HEINZ HOPF

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Heinz Hopf, the great Swiss mathematician, died on 3 June, 1971, at the age of 76. He will be deeply mourned by his many friends, colleagues and students, and indeed by the entire mathematical community.

Heinz Hopf was born in Breslau (now Wroclaw, Poland) on 19 November, 1894. He served in the German army during the First World War and then attended the Universities of Berlin, Heidelberg, and Göttingen. In Berlin he became a student of Erhard Schmidt who provided him with his first great stimulus towards topology. Hopf studied Schmidt's own proof of the Jordan curve theorem and, under Schmidt's influence, he read and absorbed Brouwer's proof of the topological invariance of the dimension of Euclidean space, and Poincaré's fundamental work on Analysis Situs. Arising out of his work at this time, Hopf developed his own ideas on the study of mappings of closed manifolds, in particular, of mappings of manifolds onto spheres, and proved (see [3]) that the Brouwer mapping degree was the sole homotopy invariant of maps $S^n \to S^n$, thus completing the famous Brouwer–Hopf Theorem. In 1925, in Göttingen, Hopf met the Russian mathematician Paul Alexandroff. This was a crucial event in Hopf's career and indeed in his entire life, since he and Alexandroff became not only close colleagues but also firm friends. Together they were to write a book which expressed the unity of algebraical and set-theoretical topology which they achieved together in their work. Indeed, Hopf has paid a very sincere tribute to the stimulus he received from Alexandroff, who, in particular in connection with his development of the idea of the nerve of a covering of a topological space, forged an important link between set-theoretical and algebraic topology. It should be recorded that Alexandroff came to Zürich in November, 1971, to attend the memorial meeting for Heinz Hopf, held under the auspices of the Eidgenössische Technische Hochschule (ETH), Zürich, and there himself delivered a most moving tribute to Hopf the man and mathematician.

In 1927, Hopf received a Rockefeller fund fellowship to spend the year at Princeton University. There he renewed his association with Alexandroff and this period, when both came under the strong influence of the Princeton topologists, in particular of Solomon Lefschetz, may be said to mark the beginning of a great epoch of topology. In 1931, Hopf was appointed Professor at the ETH, Zürich, to succeed Hermann Weyl. This was indeed a bold appointment since Weyl was, of course, a great universalist and it was a formidable undertaking for a young mathematician to succeed a man of such eminence and distinction. One can now assert with complete confidence that the appointment was an inspired one. Hopf remained at the ETH for the rest of his professional career,† that is, until his retirement in 1965. During this period,

†He became a naturalized Swiss citizen—one of the very few to be permitted citizenship without a mastery of Schweizer Deutsch!

[BULL. LONDON MATH. SOC., 4 (1972), 202–217]

Zürich flourished as one of the leading mathematical centres of the world. Hopf exercised an enormous influence on mathematics and the teaching of mathematics at all levels. He attracted very many students who have themselves now established firm reputations in the mathematical community. Visitors came to Zürich to work with him and he inspired many others through the vividness and lucidity of his writings. His home in Zollikon was a place where all were assured of a warm welcome, where, together with his wife Anja, he created an atmosphere of such gentle, all-pervasive friendliness and informality that it was difficult to realize one was in the presence of one of the really great mathematicians of all time. The death of his dearly beloved wife in 1967 was a blow from which Heinz Hopf never recovered.

Many topologists who were young men at the end of the Second World War will have reason to be extremely grateful to Heinz Hopf for his advice and encouragement. Particularly memorable for us was his attendance at the Oxford conference of young topologists in 1953, organized by Henry Whitehead. There he was held in the respect due to a great elder statesman, but his participation was boyish, enthusiastic and wholehearted. He was, characteristically, always asking questions and suggesting new problems that we might be working on. This was typical of his modesty and generosity of spirit, that he would always share his ideas with others and delight if they adopted them themselves. If they were sensible people, they would certainly do so! The present writer has particular reason to be grateful for the encouragement Hopf gave him in connection with a tract he wrote on homotopy theory. In his letter acknowledging receipt of a complimentary copy, Hopf supported the idea of calling on the reader to consult original texts. This, said Hopf, was an excellent prescription, for there is much to be gained from the original texts " wenn sie überhaupt lesbar sind " (provided that they are at all readable). Pre-eminent among the readable texts of the time were those of Heinz Hopf himself.

