HANS ARNOLD HEILBRONN 1908–1975

HANS ARNOLD HEILBRONN

J. W. S. CASSELS AND A. FRÖHLICH†

Hans Arnold Heilbronn was born in Berlin on 8 October, 1908. No doubt, his home and upbringing was one typical of the cultured German-Jewish middle class, which had in those days been thoroughly assimilated into German life. In many small ways Heilbronn's habits, his directness, his correct manner—in the true sense of the word—his strong, genuine sense of propriety, his apparent stiffness, as well as his accent, bore witness to his German background.

From 1914 to 1926 the boy attended the Realgymnasium Berlin-Schmargenhof, a school comparable to an English grammar school, the prefix " Real " indicating emphasis on the sciences and on modern languages, rather than the classics. In 1926 he entered university, reading mathematics, physics and chemistry, but evidently his interests were veering more and more towards mathematics. As is customary in Germany, the young student moved around, first attending his home university in Berlin, then going on to Freiburg and ending up in Göttingen, at that time the undisputed centre of German mathematics. One would suppose that he fitted well into the accepted pattern of student life in the Germany of those days. It was only the rebels who abhorred duelling—Heilbronn carried a duelling scar from his Göttingen days throughout his life.

In 1930 the budding mathematician became assistant to E. Landau, the leader of a flourishing school of analytic number theory in Göttingen. In 1933 Heilbronn was awarded the D.Phil., the contents of the doctoral dissertation being **[2]** in our publications list. By the end of 1933 he had published six papers, some jointly with Landau, and he had already made a mathematical name for himself, as was to become evident when practical assistance was needed after Hitler's rise to power.

By all accounts Landau was a rather formidable man and a demanding taskmaster. Heilbronn seems, however, to have managed very well. Landau came to have a very high opinion of him, and, judging by letters he wrote on his behalf in 1933, considered him his star pupil. On the other hand, those years with Landau were formative ones for Heilbronn. Throughout his life his principal research interest remained with number theory, largely analytic. In the shorter term, Landau also clearly exerted an influence on Heilbronn's mathematical style—witness his early papers.

This may be the right place to record an anecdote which, many years ago, Heilbronn told one of us (A. F.), who was then his research student. Heilbronn had been rather negative in reply to a conjecture put forward by A. F., but then, just as A. F. was leaving the room, he added: " You must never take too much notice of pessimistic comments from your supervisor, or from any other mathematician, however great ". (Advice for which A. F. has always been grateful—incidentally, he did prove his conjecture.) Heilbronn then recalled that he, when a student of Landau, had once raised the question of improving the estimates for trigonometric sums along the lines on which the Russian mathematician Vinogradov soon afterwards succeeded—only to be greeted with the answer from Landau that this was hope-

† This obituary has previously appeared in the Biographical Notices of the Royal Society. It was prepared by Professor J. W. S. Cassels and Professor A. Fröhlich, and we are grateful for permission to reproduce it.

[BULL. LONDON MATH. SOC., 9 (1977), 219–232]

less. Heilbronn added: " I might not have got anywhere, but at that time I never even tried ".

In 1933, after the Nazis had come to power, Heilbronn, like Jewish scientists all over Germany, lost his position. In November he came to Cambridge, supporting himself for the time being. " There is no possibility for me to continue scientific work in Germany ", he writes in a letter to the Academic Assistance Council (about which more later), and " I have means for half a year ". In mid-December Heilbronn received an invitation from H. R. Hassé, Head of the Department of Mathematics in the University of Bristol, offering hospitality and financial support, and on 16 January, 1934 he came to Bristol, taking up residence at a students' hall of residence, Wills' Hall, " where the food is excellent (if you don't compare it with Trinity of course) ", as he writes in a letter to H. Davenport.

These few sentences do scant justice to the efforts which had gone into finding some place for Heilbronn to go to. It is fascinating to take a look behind the scenes. The pivot of all the activity on behalf of refugee scientists was the Academic Assistance Council (A.A.C.). This is not the place to record all the good it did. One has to remember that Heilbronn was just one of many.

The first mention of Heilbronn's name in the records of the A.A.C. is in a letter from Harold Davenport, dated 31 August, 1933. Davenport had just returned from a visit to Germany, having spent May–July in Göttingen and having, as he explains, come into contact with most mathematicians there. He singles out Heilbronn as specially deserving support, being " one of the most promising young German mathematicians ". (Incidentally, this confirms that the two knew each other by then —whether this was their first meeting is not known.) In October 1933 there follows a letter of support from Hardy. In the meantime Mordell had been active in Manchester and the Vice-Chancellor writes from there offering accommodation for Heilbronn, but asking for financial support from the A.A.C. Not unexpectedly the A.A.C., notoriously short of funds, had to say no. Similar regretful negative replies went to Hardy, and to Heilbronn himself. Then, on 8 November, Hassé from Bristol says: " I am considering the possibility of inviting a young German Jewish mathematician to take a temporary position ". He mentions Heilbronn, following a recommendation by Harold Bohr, and adds that he has asked the Jewish community in Bristol to help raise £200 p.a. " Can the A.A.C. help? " Reply on 10 November: " Our annual grant is £182 "—plus a conditional offer of considering a small supplement if most of the £182 can be raised locally. On 29 November another letter from Hassé: " The Jewish community will give £75 for three years "—so he needs another £107. No reply recorded. Then on 11 December, Hassé once more with better news: " I now have promises for a further £65 p.a. mainly from university colleagues ", with hopes for more. It is not clear where the balance came from, but in any case Heilbronn got his invitation.

Heilbronn's stay in Bristol extended over nearly a year and a half. During this period he sprang into mathematical prominence with his proof of the Gauss conjecture on class numbers of imaginary quadratic fields (see [7]). The scholarship in Bristol was, of course, only a temporary one, and one sees the ever-helpful Hassé raising the question of a more permanent position for the refugee, although, as he explains, he would like to keep him in Bristol for good. It seems that at the end of 1934 Mordell came forward with the offer of a two-year scholarship in Manchester, and Heilbronn did indeed spend some time in Mordell's department. In May 1935, however, he was awarded the Bevan Fellowship at Trinity and some time afterwards he moved to Cambridge. It is fairly clear that Hardy was the prime mover behind the award. There are several letters of Hardy's in the A.A.C. files singling out Heilbronn for special support, and this to be given " here " (i.e. in Cambridge) " or in Oxford, not Canada or Australia ". Hardy continued to keep a benevolent eye on Heilbronn over many years, in particular again intervening on his behalf in 1940, when the need arose.

