not wish to work on.

During the decades that followed this four colour problem appears to have attracted increasing attention among mathematicians. In 1879 Kempe published a paper " On the geographical problem of the four colours " (*American Journal of Mathematics*, vol. 2 (1879), pp. 193–200), which purported to contain a proof of the four-colour theorem. This proof appears to have been accepted as valid by all concerned until it was refuted by Heawood in his first paper [**1**].

Before going on with the account of Heawood's work on the colouring of maps it will be helpful to have the following brief explanatory remarks.

1. The term *colouring* of a map will mean a colouring of the divisions of the map in which no two divisions with a common line of frontier (neighbouring divisions) have the same colour; two divisions without a common line of frontier but with one or more common frontier points may be coloured alike.
2. Heawood and his contemporaries were exclusively concerned with maps containing only a finite number of divisions. Today we know that if the four colour theorem is true for all such maps then it is true also for all maps containing infinitely many divisions. *In all that follows all maps are assumed to contain only a finite number of divisions.*
3. The four colour conjecture is true for all maps if it is true for all maps on the sphere in which all the divisions are simply connected and any two divisions either meet in one frontier line only or in one frontier point only or not at all (the reader can easily verify this), such maps may be called *simple.*
4. A map need not cover the whole sphere, and all its divisions together need not form a simply connected region.

Kempe's fallacious proof was in essence as follows: If the answer to the four colour problem is in the negative then there exists a simple map on the sphere with the property that five colours are needed for its colouring and that if any one of its divisions is ignored, then the remainder can be coloured with four colours. Let M denote such a map. It follows from Euler's equation for polyhedra that any map contains divisions with fewer than six neighbours; let z denote such a division of M. It follows at once from the minimal property of M that z has at least four neighbours. Suppose first that z has four neighbours a, b, c, d, and that they occur around z in this cyclic order. The divisions of M different from z can be coloured with four colours, but there is no colouring with four colours in which two or more of a, b, c, d are coloured alike. Let it be assumed that in some colouring C of the divisions of M other than z with four colours, the colours of a, b, c, d are red, blue, green and yellow, respectively. Consider the map formed by all the red and all the green divisions; this map covers one or more connected regions, and the divisions of M comprising such a connected region are called a red-green *Kempe chain* with respect to C. If the colours red and green are interchanged for the divisions comprising a red-green Kempe chain, and the colours of all the other divisions are left unchanged, then a new satisfactory colouring of the divisions of M different from z is obtained. If red and green are interchanged in the red-green chain to which a belongs then the colour of c is changed from green to red, for otherwise M could be coloured with four colours by giving z the red colour. Therefore a and c belong to the same red-green chain. This chain and z together separate b from d, therefore b and d do not belong to the same blue-yellow chain. Consequently a four-colouring of M is obtained by interchanging the colours blue and yellow in the blue-yellow chain to which b belongs and colouring z blue. But this contradicts the hypothesis that M is not four-colourable, therefore M contains no division with fewer than five neighbours. It follows that M contains a division with exactly five neighbours, z' say. Suppose that z' has the five neighbours a', b', c', d', e' and that they occur around z' in this cyclic order. The divisions of M other than z' can be coloured with four colours, but only in such a way that each of these colours occurs around z' at least once. Hence it may be assumed that the divisions of M other than z' have been coloured with four colours so that the colours of a', b', c', d', e' are respectively blue, red, green, yellow, red. If a' and c' belong to different blue-green chains, interchanging the colours in either, a' and c' become both green or both blue, afterwards z' can be coloured blue or green. If a' and c' belong to the same blue-green chain, see if a' and d' belong to different blue-yellow