A particularly significant friendship at the time of which we are now speaking was that between Hopf and Henry Whitehead. These two men were giants of topology in Europe in this period, and their close and intimate friendship must have been of enormous help to the entire development of algebraic topology during the period from 1946 to the untimely death of Henry Whitehead in 1960. Indeed, it was a particularly sad occasion in 1960, when Henry Whitehead died suddenly in Princeton, a few weeks before he was due to give one of the main addresses at a conference on algebraic topology and differential geometry organized by Hopf in Zürich.

Many honours were showered on Heinz Hopf. The first of these, in 1947, was an honorary doctorate from Princeton University bestowed on Hopf on the occasion of the 200th anniversary celebration of the University. This was followed by many other honours. Hopf became a foreign member of the National Academy of Sciences and of the Accademia dei Lincei; he received an honorary doctorate from the Sorbonne; and his last honour was the award of the Lobatchevski prize of the University of Moscow. He was, from 1955 to 1958, President of the International

Mathematical Union. The London Mathematical Society honoured him with honorary membership† in 1956.

Hopf was also honoured by his own institution in 1964 by the publication of a Festschrift, *Selecta Heinz Hopf*, edited by his student and close friend Beno Eckmann. This Festschrift contained reproductions, sometimes in slightly abbreviated form, of many of Hopf's greatest papers, together with a comprehensive bibliography. We will refer to the Selecta again later.

Hopf's mathematical research was marked by astonishing insight and informed, intensive, wide-ranging mathematical curiosity. His papers are at the same time consummations of research achieved and fruitful sources of future inquiry. Hopf did not publish fragments. He waited until the problem he was studying had achieved substantial maturity in his thoughts and he then wrote down his results in such a way as to reveal, often explicitly, the route by which he had arrived at the formulations adopted and the reasons for the choices made. Hopf was not a perfectionist in the sense of one who refuses to publish until he is convinced that he has the best possible formulation. He was a far too modest man to think in such terms. On the other hand, his publications, like his lecturing and teaching style, are redolent of the attitude and personality of a born teacher. He regarded teaching and the spreading of mathematical ideas as an integral part of the mathematician's trade.

Also characteristic of Hopf's approach to mathematical research was his interest in explicit problems. There was always in his work a very concrete foundation—a question to which Hopf wanted to find the answer. And yet, such was his genius that his methods of solution, the very context in which he set the problems he studied, became established as primary areas of mathematical research. Thus, the generality of his ideas sprang not from a deliberate intention to achieve the abstract formulation, but from the inherent importance of the problems he considered and the mathematical significance of the structures and concepts which he developed in order to solve those problems.

A complete and comprehensive account of Hopf's contributions to topology, algebra and geometry would fill many issues of the *Bulletin*. Fortunately we have, in the *Selecta Heinz Hopf*, Springer-Verlag (1964), dedicated by the ETH, Zürich, to Hopf on his 70th birthday, a collection of some of his most important papers, together with a complete bibliography. Thus the student may conveniently turn to Hopf's original papers to read his work—and he is sure to find it a most rewarding experience. As we have said, Hopf wrote with limpid clarity and elegant simplicity, always explaining to the reader the underlying motive for the direction his argument was taking. Moreover, it is a revelation, on reading Hopf's papers, to discover how many basic ideas of algebraic topology and homological algebra stem from Hopf's

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†Several of the biographical details in this notice have either been obtained by consultation with Beno Eckmann or gleaned from Eckmann's moving appreciation of Heinz Hopf published in the Neue Zürcher Zeitung of June 18, 1971. The author would like to express his gratitude to Professor Eckmann.

genius. Here we will be content to refer to some of his most outstanding contributions; the order in which we present these contributions will be chronological.