Heilbronn soon put down roots in Cambridge. He brought his parents and his sister over from Germany. Their house in Cambridge remained his true home base for many years, even after the war when he had taken up a chair in Bristol. Patrick Duval, in a recent letter, probably best sums up Heilbronn's personality and life pattern in those days.

> " I met him when he first came to Cambridge, some years before the war: probably we were introduced by Davenport, but I'm not sure of that. He was I think rather shy, and not easy to get to know quickly. My mother lived in Cambridge then, and he went to tea with her once while I was abroad (it must have been 1935 I suppose); she wrote to me quoting Dr. Johnson, 'His intellect is as exalted as his stature; I am half afraid of him, yet he is no less amiable than formidable '. This seems to me to give a very good impression of him in those days. After he got his parents and sister out of Germany, and settled them in a house in Chesterton Road, we became very friendly. I was often at their house, and they used to come to ours. His mother was very musical, collected a circle of refugee musicians, and used often to give musical parties; she herself was an excellent pianist. I was at many of these parties; they rather bored Hans, who had no interest in music, but he was present at many of them, as a very correct and attentive host ".

Heilbronn's complete lack of gift or sympathy for music was well known among his friends, although for a short while in 1951 there seems to have been an attempt to redeem himself when he writes to Harold Davenport:

> " My latest vice is playing simple tunes on a children's flute. It made my family very depressed at times, but the dog tolerated it stoically."

The Cambridge period also saw the beginning of the close collaboration with Harold Davenport. Altogether ten papers appeared under their joint names, the first in 1936 and the last in 1971 after Davenport's death. Being geographically separated most of the time after 1937, this collaboration was partly carried on by numerous letters and postcards. Heilbronn and Davenport became lifelong friends and their correspondence covers not only mathematics, but personal news, university politics, the British government, administrative problems and much else besides. Anne Davenport recalls:

> " For many years, starting before the war and going on through the war when possible and after, till at least 1954—I think till 1957 or '58—Heilbronn and Harold went on a week's walking holiday—usually in the Easter vacation. They used to go off by car till they found a suitable place and they then spent the days walking or maybe rowing. Once they discovered Hartland, Devon, they often returned there. The walks got shorter with the years. All kinds of things were discussed—I think a detective story was planned—but there were also long companionable silences."

Heilbronn and Davenport had many interests in common and showed a similar outlook in many spheres. Both were staunchly conservative. Later on, in the late fifties and early sixties, they were to launch jointly an energetic campaign against indiscriminate, rapid university expansion, being worried by the dangers of falling standards and the difficulties of recruiting sufficient new academic staff in mathematics of the right quality.

To return to events in chronological order, after the beginning of the war, Heilbronn threw himself with characteristic vigour into the task of organizing Trinity's A.R.P. Fire Service, and he even succeeded, in order to facilitate fire fighting, in getting a door opened between Trinity and St. John's. To quote the junior bursar of Trinity in a letter written on his behalf in 1940, as Heilbronn was legally a foreigner this was " the only effective expression of his loyalty to his country which circumstances allowed him ".

In 1940 when the Bevan Fellowship expired, the College Council at Trinity resolved to continue Heilbronn's salary and rights. Other events, however, intervened. Heilbronn had actually applied for British citizenship in April 1939, but because of the war his naturalization was to be delayed until October 1946. Thus, when the crisis of 1940 came, he, like many other staunchly anti-Nazi refugees, found himself interned as an " enemy alien ". This he deeply resented. Like many other " friendly enemy aliens " he had to wait many months for his freedom. Again the ubiquitous Academic Assistance Council acted on his behalf, as on that of many other refugee scientists. We mentioned Hardy's intervention earlier. On his release he enlisted in the Pioneer Corps and was subsequently transferred to the Royal Corps of Signals and in 1943 to Military Intelligence.

Heilbronn served with the British army until autumn 1945. In the autumn of that year he took up a post at University College London where Davenport had become professor. Not surprisingly there is no record of creative work by Heilbronn during his army service. He did, however, find his way back into active research very quickly. His first postwar paper **[21]** jointly with Davenport was submitted in March 1946.

In 1946 Heilbronn returned to Bristol, first as reader and from 1949 as professor and head of department. He made his home in Bristol at the Hawthorn Hotel, just across from the Royal Fort where the Department of Mathematics was then housed. During the vacations—at least in the first years—he tended to return to the house in Chesterton Road, Cambridge. Patrick Duval's description of Heilbronn as he was in the early years in Bristol cannot be bettered.

> " During the two years I was there [in Bristol], he became a very close family friend. He was still rather stiff and formal in external manner. I remember the shock it gave our colleagues when my wife and I both automatically called him Hans; no one had dreamed of such a familiarity; but by the time we left the whole Department was following our example; and I believe basically he was glad of this, and I'm sure the whole atmosphere became much easier as a result. I have a feeling he wanted to be popular, but found it uphill work. He delighted in hospitality, and used to give good dinner parties.
>
> " He was very keen on all kinds of sport, especially rowing and tennis. In interviewing students for admission he always asked about their sporting interests, and was unfavourably impressed if they had none."

Heilbronn's fondness of children is stressed in a letter from Anne Davenport, and she also recalls his formality of manner.

> " When at University College London [in 1945] Hans stayed regularly at least one night a week with us. It was still ' Heilbronn ' and ' Davenport '. I got tired to asking ' Will you have . . . Dr. Heilbronn ' and when it became Professor Heilbronn I went on strike. I announced that I was going to call him Hans—he

was rather taken aback and asked what he was to do. Only then did the two men get onto first name terms."

During his tenure of the chair of pure mathematics, Heilbronn built up an excellent department in Bristol. He seems to have had his ear to the ground; for example, using his long stays in Cambridge to get to know of the brightest graduates and using his mastery of committees to secure Bristol lectureships for them. Among the appointments for which he was responsible were those of J. C. Shepherdson (at least indirectly), D. A. Burgess, C. Davis, C. Hooley, J. M. Marstrand and E. R. Reifenberg.