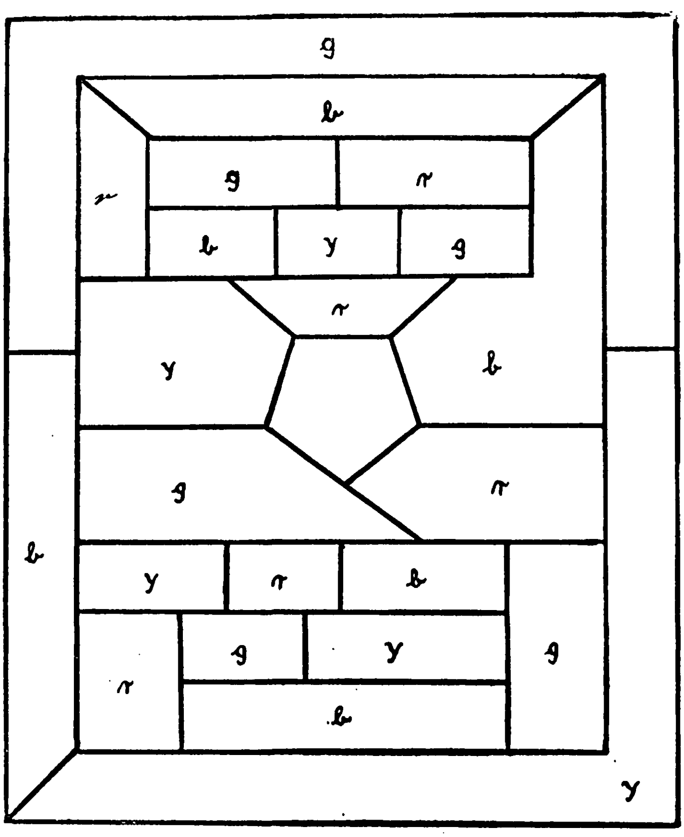

FIG. 1

chains; if they do, interchanging the colours in either chain, a' and d' become both yellow or both blue. If a' and c' belong to the same blue-green chain, and a' and d' belong to the same blue-yellow chain, the two chains cut off b' from e', so that the red-yellow chain to which b' belongs includes neither d' nor e', and the red-green chain to which e' belong-includes neither b' nor c'. " Thus, interchanging the colours in the reds yellow chain to which b' belongs, and in the red-green chain to which e' belongs, b' becomes yellow and e' green, a', c' and d' remaining unchanged." Afterwards z' can be coloured red, and so M is four-colourable. This contradiction proves the four colour theorem.

Kempe's proof seems to have been accepted for over ten years ; for example, Cayley proposed the problem to the London Mathematical Society in 1878 and did not subsequently raise it again, and Heawood in the preface of his paper of 1890 wrote : ". . . . The present article does not profess to give a proof of this original Theorem; in fact its aims are so far rather destructive than constructive, for it will be shown that there is a defect in the now apparently recognised proof. . . ." In order to appreciate this example of Heawood's penetration and ingenuity, the reader should pause here and try to discover the defect in Kempe's argument, which is contained in the sentence in quotation marks.

Heawood first explains Kempe's argument and then writes : " But, unfortunately, it is conceivable that though *either* transposition would remove a red, *both* may not remove both reds. *Fig.* 1 is an actual exemplification of this possibility, where either transposition prevents the other from being of any avail, by bringing the red and the other division into the same chain, so that Mr. Kempe's proof does not hold. . . ."

Besides the four colour problem the following topics were treated in Heawood's first paper.

1. *The five colour theorem.* Heawood proved rigorously that any map drawn on the plane or on the sphere can be coloured with at most five colours. His proof does not assume that the maps considered are simple. It goes as follows : Suppose that there exist maps on the plane or on the sphere which cannot be coloured with five colours. Let M denote one of them containing a minimal number of divisions : M is not assumed to be simple. All divisions of M clearly have at least five neighbours, and it follows from Euler's equation that some have exactly five. Let z denote a division of M having the divisions a, b, c, d, e as neighbours. Two neighbours of z can be selected which have no common frontier line, because it is impossible to draw a map with five divisions each a neighbour of the other four. Suppose that a and c are not neighbours. Let M^* denote the map obtained from M by uniting a, c and z into one division. By the minimal property of M, M^* can be coloured with five colours. Any five colouring of M^* automatically furnishes a five-colouring of the

divisions of M different from z in which a and c have the same colour. But z is then adjacent to at most four different colours, therefore M can be coloured with five colours. This contradiction proves the five colour theorem.

Heawood's invention of reducing the number of divisions by deleting separating frontier lines has been applied *inter alia* by G. D. Birkhoff, Franklin and Winn. Their work has established the truth of the four colour theorem for all maps with at most 35 divisions.