In 1925, Hopf came under the influence of Emmy Noether in Göttingen, and was perhaps the first topologist to appreciate the significance of her point of view that the proper objects to describe homology relations in a simplicial complex were not numbers (Betti numbers, torsion coefficients) but algebraic structures, and, in particular, abelian groups. This appreciation may, without being fanciful, be regarded as the precursor of the functorial approach which now imbues the whole of algebraic topology, and which is so obvious to the present generation of topologists as to need no explicit justification. In his paper **[12]**, Hopf used for the first time Emmy Noether's conceptual framework for homology theory to prove a generalization of the Euler–Poincaré formula; Lefschetz had already obtained this generalization in a special case.† We consider a linear chain-map $\phi : C \to C$, where C is a chain-complex over $\mathbb{Q}$ such that $C_r = 0$, $r < 0$, $r > n$. Then ϕ induces a homology homomorphism $\phi_* : H(C) \to H(C)$, and the Hopf Trace Formula asserts that

$$\sum_{r=0}^{n} (-1)^r \operatorname{tr} \phi_r = \sum_{r=0}^{n} (-1)^r \operatorname{tr} \phi_{*r},$$

where $\operatorname{tr} \psi$ is the trace of the linear map ψ. The Euler–Poincaré formula is obtained by taking $\phi = 1$. Moreover, the Lefschetz fixpoint theorem readily follows, given the apparatus of barycentric subdivision and simplicial approximation: if X is a compact polyhedron and $f: X \to X$ is a continuous function without fixpoint, then

$$\sum_{r=0}^{n} (-1)^r \operatorname{tr} f_{*r} = 0.$$

Here f_{*r} is the homology homomorphism induced by f and $\operatorname{tr} f_{*r}$ is, of course, a rational integer, so that $\sum_{r=0}^{n} (-1)^r \operatorname{tr} f_{*r}$ is an integer called the *Lefschetz number* of f. Since, for X connected, $\operatorname{tr} f_{*0} = 1$, we get as an immediate consequence the fixpoint property for maps of contractible compact polyhedra. It should be observed how the concept of homology groups (rather than Betti numbers which are invariants of those groups) leads to an easy and natural description of the Lefschetz number.

In 1925, also, as previously mentioned, Hopf met Alexandroff in Göttingen and they were to become life-long friends. In 1927/28, they spent the year together in Princeton, and enjoyed stimulating contacts with Veblen, Lefschetz and Alexander. Hopf identified Lefschetz as the single most important influence on his mathematical development at that time. Lefschetz had introduced his "product method" for studying maps $f: X \to Y$ of n-dimensional manifolds. Hopf discovered the "Umkehrungshomomorphismus" $R(f)$ which maps the intersection ring $R(Y)$ of Y into that of X. As we now understand, Hopf was really doing cohomology, so that

†Hadamard had, much earlier, observed the theorem on the existence of vector fields on a manifold without making explicit mention of the Euler characteristic.

$R(f)$ was effectively, the induced map of the cohomology ring of Y into the cohomology ring of X. However, this point of view was not to surface until 1935; then it was a revelation that, while the intersection ring was isomorphic to the cohomology ring, the intersection was defined only for manifolds while the cohomology ring could be defined for arbitrary polyhedra. Hopf used the Umkehrungshomomorphismus to obtain an important result relating to the mapping degree of the map $f: X \to Y$ between two n-dimensional manifolds, and proved thereby **[16]** that, if the degree of f is non-zero, then f maps the intersection ring $R(X)$, with rational coefficients, onto $R(Y)$. He subsequently applied the same idea to other problems relating to maps between manifolds of different dimensions. The best-known and most exciting application, to generalized group-manifolds, will be discussed below.

In 1931, Hopf again used the intersection ring to establish one of the most significant results in the history of homotopy theory. He demonstrated **[18]** that there exist essential maps from S^3 to S^2, indeed, that there are infinitely many distinct homotopy classes of such maps. The language of homotopy groups was not yet available; it was in 1932 that Cech gave a definition of such groups at the Zürich International Congress, but it was not, in fact, until 1935 that they became an established tool when Hurewicz was the first to describe applications of the homotopy groups to classification problems of algebraic topology, thus effectively to what we now know as obstruction theory. In the language of modern homotopy theory, Hopf defined a numerical invariant γ attached to elements of the homotopy group $\pi_3(S^2)$ and showed that γ can take any integer value. We now know—it is an easy deduction from Hopf's methods—that γ is an isomorphism of $\pi_3(S^2)$ onto $\mathbb{Z}$. It is particularly significant to note that Hopf gave, in this work, the first demonstration of the existence of maps which are essential but *homologically* trivial, that is, their essentiality could not be detected by means of the induced homology homomorphism. That this is today a commonplace is largely due to the inspiration of Hopf's pioneer work.