During this Bristol period Heilbronn exerted a major influence on mathematicians of the younger generation, which goes well beyond what has become apparent in the printed word. Some measure of his success is the number of former colleagues or students who now hold senior positions.

J. C. Shepherdson, who was to succeed Heilbronn to the chair of pure mathematics and who has made his name in mathematical logic, actually wrote his first paper on a number theoretic problem, suggested by Heilbronn (Shepherdson 1947). This was subsequently superseded by results of H. B. Mann.

One of us, as his research student in the late forties, was introduced by Heilbronn into algebraic number theory, in those days a topic which distinctly lacked popularity in Great Britain. Heilbronn's suggestion of this thesis subject was based on his idea that one could gain information on the ideal class group by viewing it as a module over the Galois group. This approach has nowadays found many adherents, but in those days it was rather novel. In this instance it led in particular to the determination of all absolutely Abelian fields whose degree was a power of prime l and whose class number was not divisible by l (Fröhlich 1954a, 1955), and to a criterion for the genus of a cyclic field of degree lr to have a class number prime to l (Fröhlich 1954b). Although Heilbronn himself never published anything on this topic of ideal class groups, it interested him deeply and he kept on coming back to it. He suggested problems in this area again to his last Ph.D. student in Toronto, T. Callahan, who has since published papers on this subject (Callahan 1974a, b, 1975).

To return to the fifties and to Bristol, many other members of the department, notably the number theoreticians, were stimulated by Heilbronn and helped by his active interest in their work. He was always accessible. His manner was most direct and his intellectual honesty did not allow him to pass over in politeness whatever he felt was below standard. But although some may have found him a bit frightening at first acquaintance, he was a kind and modest man, who entirely lacked pomposity or aggressiveness.

In the late fifties developments in universities began to worry Heilbronn, and he became increasingly upset by what, in his eyes, were wrong decisions, both nationally and locally. As already noted earlier, he was entirely out of sympathy with academic policy and government planning in the Robbins period, and he spent a great deal of time and energy in trying to convince policy makers and colleagues of the dangers he saw ahead. It must have been a very frustrating time for him. In Bristol University he had always been a vigorous champion of his department. He had fought many battles, won some and lost some. Late in 1962 he began to feel that he had been let down, and with characteristic uncompromising directness he submitted his resignation in April 1963. At that stage he had no definite idea where he would go, but this did not seem to worry him at all. He left the university in March 1964.

Heilbronn's decision was deeply regretted by all his departmental colleagues, but the effect of his resignation on himself was wholly to the good. He behaved like a man suddenly released from captivity. Whereas during the preceding few years, it had become increasingly difficult to have a mathematical conversation with him, in the face of his preoccupation with policy problems and administrative irritations, he suddenly wanted to talk again and listen again, and he was once more full of ideas.

The year 1964 also saw a new departure in Heilbronn's personal life. In March he married Mrs. Dorothy Greaves, who had been widowed some years earlier and whom he had met in Cambridge. They both shared in particular a passion for bridge. This had indeed become Heilbronn's principal relaxation and one was told that he was an expert at the game.

After leaving Bristol Heilbronn, with his wife, first spent some time at the California Institute of Technology at Pasadena, following an invitation from Olga Taussky-Todd, an old friend and colleague. Later that year the couple settled down in Toronto, where he had accepted the offer of a chair at the university. Heilbronn had spent some time 24 years earlier in Canada, then as visiting professor at the University of British Columbia, Vancouver. He now found it quite easy to settle down permanently in Canada and he eventually took out Canadian citizenship in 1970. The couple left their mark, both among their friends at bridge and among their friends in the university, and the many mathematical visitors who came through and experienced their generous hospitality.

Mathematically Toronto gained a great deal from Heilbronn's presence. He found some number theoreticians in Toronto, notably John Chalk whom he had first met in 1945 at University College. With great vigour and drive he built up an active research school, supervising a number of Ph.D.s, and running seminars and study groups at which he took the major part in presenting new developments in his own way. According to the participants, the lecture courses he gave on various classical topics or recent advances in algebraic number theory contained new ideas as regards approach and presentation, but none of this was ever published.

Heilbronn also took an active part in the wider mathematical life of Canada. As the delegate of the Canadian Mathematical Congress to the International Mathematics Congress in Nice (1970) he officially presented the invitation for the next International Congress to meet in Vancouver in 1974. He was an active and conscientious member of the committee that planned the academic programme for Vancouver. Only ill health prevented him from taking a more active part in the running of the 1974 Congress itself.

Heilbronn's robust health took its first setback in the fifties, when he developed a peptic ulcer. When, after some time, he was categorically forbidden wine, he distributed his excellent cellar among his friends. In November 1973 he had a heart attack. Although he did recover, he nevertheless lost some of his driving energy, and his physical activities had to be curtailed. He continued, however, to play an active role in the life of the department in Toronto, and was scarcely absent from his office for even a day. He had become intensely interested in the history of the transcontinental railroad system in Canada, and in summer 1974 he and one of us, together with our wives, travelled across the continent to Vancouver by rail, Heilbronn at each stage explaining some interesting local aspect or anecdote connected with the original construction. On 28 April, 1975 Heilbronn died during an operation to implant a pacemaker.

Heilbronn was a member of the London Mathematical Society, serving on its council for many years. Together with Davenport he oversaw the investments of the Society over a considerable period. He was President of the Society from 1959 to 1961. He was elected a fellow of the Royal Society in 1951. He was also a fellow of the Cambridge Philosophical Society and a member of the American Mathematical Society. Once in Canada he took an active part in the business of the Canadian Mathematical Congress, as already noted above, in connection with the International Congress of Mathematicians, and he was editor of the *Canadian Journal of Mathematics* 1967–69. In 1967 he was elected to the Royal Society of Canada and during 1971–73 he was a member of its council.

HEILBRONN'S MATHEMATICAL WORK

(*a*) *Göttingen period* (papers [**1**]–[**6**])

During this period Heilbronn was a pupil of Landau and latterly his assistant.