2. *The number of colours required for maps on higher surfaces.* Heawood proved that for all $h \geqslant 2$ any map drawn on any surface of connectivity h can be coloured with at most $[3\frac{1}{2}+\frac{1}{2}\sqrt{(24h-23)}]$ colours. He also showed that a map consisting of seven divisions, each a neighbour of all the others, can be drawn on the torus ($h = 3$, see *Fig.* 2). In the case of the other surfaces he left open the question whether there exist maps on them requiring the full number of $[3\frac{1}{2}+\frac{1}{2}\sqrt{(24h-23)}]$ colours, he conjectured that this was the case. It has since been proved (principally by G. Ringel) that Heawood's guess is correct for unorientable surfaces for all h except $h = 3$ (Möbius strip and Klein bottle) and for orientable surfaces for the majority of values of h. The phrasing of Heawood's remarks indicates that he may have believed that all the maps requiring the full number of $[3\frac{1}{2}+\frac{1}{2}\sqrt{(24h-23)}]$ colours contain this

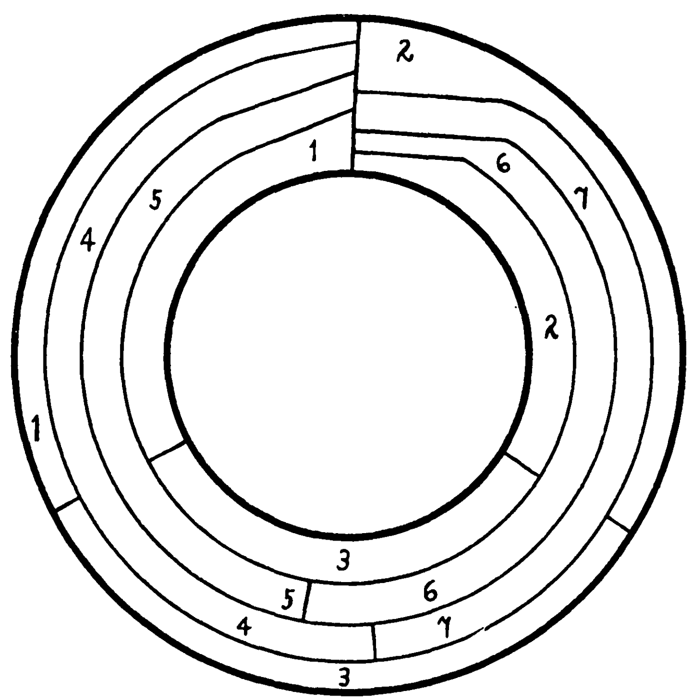

FIG. 2

many mutually neighbouring divisions, and it has since been proved that this is actually the case for all h except possibly $h = 2$ and $h = 4$.

3. *The problem of the "colonies".* In actual maps a country sometimes consists of several detached portions which all have to have the same colour; the same is the case for a country and its colonies and for seas and lakes. Heawood proved that if r is the maximum number of detached portions of a country or an empire, then for $h \geqslant 1$ the map drawn on any surface of connectivity h can be coloured with at most

$$\tfrac{1}{2}(6r+1+\sqrt{\{24h+(6r+1)^2-72\}})$$

colours, except possibly when $h = r = 1$. For $h = 1$ and $r = 2$, *i.e.* the plane or the sphere and each country consisting of at most two detached portions, this gives 12 colours. Heawood discovered the ingenious example of a map consisting of 12 countries, each in two portions, which needs 12 colours for its colouring shown in *Fig.* 3.

"Map-colour theorems" was an epoch-making contribution to the theory of map colouring and the starting impulse for subsequent investigations up to the present because it introduced new aspects, namely maps on the higher surfaces and maps with colonies, and because it settled many of the new and at first sight more difficult questions, while the four colour conjecture emerged as the central unsolved problem.