Hopf's definition of the invariant γ proceeded essentially as follows. We triangulate and orient S^3 and S^2 and assume f to be a simplicial map from S^3 to S^2. Let p, q be interior points of distinct 2-simplexes of S^2. Then $f^{-1}(p), f^{-1}(q)$ are 1-cycles on a suitable sub-division of S^3 and so have a linking number in S^3, which we call $\gamma(f)$. Hopf modified Brouwer's famous proof of the topological invariance of the mapping degree to show that $\gamma(f)$ was independent of the choice of triangulation of S^3 and S^2, and was indeed a homotopy invariant of f. He further showed that if $u: S^3 \to S^3$ is a map of degree d, then

$$\gamma(fu) = d\gamma(f); \tag{1}$$

and, if $v: S^2 \to S^2$ is a map of degree d, then

$$\gamma(vf) = d^2\gamma(f). \tag{2}$$

Hopf then produced a map $f_o: S^3 \to S^2$ with $\gamma(f_o) = 1$; from this and (1) above, the existence of maps of arbitrary *Hopf invariant* follows immediately. The map f_o is the celebrated *Hopf fibration* or principal *Hopf bundle*, which we may express as follows. We regard S^3 as the space of two complex variables (z_1, z_2) such that $|z_1|^2 + |z_2|^2 = 1$, and we regard S^2 as the complex projective line, with coordinates the ratios $[z_1, z_2]$ where, of course, not both of z_1, z_2 are zero. Then we define $f_o\colon S^3 \to S^2$ by

$$f_o(z_1, z_2) = [z_1, z_2]. \tag{3}$$

It is plain that $f_o^{-1}[z_1, z_2]$ is the great circle of points $(z_1 \mathrm{e}^{i\alpha}, z_2 \mathrm{e}^{i\alpha})$ and Hopf showed that any two such great circles link with linking number (for suitable orientations of S^3, S^2) equal to $+1$. We now recognize f_o as the bundle projection of a principal S^1-bundle.

In [**26**] Hopf generalized his argument in order to be able to consider maps f from S^{2n-1} to S^n. Now the points p, q are interior points of principal n-simplexes of S^n and their counter-images $f^{-1}p$, $f^{-1}q$ are $(n-1)$-dimensional cycles on a suitable subdivision of S^{2n-1}. Thus they have a linking number ω and again this linking number ω depends only on the homotopy class of f so that we may set

$$\gamma(f) = \omega(f^{-1}p, f^{-1}q). \tag{4}$$

However, it is immediately clear that ω is zero if n is odd, so that we should immediately restrict ourselves to the case of n even. The analogues of (1) and (2) above again hold; moreover, Hopf also pointed out that the definition of f_o might be adapted to lead to maps

$$f_o : S^7 \to S^4, \qquad f_o : S^{15} \to S^8, \tag{5}$$

such that $\gamma(f_o) = 1$. These (together with (3)) constitute the famous *Hopf fibrations* and are obtained by replacing the complex numbers, in (3), by the quaternions and octonions (Cayley numbers) respectively. For the general case of n even, Hopf did not obtain maps of invariant 1; however, he did obtain maps f with $\gamma(f) = 2$; this, of course, is sufficient, with (1), to establish that γ can take any even value. Today it is easy to see that the *Whitehead product* map $w : S^{2n-1} \to S^n$, n even, yields $\gamma(w) = \pm 2$. However, Hopf's method of obtaining maps f with $\gamma(f) = 2$ was based on a very beautiful geometrical idea and has led to substantial generalization and exploitation. Hopf described a map $g_o : S^{n-1} \times S^{n-1} \to S^{n-1}$, n even, of type $(1, 2)$. Here we say that the type of $g : S^{n-1} \times S^{n-1} \to S^{n-1}$ is (k, l) if g, restricted to $pt \times S^{n-1}$, has degree k, and, restricted to $S^{n-1} \times pt$, has degree l. Now we may express S^{2n-1} in a natural way as the union of $V^n \times S^{n-1}$ and $S^{n-1} \times V^n$, where V^n is an n-ball bounded by S^{n-1}; moreover the intersection of $V^n \times S^{n-1}$ and $S^{n-1} \times V^n$ is $S^{n-1} \times S^{n-1}$. Now the sphere S^n may be regarded as the union of two hemispheres B_+^n, B_-^n intersecting in the equator S^{n-1}. The map $g : S^{n-1} \times S^{n-1}$ extends to