In his first paper [**1**] he shows that the number $P(\xi)$ of primes represented by an integral polynomial $f(x)$ for integral x, $0 < x < \xi$, satisfies the estimate $P(\xi) = O(\xi/\ln \xi)$. Previously only Nagell's weaker $P(\xi) = o(\xi)$ was known. For this he used a formulation of Brun's sieve which had recently been given by Landau and doubtless the problem was suggested by him. Subsequent workers have obtained more information about the constant implied by the O (see Halberstam-Richert 1974, especially §5, 6), but presumably the O cannot be replaced by o.

The second paper [**2**] is Heilbronn's doctoral dissertation. In 1930 Hoheisel had proved the existence of a $\theta < 1$ such that for all large x there is a prime p such that $x < p < x+x^{\theta}$; and indeed that the number of such p is asymptotically $x^{\theta}/\ln x$. Heilbronn's method is simpler than Hoheisel's and he applied it also to primes in arithmetic progressions and to estimates of the sum of the Möbius function. The value for θ obtained by Hoheisel namely $1-(3300)^{-1}+\varepsilon$ is very near 1 and Heilbronn's is distinctly better: $1-(250)^{-1}+\varepsilon$. Subsequently improved values of θ were obtained by a number of workers, the main milestones being $\frac{5}{8}+\varepsilon$ (Ingham 1937), $\frac{3}{5}+\varepsilon$ (Montgomery 1971, ch. 14, where there is an historical discussion) and $\frac{7}{12}+\varepsilon$ (Huxley 1972a, b, ch. 28), the best to date: for primes in arithmetic progressions, see also Gallagher (1972).

The three papers written together with Landau [**3**], [**4**] and [**5**] deal with applications and improved proofs of Tauberian theorems, in particular what is now known as the Landau–Ikehara theorem (for the modern context see H. R. Pitt 1958, especially §6.1). The only other paper from the Göttingen period is a simple proof of a strong form of Cauchy's theorem [**6**].

(*b*) *The class-number of imaginary quadratic fields* (papers [**7**], [**8**], [**16**], [**19**])

Soon after his arrival in Britain, Heilbronn made a considerable stir in the mathematical world by proving a long-standing conjecture of Gauss. This was that

$$h(d) \to \infty \qquad \text{(i)}$$

as $d \to -\infty$ where $h(d)$ is the class-number of the quadratic number-field of discriminant† d. Already before 1918 Hecke had shown that (i) would follow from an extended Riemann hypothesis. In 1933 Deuring had proved the unexpected result that the *falsity* of the (ordinary) Riemann hypothesis implies that $h(d) \geqslant 2$ for all large enough d and in a manuscript which was available to Heilbronn but only published subsequently Mordell (1934) had extended Deuring's argument to show that the falsity of the Riemann hypothesis implies (i). What Heilbronn does in [**7**] is to

† So $d < 0$ means that the field is imaginary. More precisely, Gauss had made the conjecture in terms of quadratic forms.

show more generally that the falsity of the generalized Riemann hypothesis used by Hecke implies (i). It follows (*tertium non datur*) that (i) is true unconditionally. Quite shortly afterwards Siegel (1935) gave the explicit estimate

$$\ln h(d) \sim \tfrac{1}{2}\ln|d| \quad (d \to -\infty)$$

together with a corresponding theorem for $d > 0$ (i.e. for real quadratic fields) involving the regulator with simple proof. An appropriate generalization to fields of any degree was obtained by Brauer (1950).

Numerical evidence strongly suggested that there are at most nine negative values of d for which $h(d) = 1$, namely

$$-3, \quad -4, \quad -7, \quad -8, \quad -11. \quad -19, \quad -43, \quad -67, \quad -163. \qquad \text{(ii)}$$

At almost the same time as [7] Heilbronn published a joint paper [8] with E. H. Linfoot (who subsequently had a distinguished career as an astronomer) showing by methods similar to those of [7] that there could be at most one further d for which $h(d) = 1$. The proof of Heilbronn and Linfoot consisted in deducing a contradiction from the hypothesis that two negative values of d with $h(d) = 1$ exist other than those given by (ii) and so gave no possibility of deciding whether or not one extra $d < 0$ with $h(d) = 1$ exists. The little known German mathematician Heegner claimed to have disproved the existence of the additional d (Heegner 1952) but the contemporary mathematical consensus was that the paper was wrong-headed and the problem remained until, practically simultaneously, Stark (1967*a*) showed that the additional d did not exist and Baker (1966) gave a general theorem which could be applied to this and to many other problems. A subsequent examination of Heegner's paper showed that it gave, in fact, a viable approach (Birch 1968, Deuring 1968, Siegel 1968, Stark 1975b). For later developments see A. Baker (1975, especially ch. 5). For an ineffective generalization of the Heilbronn–Linfoot paper see Tatuzawa (1951).

The two papers [7] and [8] just discussed were both, apparently, written after Heilbronn left Germany in the brief period when he was at Bristol. The later and minor paper [19] supplies a neat proof of a formula which plays a key role in the generalization by Siegel (1935) of [7].

Paper [16] comes into the same circle of ideas. It neatly disproves a conjecture of Chowla about iterated character sums which, by the result of Hecke mentioned above, would have given good estimates for the class-number of imaginary quadratic fields. (For later related work see Selberg & Chowla 1967 and Bateman *et al.* 1975.)

(*c*) *Analytic number theory* (papers **[9]**–**[15]**, **[18]**, **[21]**, **[40]**)

In 1934 and 1935 the Russian mathematician I. M. Vinogradov published a number of difficult and obscure papers giving vastly superior estimates in Waring's problem to those which had been obtained by Hardy and Littlewood. In [9], which appeared shortly after Heilbronn had taken up his fellowship at Trinity, he simplified Vinogradov's method substantially and somewhat improved the estimates generally. This was followed shortly by four papers with Davenport on specific problems of the Waring type (**[13]** is a technical lemma; **[12]**, on sums of fourth powers, was subsequently superseded by Davenport 1939; **[14]** is on sums of two cubes and a square, cf. Roth 1949; and **[15]** is on the sum of a prime and a kth power).

Schnirelmann had shown that there is some constant c such that every sufficiently large integer n is the sum of at most c prime numbers and in **[18]** Heilbronn, Landau & Scherk show that $c = 71$ is an admissible value. This was a very substantial improvement over existing estimates but was rapidly superseded by Vinogradov

(1937) who showed that every sufficiently large odd integer is the sum of at most three primes (" Goldbach's theorem ", see also Vinogradov 1954, 1971).