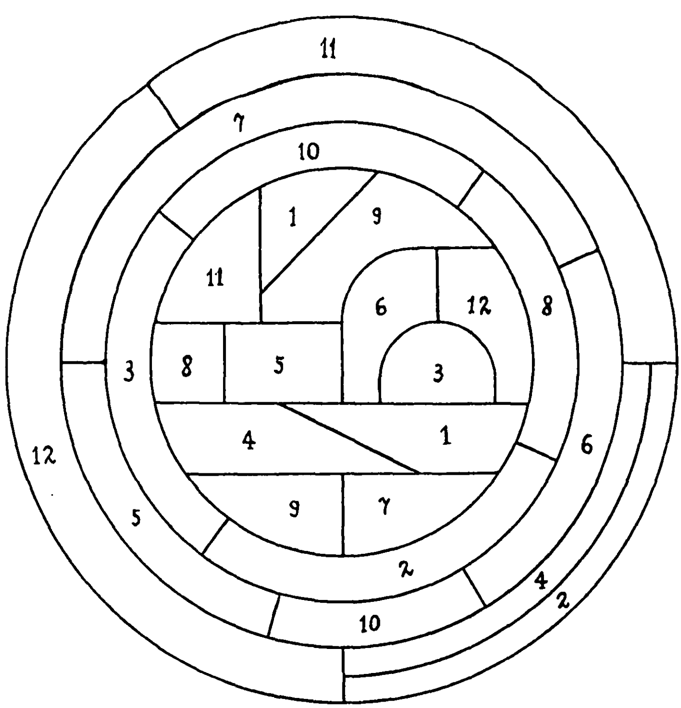

FIG. 3

" *On the four-colour map theorem* " [4] is a sequel to " *Map-colour theorem* " and is concerned with the four colour problem exclusively. The maps are supposed to be drawn on a sphere, and it is assumed that they are simple, and that at most three divisions concur at a point. If the four colour theorem were true for all such so-called ' *standard* ' maps then it would be true for all maps because any simple map can be made standard, if necessary, *e.g.* by adding new small circular divisions where more than three divisions concur, and if the new map can be coloured with four colours then so can the original map. Most of the assertions stated in [4] are not actually proved, only made plausible, but they have since been proved rigorously by other writers, which indicates that Heawood was in possession of all the necessary proofs but did not choose to include them.

The following map-colour theorems are stated in [4] :—

1. *If each division of a standard map has an even number of neighbours, then the map can be coloured with three colours.*

2. *If the number of neighbours of each division of a standard map is a multiple of* 3, *then the map can be coloured with four colours.*

Further, each of the following two properties is stated to be a necessary and sufficient condition for a standard map to be four-colourable :—

I. The frontier lines can be partitioned into three mutually disjoint classes in such a way that no two in the same class meet at a point.

II. The numbers $+1$, -1 can be so distributed among the corner points of the map (points of concourse of three frontier lines or points of concurrence of three divisions) that each corner point is assigned one of the two numbers, and around each division the sum of the numbers assigned to the corner points is a multiple of 3.

I. was first discovered by Tait. Such a partitioning of the frontier lines is nowadays called a factorisation into three 1-factors. II. was not known before Heawood's paper. Such distributions of two signs or numbers among the corner points are nowadays called " Heawood's congruences ". They have been used in investigations of the four colour problem with the help of computing machines. Most of the paper is devoted to pursuing a proof of the four colour theorem along the lines indicated by 2., I. and II. The problem is ingeniously illuminated from different directions in this way, but it is of course recognised to remain unsolved.

Heawood's remaining papers on the four colour problem [**8, 9, 10, 11, 12**] appeared between 1932 and 1950; during the 35 years between [4] and [8] he published three papers on other subjects.

It is easily proved that the four colour theorem is true for all maps, if it is true for all standard maps which cover the sphere completely and in which each division has at least five neighbours; Heawood calls such maps "*normal*". [8] to [12] are concerned with proving that II. holds for normal maps; they represent successive steps in this direction made with great tenacity, inventiveness and penetration.

If the corner points of a map are labelled $x_1, x_2, .., x_n$ then II. is equivalent to the solution of a system of congruences of the form

$$x_p+x_q+x_r+... \equiv 0 \pmod 3$$

in which there is one congruence corresponding to each division—the variables occurring in it are the corner points adjacent to the division—with $x_i \not\equiv 0 \pmod 3$ for $i = 1, ..., n$. Each of the variables occurs in exactly three of the congruences. "But this particular set of congruences is only one of a greatly extended system, differing from the above only in having on the right 0 or 1 or 2, and not necessarily zero. Moreover any distribution of $+1$ and -1 at the corner points of the map will necessarily correspond to the solution of *one* such system."

In [8] it is "shown in typical instances, how few are the cases out of the immense number of such systems related to a given map in which solution is impossible; and some grounds are shown for believing that, when there are more than a very limited number of divisions, such exceptions do not occur at all." For the dodecahedron it is shown that all but 8 of the 1835 essentially different congruence systems, obtainable by assigning one of 0, 1, 2 to each division, have solutions. For the normal map covering the sphere and consisting of two hexagons opposite to each other, each surrounded by six pentagons, the number of essentially different congruence systems is 68,369, and the number of failures out of these is shown to be 11. The proportion of failures is shown to diminish if one of the hexagons in this map is divided into two adjacent pentagons, and to diminish still further if both the hexagons are so divided.

In [9] it is proved that if the numbers 1 and 2 are assigned to two adjacent divisions of a normal map and 0 to all the remaining divisions, then the system of congruences has no solution with $x_i \not\equiv 0 \pmod 3$ for $i = 1, ..., n$. The persistence of some of the failures registered in [8] was thus inevitable. The proof of failure with this assignment is based on Heawood's famous "doctrine of residuals" developed in this paper.