$$g_+ : V^n \times S^{n-1} \to B_+^n, g_- : S^{n-1} \times V^n \to B_-^n,$$

and hence, putting these extensions together, the map $g : S^{n-1} \times S^{n-1} \to S^{n-1}$ extends to a map $H(g) : S^{2n-1} \to S^n$. Hopf showed that if g is of type (k, l), then

$$\gamma H(g) = kl; \tag{6}$$

in particular,

$$\gamma H(g_o) = 2. \tag{7}$$

The construction H may be immediately generalized† to associate a map

$$H(g) : S^{p+q-1} \to S^r \qquad \text{with a map} \qquad g : S^{p-1} \times S^{q-1} \to S^{r-1}$$

and is known as the *Hopf construction*; it plays a crucial role in homotopy theory, for example, in the definition of the celebrated J-homomorphism of G. W. Whitehead. Here we may specialize to $J : \pi_n(SO(q)) \to \pi_{n+q}(S^q)$, where $SO(q)$ is the special orthogonal group operating on Euclidean space $\mathbb{R}^q$. Thus a map $S^n \to SO(q)$ may be regarded as a map $g : S^n \times S^{q-1} \to S^{q-1}$ and the Hopf construction produces a map $H(g) : S^{n+q} \to S^q$.

There is a wealth of connection between this work of Hopf and subsequent work in homotopy theory. We mention just a few salient features. Steenrod was the first to give an invariant definition of the Hopf invariant and generalized it in the direction of the *functional cup-product* and, more generally, functional cohomology operations. Steenrod's interpretation may be carried a stage further, using J. H. C. Whitehead's idea of attaching a cell to a space. Since it is in this latter form that much of the work on the Hopf invariant has been done, we give this formulation. A map $f : S^{2n-1} \to S^n$ may be used to attach a $2n$-cell to S^n to produce the space C_f. Then the additive cohomology of C_f has $\mathbb{Z}$ in dimensions n and $2n$, with generators α and β. The Hopf invariant is given by $\alpha^2 = \gamma(f)\,\beta$. In the case of the Hopf maps

$$f_o : S^{2n-1} \to S^n, n = 2, 4, 8,$$

the space C_{f_o} is the complex (resp. quaternionic, Cayley) projective plane, and its cohomology ring structure follows from Poincaré duality. The question of the existence of a map f of Hopf invariant 1 is thus the question of the existence of a space C whose cohomology ring is a polynomial ring in a variable x, of dimension n, truncated at x^3. Many partial results were obtained, by G. W. Whitehead, J. Adem, H. Toda and others, before J. F. Adams definitively answered the question in 1960 by proving that the known values of n were the *only* values of n for which maps f with $\gamma(f) = 1$ existed; subsequently, Adams and Atiyah produced in 1966 a very simple and elegant proof of this result using complex K-theory—by this method the problem was reduced to the fact of elementary number theory that if $2^m | 3^m - 1$ then $m = 1, 2$ or 4.

The relation of the Hopf invariant to the Freudenthal suspension, which is a homomorphism $\pi_r(S^n) \to \pi_{r+1}(S^{n+1})$ was first noticed by Freudenthal himself. This relation has led to generalizations of the Hopf invariant and to the suspension, and to the embedding of these two homomorphisms in a basic exact sequence also involving the Whitehead product. Moreover, the generalized suspension leads to the fundamental notion of the stable homotopy category and stable cohomology operations. The

†It may be, and has been, generalized much further than this!

Hopf invariant problem may be formulated stably, when it appears as a mod 2 problem. There is also a mod p version for any odd prime p.

In 1935, a topology conference was held in Zürich†, presided over by Elie Cartan. In his presidential address, Cartan drew attention to the fact that, if one took any Lie group G in one of the four main series of simple Lie groups, then the intersection ring of G over the rationals was that of the product of odd-dimensional spheres; thus we may write

$$G \overset{\mathbb{Q}}{\simeq} S^{r_1} \times S^{r_2} \times \ldots \times S^{r_k}, \qquad r_i \text{ odd.} \tag{8}$$

For example,

$$SO(n) \overset{\mathbb{Q}}{\simeq} S^3 \times S^7 \times \ldots \times S^{4m-1}, \qquad n = 2m+1,$$

$$SO(n) \overset{\mathbb{Q}}{\simeq} S^3 \times S^7 \times \ldots \times S^{4m-1} \times S^{n-1}, \qquad n = 2m+2.$$