The papers **[10]** and **[11]** both written in collaboration with Davenport go in a rather different direction. Let $Q(x, y) = ax^2+bxy+cy^2$ be a definite quadratic form with integral coefficients a, b, c and discriminant $d = 4ac-b^2 < 0$. The Epstein zeta-function is defined by

$$\zeta(s, Q) = \sum_{\substack{(x,y)\neq(0,0)\\ x,y \text{ integral}}} Q(x, y)^{-s}.$$

When the class-number† $h(-d) = 1$ then $\zeta(s, Q)$ is indeed a Dedekind zeta-function with an Euler product; in particular it can have no zeros in $\Re s > 1$ (and a " generalized Riemann Hypothesis " conjectures that the non-trivial zeros are all on $\Re s = \frac{1}{2}$). Potter and Titchmarsh had shown that, whatever the value of $h(-d)$, there are always infinitely many zeros on $s = \frac{1}{2}$. In these two papers it is shown that if $h(-d) \neq 1$ there are always infinitely many zeros in $\Re s > 1$ and so any presumed Riemann hypothesis fails spectacularly (for later information on the zeros of Epstein's zeta-functions see Stark 1967b). In **[10]** Davenport and Heilbronn also consider the function

$$\zeta(s, a) = \sum_{n=0}^{\infty} (n+a)^{-s}$$

defined for any real $a > 0$. For $a = 1$ this is Riemann's zeta-function and so has no zeros in $\Re s > 1$. They show that if $a \neq 1$ is rational or if a is transcendental then again there are infinitely many zeros in $\Re s > 1$. (Their method was extended by Cassels (1961) to cover the remaining case: a algebraic.)

All the papers mentioned so far in this section date from Heilbronn's Trinity period. **[21]** was written with Davenport after the war and is rather different in nature. Let $\lambda_1, \ldots, \lambda_5$ be non-zero real numbers. Then it is shown that

$$\sum_{j=1}^{5} \lambda_j x_j^2$$

takes arbitrarily small values for integral values of the variables (not all zero). When the ratios λ_i/λ_j are all rational this is a special case of a classical theorem of Meyer so they can assume that, say, λ_1/λ_2 is irrational. This is dealt with by an elegantly simple modification of the Waring problem techniques. The result was the beginning of a large programme by Davenport and others on small values taken by quadratic forms (and, later, by forms of higher degree); but in this Heilbronn took no part.

At the very end of his life Heilbronn returned to the zeros of ζ-functions in **[40]**. If $\zeta_C(s)$ is the zeta-function of the algebraic field C then it is classical that the residue κ at $s = 1$ satisfies $\kappa^{-1} = O(\ln|d|)$, where d is the absolute discriminant, unless there is a real zero s_0 with $\frac{1}{2} < s_0 < 1$ (which would, of course, contradict the generalized Riemann hypothesis). In certain cases L. Goldstein and J. Sunley had obtained effective estimates for κ by essentially algebraic techniques. In **[40]** using techniques of Brauer and the analytic theory of the Artin L-function Heilbronn shows quite generally that the problem of the existence of such an s_0 reduces to the corresponding problem for quadratic number fields. Following up this approach, Stark (1974, 1975a) obtains the striking consequence that the Brauer–Siegel theorem about the magnitude of the class numbers of algebraic number fields can be made effective in a large number of cases.

† We have written $h(-d)$ for the $h(d)$ of the papers so as to preserve conformity with the notation used earlier.

(*d*) *Euclidean algorithm*

In **[20]**, written during the Trinity period, Heilbronn showed that there are only finitely many real quadratic fields with a euclidean algorithm. Partial results had been obtained by a number of mathematicians: in particular Erdős and Chao Ko had disposed of the case when the discriminant is prime and Heilbronn's proof is based on theirs. The estimates were such that it was hopeless to attempt to enumerate all the euclidean real quadratic fields, or at least it was not attempted.

After the war Davenport invented a new approach based on ideas of the Geometry of Numbers which gave a much better estimate of the discriminants of euclidean real fields and so enabled them all to be determined. Davenport's geometrical approach also showed the finiteness of the number of euclidean fields of two other types: non-totally real cubic fields and totally complex quartic fields (i.e. whenever the group of units has rank 1). (See Davenport 1950a, b, 1951 or Cassels 1952 for a slightly different treatment.) It appears that these are the only types of field that can be dealt with by geometric methods. At about the same time Heilbronn returned to the problem and showed (**[26]**, **[27]**) that his method applied to cyclic cubic fields (which, being totally real, are not covered by Davenport's results) and, more generally, for a wide class of cyclic fields. (For numerical results see Smith 1969.) This approach has not been carried further and it apparently remains an open question in general whether a given type of algebraic number field contains infinitely many fields with a euclidean algorithm. (It should perhaps be remarked that the term "euclidean algorithm" here refers always to that using the norm function. The general type of algorithm discussed recently by Samuel and others raises entirely different problems.)

(*e*) *Geometry of numbers* (papers **[22]**, **[23]**)

Both of these were written with Davenport. The first is rather special and discusses the minimum of $|(\alpha x+\beta y)(\gamma z+\delta t)|$ for given real α, β, γ, δ and for integral x, y, z, t satisfying $xt-yz = \pm 1$. It is shown that the first two minima are "isolated" and correspond to the first two minima of the Markoff chain but that then the spectrum has a limit point. The methods are more or less standard.

The result of **[23]** is more interesting, though the techniques are equally standard. Let ξ, η be linear forms in variables u, v with determinant $\Delta \neq 0$ and let λ, μ be real numbers. Then it is shown that there are integral values of the variables such that

$$\xi+\lambda \geqslant 0, \quad \eta+\mu \geqslant 0,$$

$$(\xi+\lambda)(\eta+\mu) < |\Delta|. \tag{iii}$$

It is shown that this statement becomes false if $|\Delta|$ in (iii) is replaced by $\kappa|\Delta|$ with any $\kappa < 1$ and there is an examination of various special cases. This theorem was generalized to n dimensions by Chalk (1947) and Macbeath (1952) gave a further generalization.