A closed circuit (closed Jordan curve) made up of frontier lines divides the map into two parts. "For convenience one of these parts is spoken of as the *outside* and the other as the *inside* of this circuit. Let one of the values $+1$, -1 be assigned to each corner point of the circuit (*i.e.* corner point of the map lying on the circuit). Let $m_1, m_2, m_3, ...$ be the amounts (mod 3) contributed to the successive congruence-sums for divisions adjacent to the circuit on the inside, by the variables associated with points along its course; $n_1, n_2, n_3, ...$ the like for those on the outside. Suppose that (either for the m's or the n's) we have a sequence of ' digits ', such as 0 1 1 2 1 2 2 2 0 0 1 0 taken cyclically, and that we ' reduce ' them according to the following rules :

(a) a ' 2 ' may be cancelled and 1 added to the adjacent digits;

(b) a ' 1 ' may be cancelled and 2 added to the adjacent digits;

(c) a ' 0 ' and the two digits adjacent to it may be replaced by their sum.

[3 is, of course, to be replaced by 0 and 4 by 1, in any such result.]

" After (a) or (b) the sequence will include one digit fewer, after (c) two digits fewer; but after any of the three, their sum will be congruent to the same value as before, *i.e.*, in the case with which we propose to deal, congruent to zero. After a succession of such steps, then, we necessarily arrive finally *either* at two digits 0 0 or 2 1 or 1 2 *or* at a single digit 0. We regard the essential alternatives for ' residuals ' as being 0 0 on the one hand, and 2 1 or 1 2 or 0 on the other, the last three being treated as equivalent. On this understanding, we are prepared to show (1) that, for any given sequence whose sum is congruent to zero, the residuals will be the same, whatever the order of the steps of ' reduction ' may be; and (2) that the residuals for the m's and the n's are necessarily the same."

The significance of this is that if one circuit C encloses another circuit C' and $+1$, -1 have been distributed among the corner points in such a way that the sums around the divisions on the same side of C as C' are all $\equiv 0 \pmod 3$, then if the sum of the inside m's for C' is zero, the sum of the inside m's for C is zero.

The doctrine of residuals is the basis of the investigations in [**10**], [**11**] and [**12**]; in these papers Heawood attempts to prove the four colour conjecture by showing, with the help of the doctrine of residuals, that if a suitable pentagonal division and its five neighbours are first ignored, then at least one of the solutions of the system of congruences related to the remaining map, with zero assigned to each division, can be extended to give a solution of the system of congruences for the whole map with zero assigned to each division. In [**10**] and in [**11**] the geometrical aspect is kept in the foreground, in [**12**] the spatial relations involved recede; at the end of this last paper, which he completed in 1947 at the age of 85, Heawood wrote :—

" It cannot be pretended that the above analysis covers all possible cases . . . and it is fitting that the author should content himself with the claim that he has shown how the map-colour theorem may be made to depend on the persistence of a form of arithmetical sequence, based on a matrix-operator whose structure is independent of all spatial relations and whose laws are sufficiently remarkable. Further that the more obvious ways in which such a sequence might be terminated fail to achieve this result. And if the analysis has not been carried to the point of demonstrating that in all possible cases such persistence is assured, he is content to leave it as a challenge to others to show exactly where the analysis fails and if possible to complete it."

Seventy-two years have elapsed since the publication of Heawood's first paper on the colouring of maps. Since then some advances have been made; for example, the four colour theorem has been proved to hold for certain kinds of maps, and several problems concerning the colouring of maps on the higher surfaces have been settled. But Heawood's first two papers on the subject contain the most important contributions up to now, his discoveries are more substantial than all later ones by all others put together. The four colour problem, which has defied all the many efforts made to solve it, is Heawood's great mathematical monument.

Acknowledgments. Most of the details of Professor Heawood's life and activities at Durham were very kindly supplied by Professor J. L. Burchnall. Professor T. W. Chaundy located and forwarded all the material on which the account of mathematical teaching at Oxford in Heawood's time is based. The Rector of Exeter College, Oxford, kindly supplied exact data concerning Professor Heawood's career at Oxford. The Registrar of the University of Durham obligingly forwarded a biography of Professor Heawood at Durham. Some facts have been taken from Sir James Duff's obituary notice on Professor Heawood in *Nature*, vol. 75, p. 368 (26 February, 1955).

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In addition to the above articles and notes Heawood also contributed answers to queries, solutions of problems, and book reviews to the *Mathematical Gazette*.