This result had been obtained independently by Pontryagin and R. Brauer, and another proof was given later by Ehresmann. Cartan pointed out that it only remained to study the exceptional Lie groups G_2, F_4, E_6, E_7, E_8; but he added that, even if a similar result were found to hold for these five exceptional cases, one would still not really know *why* the result was true—this is always a defect of proof by enumeration of cases. Hopf responded to the challenge and solved the problem completely in 1939, his paper in fact appearing [**40**] in 1941‡. The title of this paper, "Über die Topologie der Gruppen-Mannigfaltigkeiten und ihrer Verallgemeinerungen", is highly significant. For Hopf showed that the phenomenon described in (8) above springs from the mere fact that G is a manifold admitting a continuous multiplication $\mu : G \times G \to G$ with two-sided unity element. In fact, it is not necessary in Hopf's argument that G be a manifold; however, it now appears likely that, if X is a polyhedron admitting such a multiplication μ, then it will have the homotopy type of a manifold. Hopf considered manifolds because he was familiar with the intersection ring of a manifold and preferred to use this rather than the equivalent cohomology ring. Hopf also referred to such a manifold as a generalization of a *group manifold*; of course, we now know, thanks to the work of Gleason and Montgomery-Zippin in the 1950's, that a group manifold does admit the structure of a Lie group.

Hopf's procedure in attacking the problem was to bring to bear the two algebraic structures on the homology, with rational coefficients, $H_*(X; \mathbb{Q})$, of a manifold X admitting a multiplication μ. On the one hand one has the intersection ring. On the other hand, the multiplication μ induces a multiplication on the graded abelian group $H_*(X; \mathbb{Q})$; this multiplication is due essentially to Pontryagin. In modern

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†This was also the year of the Moscow conference, which saw the birth of cohomology theory and the description of the Stiefel–Whitney classes in connection with the vector field problem on manifolds. It was also the year that Hurewicz wrote his key papers on applications of the homotopy groups.

‡There was a brief announcement [**36**] in 1939. The paper [**40**] was originally accepted in 1939 by *Compositio Mathematica*, but was transferred to the *Annals* when *Compositio* was obliged to cease publication.

terminology we have the cohomology ring $H^*(X; \mathbb{Q})$, which is a graded commutative $\mathbb{Q}$-algebra. Then $\mu : X \times X \to X$ induces a *diagonal map*

$$\mu^* : H^*(X; \mathbb{Q}) \to H^*(X; \mathbb{Q}) \otimes_{\mathbb{Q}} H^*(X; \mathbb{Q}),$$

which is a homomorphism of graded rings. If μ is homotopy-associative (homotopy-commutative), then μ^* is associative (commutative) in an abvious sense. The structure we are here attributing to $H^*(X; \mathbb{Q})$ is now called a *graded commutative Hopf algebra* over $\mathbb{Q}$; if A is such an algebra, so that $A = \bigoplus_{n \geq 0} A_n$, then A is said to be *connected* if $A_0 = \mathbb{Q}$ (certainly the case if X is connected and $A = H^*(X; \mathbb{Q})$), *coassociative* if the diagonal map $A \to A \otimes_{\mathbb{Q}} A$ is associative, *cocommutative* if the diagonal map is commutative. The fact that $\mu : X \times X \to X$ has the base point as two-sided unity is reflected in the axiom on the Hopf algebra A that there is a *counit* $\eta : A \to \mathbb{Q}$ such that $A \xrightarrow{\Delta} A \otimes_{\mathbb{Q}} A \xrightarrow{1 \otimes \eta} A \otimes_{\mathbb{Q}} \mathbb{Q}$ and $A \xrightarrow{\Delta} A \otimes_{\mathbb{Q}} A \xrightarrow{\eta \otimes 1} \mathbb{Q} \otimes_{\mathbb{Q}} A$ are the canonical isomorphisms, where Δ is the diagonal map. (Of course, the structure of A as a $\mathbb{Q}$-algebra involves the *unit* $\varepsilon : \mathbb{Q} \to A$.) Hopf showed that, as a $\mathbb{Q}$-algebra, A must be isomorphic to an exterior algebra on odd-dimensional generators. This explains the phenomenon cited by Cartan. Moreover, if A is coassociative then the generators of the exterior algebra will be *primitive* elements of the Hopf algebra, that is, elements x such that $\Delta x = x \otimes 1 + 1 \otimes x$.