(*f*) *Miscellaneous*

Paper **[17]** proves an inequality which is, apparently, motivated by problems of additive number theory and **[24]** discusses a generalization of harmonic functions to real-valued functions defined on the integral two-dimensional lattice.

In **[25]** it is shown that for every $\eta > 0$ there is a $C(\eta) > 0$ with the following property: for every real θ and for every integer $N \geqslant 1$ there exist integers n, g such that

$$1 \leqslant n \leqslant N; \quad |n^2\theta - g| \leqslant C(\eta)N^{-\frac{1}{2}+\eta}.$$

A weaker form with $\frac{2}{5}$ instead of $\frac{1}{2}$ in the exponent had been obtained already in 1927 by Vinogradov, and Heilbronn's proof is similar in general shape but depends on a more delicate analysis of the remainder term. Heilbronn raises the question whether the $\frac{1}{2}$ in the exponent is best possible but no progress has been made here. The paper did, however, spark off a series of generalizations in other directions (Davenport 1967, several papers by Danicic, e.g. 1959, 1967, and Cook, e.g. 1972, 1975, Liu 1974).

Paper **[28]** arose from work of Fröhlich (1954b) on cyclic fields whose degree is a power of a prime l. He obtained necessary and sufficient conditions for the class number of the principal genus to be prime to l. These conditions are in terms of the mutual congruence behaviour of the discriminant prime divisors and in **[28]** Heilbronn deals with the density of, say, pairs of primes with given mutual congruence behaviour. The actual consequences for cyclic fields have never been stated explicitly.

Paper **[29]** answers a question posed by Kuratowski by showing that if $f(x)$ is a complex-valued function defined on an open set A of the complex plane then there is a regular function $F(x)$ and a continuous function $g(x)$, both defined on A, such that $f(x) = F(x)e^{g(x)}$ and there is some discussion of the problem of when every homotopy class of maps from one set A of the complex plane to another set contains a map given by a regular function.

Paper **[31]** is similarly a response to a problem. It is well known that every rational $x > 0$ can be put in the shape

$$x = {y_1}^2 + {y_2}^2 + {y_3}^2 + {y_4}^2,$$

where $y_1, \ldots, y_4$ are rational. Heilbronn shows by a rather elaborate argument that the y_j can be chosen in such a way that they are continuous functions of x: indeed there are integral functions $f_j(x)$ such that $\sum(f_j(x))^2 = 1$ and $y_j = x^{\frac{1}{2}} f_j(x)$ is rational whenever x is rational. There is a rather weaker theorem for sums of higher powers.

Paper **[32]** with Erdős discusses the number F of solutions of a congruence

$$e_1 a_1 + \ldots + e_k a_k \equiv N \pmod{p},$$

where $a_1, \ldots, a_k$, N are given and $e_1, \ldots, e_k$ take only the values 0, 1. It is shown that $F > 0$ provided that $k > 3(6p)^{\frac{1}{2}}$ and that $F \sim p^{-1} 2^k$ provided that k is large compared with $p^{2/3}$. (For the latest state of the problem, see Olsen 1975.)

Paper **[36]** is concerned with the length $n(a) = n_N(a)$ of the continued fraction for a/N where the integer $N > 1$ is given and a is prime to N. It is shown that the average value of $n(a)$ is asymptotic to $12\pi^{-2}$ (ln2)(ln N). The error term was subsequently improved by Porter (1975). The detailed argument of **[36]** was used by Manin (1972) for the computation of his "modular symbols". See also Yao & Knuth (1975).

Although **[38]** was published only in 1969 the work which it records had been done much earlier. Davenport and Heilbronn had obtained estimates for the numbers of cubic fields (for the two cases: totally real and otherwise) whose discriminants were below a given bound but, as they could not obtain results as precise as they had hoped, the work was put on one side. When Davenport was on the point of death Heilbronn must have felt that the results should be recorded and wrote **[38]**. This impelled him to consider the problem again and the more precise results of **[39]**, although published over the joint names, were not in fact discovered until after Davenport's death. There is one intriguing result for the non-cyclic cubic fields which might possibly hint at a quite general phenomenon. Except for the finite number of primes dividing the discriminant a rational prime p can have one of three possible decomposition patterns in such a field k. It is shown that (roughly speaking) the relative densities of the fields k in which a given prime p has each of these three behaviours are the same as the densities of the set of primes p when the field k is kept fixed.

The other papers call for no special comment.

We are most grateful to Mrs. Anne Davenport for her help, and in particular for the loan of the collection of letters which Harold Davenport had received from Heilbronn, and to Miss Esther Simpson, O.B.E., secretary to the Society for the Protection of Science and Learning Limited, formerly the Academic Assistance Council, for information and for her kind permission to have access to the file of Hans Heilbronn in the Society's archives. We also wish to acknowledge the assistance of the Department of Western MSS at the Bodleian Library, where these archives are now housed. We received helpful comments and information from Mrs. D. Heilbronn, Dr. T. Callahan, Professor J. H. Chalk, Professor H. S. M. Coxeter, Dr. P. Du Val and Professor J. C. Shepherdson, and we wish to express our thanks to all of them.

The photograph was supplied by Mrs. D. Heilbronn.

References

Baker, A. 1966 " Linear forms in the logarithms of algebraic numbers ", *Mathematika* 13, 204–210.
Baker, A. 1975 *Transcendental number theory* (Cambridge University Press).
Bateman, P. T., Purdy, G. B. and Wagstaff, S. S., Jr., 1975 "Some numerical results on Fekete polynomials ", *Math. Comp.*, 29, 7–23.
Birch, B. J. 1968 Diophantine analysis and modular functions. *Proc. Bombay Colloquium on Algebraic Geometry* (Tata Institute), pp. 35–42.
Brauer, R. 1950 " On the zeta-functions of algebraic number fields II ", *Am. J. Math.*, 72, 739–746.
Callahan, T. H. 1974a " The three-class groups of non-Galois cubic fields—I ", *Mathematika* 21, 72–89.
Callahan, T. H. 1974b " The three-class groups of non-Galois cubic fields—II ", *Mathematika* 21, 168–188.
Callahan, T. H. 1975 " Dihedral extensions of order $2p$ with class number divisible by p ", *Can. J. Math.* (to appear).
Cassels, J. W. S. 1952 " The inhomogeneous minimum of binary quadratic, ternary cubic and quaternary quartic forms ", *Proc. Camb. Phil. Soc.* 48, 72–86 (Addendum: 519–520).
Cassels, J. W. S. 1961 " Footnote to a note of Davenport and Heilbronn ", *J. Lond. Math. Soc.*, 36, 177–184.
Chalk, J. H. H. 1947 " On the positive values of linear forms I, II ", *Quart. J. Math.*, 18, 215–227; 19, 67–80.
Cook, R. J. 1972 " The fractional parts of an additive form ", *Proc. Camb. Phil. Soc.* 72, 209–212.
Cook, R. J. 1975 " On the fractional parts of a set of points III ", *J. Lond. Math. Soc.* 9, 490–494.
Danicic, I. 1959 " On the fractional parts of θx^2 and ϕx^2 ", *J. Lond. Math. Soc.* 34, 353–357.
Danicic, I. 1967 " The distribution (mod 1) of pairs of quadratic forms with integer variables ", *J. Lond. Math. Soc.* 42, 618–623.
Davenport, H. 1939 " On Waring's problem for fourth powers ", *Ann. Math.* 40, 731–747.
Davenport, H. 1950a " Euclid's algorithm in cubic fields of negative discriminant ", *Acta Math., Stockh.* 84, 159–179.
Davenport, H. 1950b " Euclid's algorithm in certain quartic fields ", *Trans. Amer. Math. Soc.* 68, 508–532.
Davenport, H. 1951 " Indefinite binary quadratic forms and Euclid's algorithm in real quadratic fields ", *Proc. Lond. Math. Soc.* 53, 65–82.
Davenport, H. 1967 " On a theorem of Heilbronn ", *Quart. J. Math.* 18, 339–344.
Deuring, M. 1968 " Imaginäre quadratische Zahlkörper mit der Klassenzahl Eins ", *Invent. Math.* 5, 169–179.
Fröhlich, A. 1954a " On the absolute class group of Abelian fields ", *J. Lond. Math. Soc.* 29, 211–217.
Fröhlich, A. 1954b " The generalization of a theorem of L. Rédei's ", *Quart. J. Math.* (2), 5, 130–140.

Fröhlich, A. 1955 " On the absolute class group of Abelian fields II ", *J. Lond. Math. Soc.* 30, 72–80.
Gallagher, P. X. 1972 " Primes in progressions to prime-power modulus ", *Invent. Mat.* 16, 191–201.
Halberstam, H. and Richert, H.-E. 1974 *Sieve methods* (Academic Press).
Heegner, K. 1952 " Diophantische Analysis und Modulfunktionen ", *Math. Z.* 56, 227–253.
Huxley, M. N. 1972a " On the difference between consecutive primes ", *Invent. Mat.* 15, 164–170.
Huxley, M. N. 1972b *The distribution of prime numbers* (Oxford University Press).
Ingham, A. E. 1937 " On the difference between two consecutive primes ", *Quart. J. Math.* 8, 255–266.
Liu, M. C. 1974 " Simultaneous approximation of two additive forms ", *Proc. Camb. Phil. Soc.* 75, 77–82.
Macbeath, A. M. 1952 " A theorem on non-homogeneous lattices ", *Ann. Math.* 56, 269–293.
Manin, Ju. I. 1972 " Parabolic points and the zeta-functions of modular curves (in Russian) ", *Izv. Akad. Nauk SSSR* (*ser. mat*) 36, 19–66.
Montgomery, H. L. 1971 *Topics in multiplicative number theory.* (Springer Lecture Notes in Mathematics no. 227).
Mordell, L. J. 1934 " On the Riemann and imaginary quadratic fields with given class number ", *J. Lond. Math. Soc.* 9, 289–298.
Olsen, J. E. 1975 " Sums of sets of group elements ", *Acta Arith.* 28, 147–156.
Pitt, H. R. 1958 *Tauberian theorems* (Tata Institute and Oxford University Press). See especially §6.1.
Porter, J. W. 1975 " On a theorem of Heilbronn ", *Arithmetika* 22, 20–28.
Roth, K. F. 1949 " Proof that almost all positive integers are sums of a square, a positive cube and a fourth power ", *J. Lond. Math. Soc.* 25, 4–13.
Selberg, A. and Chowla, S. 1967 " On Epstein's zeta-function ", *J. reine angew. Math.* 227, 86–110.
Shepherdson, J. C. 1947 " On the addition of elements of a sequence", *J. Lond. Math. Soc.* 22, 85–88.
Siegel, C. L. 1935 Über die Classenzahl quadratischer Zahlkörper ", *Acta Arith.* 1, 83–86.
Siegel, C. L. 1968 " Zum Beweise des Starkschen Satzes ", *Invent. Math.* 5, 180–191.
Smith, J. R. 1969 " On Euclid's algorithm in some cyclic cubic fields ", *J. Lond. Math. Soc.* 44, 577–582.
Stark, H. M. 1967a "A complete determination of the complex quadratic fields of class number one", *Mich. Math. J.* 14, 1–27.
Stark, H. M. 1967b " On zeros of Epstein's zeta function ", *Mathematika* 14, 47–55.
Stark, H. M. 1974 " Some effective cases of the Brauer–Siegel theorem ", *Invent. Math.* 23, 135–152.
Stark, H. M. 1975a " The analytic theory of algebraic numbers ", *Bull. Am. Math. Soc.* 81, 961–972.
Stark, H. M. 1975b " Class-number of complex quadratic fields ", *Modular functions of one variable I* (Antwerp summer school), 153–175. (Springer Lecture Notes in Mathematics no. 320.)
Tatuzawa, B. 1951 " On a theorem of Siegel ", *Jap. J. Math.* 21, 163–178.
Vinogradov, I. M. 1937 " Representation of an odd integer as the sum of three primes ", *Dokl. Akad. Nauk SSR* (*Comptes Rendus*) 15, 169–172.
Vinogradov, I. M. 1954 *The method of trigonometric sums in the theory of numbers.* (Interscience).
Vinogradov, I. M. 1971 *The method of trigonometric sums in the theory of numbers* (in Russian). Moscow: Nauka.
Yao, A. C. and Knuth, D. E. 1975 " Analysis of the subtractive algorithm for greatest common divisors ", *Proc. Nat. Acad. Sci. U.S.A.* 72, 4720–4722.