Today, the theory of Hopf algebras is an important part of modern algebra. It has also fed back extensively into topology. For example, Browder has used the Hopf algebra structure in the cohomology to elucidate further the homology relations between fibre F, total space E, and base space B, where all spaces are Hopf spaces (i.e., spaces with continuous multiplication) and the maps $F \to E \to B$ are H-maps, that is, homorphisms up to homotopy. At first, Hopf's generalization to spaces X admitting $\mu : X \times X \to X$ with two-sided unity seemed only to bring in the two examples S^7, $\mathbb{R}P^7$ of such manifolds which are not topological groups.† However, very recently, powerful new techniques, involving in particular the method of *localization*, have revealed a vast store of *Hopf spaces*, none of which is a Lie group. Some of these have the homotopy type of a topological group, showing that the solution of the homotopy version of Hilbert's Fifth Problem is negative. We would now interpret (8) as saying that G and $S^{r_1} \times S^{r_2} \times \ldots \times S^{r_k}$ have the same *rationalizations*; we know further that they have the same localizations at almost all primes. This observation is relevant to much of the current work on Hopf spaces. It is a pleasure to be able to record that Hopf was able to participate in a conference organized in his honour at Neuchâtel in August, 1970, when some 27 invited papers were given all devoted to recent work in Hopf spaces and Hopf algebras. In the published proceedings of that conference (Springer Lecture Notes No. 196) James listed 347 papers relevant to the study of Hopf spaces!

†Plainly, the existence of continuous multiplications on S^1, S^3, S^7 explains, as described earlier, the existence of maps $S^3 \to S^2$, $S^7 \to S^4$, $S^{15} \to S^8$ with Hopf invariant 1. One applies the Hopf construction to the multiplications.

In 1941, and indeed much earlier, Hopf had been studying the influence of the curvature of a Riemannian manifold on its topology. His student Preissmann had shown that, if the curvature of a closed Riemannian manifold was nowhere positive then one could associate with two geodesic loops u and v, which represented commuting elements of the fundamental group, a surface of the type of a torus spanned by u and v. Hopf noted that what lay behind this " product " was a purely algebraico-topological phenomenon, the influence of the fundamental group on the second homology group [**44**]. Of course, the fundamental group entirely determines the first homology group, which is just the fundamental group abelianized. On the other hand, the fundamental group obviously does not determine the second homology group; however, Hopf showed that it does determine the second homology group modulo the subgroup of spherical cycles. Let K be a connected polyhedron and let

$$\theta_n : \pi_n K \to H_n K$$

be the Hurewicz homomorphism. Then the image of θ_n is called the subgroup of spherical n-cycles and Hopf showed that $\pi_1 K$ entirely determines the quotient of $H_2 K$ by the image of θ_2, that is, the cokernel of θ_2. In fact, Hopf gave an explicit formula, now known as the *Hopf formula*, for the cokernel of θ_2 in terms of a free presentation of $\pi = \pi_1 K$. Thus let π be represented as F/R, where F is a free group and R a normal subgroup of F; such a representation of a group π may well arise from a presentation of π by means of generators and relators. The Hopf formula then reads

$$\text{coker} \quad \theta_2 = H_2 K/\theta_2 \pi_2 K \cong (R \cap [F, F])/[F, R]. \tag{9}$$

Here $[F, F]$ is the commutator subgroup of F and $[F, R]$ is the subgroup of F generated by commutators $f^{-1} r^{-1} fr$, $f \in F$, $r \in R$. In particular, we see that if $\pi_2 K = 0$, then $H_2 K \cong (R \cap [F, F])/[F, R]$.

Formula (9) breaks new ground in expressing a function of a group π by means of a free presentation of π; for it is by no means clear *ab initio* that the group

$$(R \cap [F, F])/[F, R]$$

depends only on π and is independent of the choice of free presentation of π. Of course, (9) contains much more information than just that, since it attaches a deep topological significance to the group $(R \cap [F, F])/[F, R]$; however, in this particular aspect which we have emphasized, it may be said to be the precursor of the basic technique of modern homological algebra.