LIST OF PUBLICATIONS BY H. A. HEILBRONN

1. " Über die Verteilung der Primzahlen in Polynomen ", *Math. Annal.*, 104 (1931), 794–799.
2. " Über den Primzahlsatz von Herrn Hoheisel ", *Math. Z.*, 36 (1933), 394–423.
3. (With E. Landau) " Bemerkungen zur vorstehenden Arbeit von Herrn Bochner ", *Math. Z.*, 37 (1933), 10–16.
4. (With E. Landau) " Ein Satz über Potenzreihen ", *Math. Z.*, 37 (1933), 17.
5. (With E. Landau) " Anwendungen der N. Wienerschen Methode ", *Math. Z.*, 37 (1933), 18–21.
6. " Zu dem Integralsatz von Cauchy ", *Math. Z.*, 37 (1933), 37–38.
7. " On the class-number in imaginary quadratic fields ", *Quart. J. Math.*, 5 (1934), 150–160.
8. (With E. H. Linfoot) " On the imaginary quadratic corpora of class-number one ", *Quart. J. Math.*, 5 (1934), 293–301.
9. " Über das Waringsche Problem ", *Acta Arith.*, 1 (1936), 212–221.
10. (With H. Davenport) " On the zeros of certain Dirichlet series ", *J. Lond. Math. Soc.*, 11 (1936), 181–185.
11. (With H. Davenport) " On the zeros of certain Dirichlet series II ", *J. Lond. Math. Soc.*, 11 (1936), 307–312.
12. (With H. Davenport) " On Waring's problem for fourth powers ", *Proc. Lond. Math. Soc.* 41 (1936), 143–150.
13. (With H. Davenport) " On the exponential sum ", *Proc. Lond. Math. Soc.*, 41 (1936), 449–453.
14. (With H. Davenport) " On Waring's problem: Two cubes and one square ", *Proc. Lond. Math. Soc.*, 43 (1937), 73–104.

15. (With H. Davenport) " Note on a result in the additive theory of numbers ", *Proc. Lond. Math. Soc.*, 43 (1937), 142–151.
16. " On real characters ", *Acta Arith.*, 2 (1937), 212–213.
17. " On an inequality in the elementary theory of numbers ", *Proc. Camb. Phil. Soc.*, 38 (1937), 207–209.
18. (With E. Landau and P. Scherk) " Alle grossen Zahlen lassen sich als Summe von höchstens 71 Primzahlen darstellen'", *Casopis*, 65 (1936), 117–141.
19. " On Dirichlet series which satisfy a certain functional equation ", *Quart. J. Math.*, 9 (1938), 194–195.
20. " On Euclid's algorithm in real quadratic fields ", *Proc. Camb. Phil. Soc.*, 34 (1938), 521–526.
21. (With H. Davenport) "On indefinite quadratic forms in five variables", *J. Lond. Math. Soc.*, 21 (1946), 185–193.
22. (With H. Davenport) " On the minimum of a bilinear form ", *Quart. J. Math.*, 18 (1947), 107–121.
23. (With H. Davenport) Asymmetric inequalities for non-homogeneous linear forms ", *J. Lond. Math. Soc.*, 22 (1947), 53–61.
24. " On discrete harmonic functions ", *Proc. Camb. Phil. Soc.*, 45 (1949), 194–206.
25. " On the distribution of the sequence $n^2\theta$ (mod 1) ", *Quart. J. Math.*, 19 (1948), 249–256.
26. " On Euclid's algorithm in cubic self-conjugate fields ", *Proc. Camb. Phil. Soc.*, 46 (1950), 377–382.
27. " On Euclid's algorithm in cyclic fields ", *Can. J. Math.*, 3 (1951), 275–268.
28. " On the averages of some arithmetical functions of two variables ", *Mathematika*, 5 (1958), 1–7.
29. " On the representation of homotopy classes by regular functions ", *Bull. Acad. Pol. Sci.*, 6 (1958), 181–184.
30. " Old theorems and new methods in class-field theory (Presidential Address)", *J. Lond. Math. Soc.*, 37 (1962), 6–9.
31. " On the representation of a rational number by four squares by means of regular functions ", *J. Lond. Math. Soc.*, 39 (1964), 72–76.
32. (With P. Erdős) " On the addition of residue classes mod *p* ", *Acta Arith.*, 9 (1964), 149–159.
33. *Collected papers of G. H. Hardy*, vol. 1, pt. 1, Additive number theory (edited). (Oxford University Press, 1966.)
34. *Algebraic number theory* (ed. J. W. S. Cassels and A. Fröhlich), Ch. VIII, Zeta functions and *L*-functions. (Academic Press, 1967.)
35. (With S. Bachmuth and H. Y. Mochizuki) "Burnside metabelian groups", *Proc. R. Soc. Lond.* A., 307 (1968), 235–250.
36. " On the average length of a class of finite continued fractions ", *Abhandlungen aus Zahlentheorie und Analysis zur Erinnerung an Edmund Landau*† (ed. P. Turán), pp. 87–96. (Berlin: VEB Deutscher Verlag der Wissenschaften, 1969).
37. (With P. Scherk) " Sums of complexes in torsion-free abelian groups ", *Can. Math. Bull.*, 12 (1969), 479–480.
38. (With H. Davenport) " On the density of discriminants of cubic fields ", *Bull. Lond. Math. Soc.*, 1 (1969), 345–348.
39. (With H. Davenport) " On the density of discriminants of cubic fields, II ", *Proc. R. Soc. Lond.* A, 322 (1971), 405–420.
40. " On real zeros of Dedekind ζ-functions ", *Can. J. Math.*, 25 (1973), 870–873.
41. " On the 2-class group of cubic fields ", *Studies in pure mathematics, papers presented to Richard Rado*, pp. 117–119. (Academic Press, 1971.)
42. (With G. H. Hardy) E. Landau (obituary). *J. Lond. Math. Soc.*, 13 (1938), 310–318.

† Also published under the title *Number theory and analysis* by Plenum Press, New York.