It was indeed in many senses the precursor of homological algebra. Hurewicz had pointed out, in his seminal papers in 1936, that if X is a connected polyhedron which is aspherical in all dimensions $\geqslant 2$ (that is, $\pi_n X = 0, n \geqslant 2$), then the homology groups of X are entirely determined by the fundamental group $\pi = \pi_1 X$. Indeed, the homotopy type of X is determined by π. Hopf realized, with Hurewicz, that this theorem generalized to the statement that if X is a connected polyhedron which is aspherical in dimensions $2 \leqslant n \leqslant k-1$, then the cokernel of the Hurewicz homomorphism $\theta_k : \pi_k X \to H_k X$ is entirely determined by π. He was thus led to seek a generalization of formula (9) above, and the very beautiful result [49] appeared in 1944. Owing to wartime conditions, there was virtually no communication at that time between European and American mathematicians, so that further developments of Hopf's theory, by Eckmann and Freudenthal in Europe, and by Eilenberg and MacLane in the U.S.A., proceeded quite independently.

Hopf, as so often, based himself on a geometrical idea (by contrast, the work of Eilenberg and MacLane was much more purely algebraically inspired). Let $C(\tilde{K})$ be the chain group of the universal cover $\tilde{K}$ of the aspherical complex K, with integer coefficients. Then the fundamental group $\pi = \pi_1 K$ operates freely on the simplexes of $\tilde{K}$ without fixpoints and hence $C(\tilde{K})$ is a free π-module. Moreover, $\tilde{K}$ is contractible, so that $C(\tilde{K})$ is acyclic, and $C(\tilde{K})_\pi = C(K)$. Now if J is a coefficient group, then we may, of course, compute the homology of K with coefficients in J from the chain group

$$C(K) \otimes J = C(\tilde{K})_\pi \otimes J = C(\tilde{K}) \otimes_\pi J \qquad (10)$$

(of course, Hopf did not explicitly invoke the tensor product). Hopf observed that any chain group which was acyclic and a free π-module could be substituted for $C(\tilde{K})$ in (10) to yield *isomorphic homology groups*; and he proceeded to describe one such chain group.

Thus let $C_n(\pi)$ be the free abelian group on $(n+1)$-tuples $(x_0, x_1, \ldots, x_n)$ of elements of π. We turn $C_n(\pi)$ into a π-module by defining

$$(x_0, x_1, \ldots, x_n)\, x = (x_0\, x, x_1\, x, \ldots, x_n\, x)$$

and $C_n(\pi)$ is then plainly a free π-module. We define a differential $\partial : C_n(\pi) \to C_{n-1}(\pi)$ by the usual simplicial boundary formula

$$\partial(x_0, x_1, \ldots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, x_1, \ldots, \hat{x}_i, \ldots, x_n), \qquad (11)$$

where $\hat{x}_i$ indicates that x_i is to be omitted. Obviously ∂ is a module map and $\partial\partial = 0$. We define an augmentation $\varepsilon : C_0(\pi) \to \mathbb{Z}$ by $\varepsilon(x) = 1$ and thus have a chain complex of π-modules

$$\ldots \to C_n(\pi) \xrightarrow{\partial} C_{n-1}(\pi) \to \ldots \xrightarrow{\partial} C_0(\pi) \xrightarrow{\varepsilon} \mathbb{Z}. \qquad (12)$$

We show that (12) is acyclic by defining a contracting homotopy

$$s\colon C_{n-1}(\pi) \to C_n(\pi),$$

which is a homomorphism of abelian groups, by the rule

$$s(x_0, x_1, \ldots, x_{n-1}) = (e, x_0, x_1, , \ldots x_{n-1}). \qquad (13)$$

Thus $C(\pi)$ may be used to replace $C(\tilde{K})$ in (10) and we have that, if K is an aspherical complex with $\pi_1 K = \pi$, and if J is a coefficient group, then

$$H_*(K; J) \cong H_*(C(\pi) \otimes_\pi J) = H_*(C(\pi)_\pi \otimes J). \qquad (14)$$

It is natural to write $H_*(\pi; J)$ for the middle group in (14), so that we may say that we have defined homology groups of π (with coefficients in J) and these groups are isomorphic to the homology groups of K. Of course we may allow J to be a (left)

π-module. Then $H_*(K;\ J)$ is to be understood as the homology of K with local coefficients J and we still have

$$H_*(K;\ J) \cong H_*(\pi;\ J), \tag{15}$$

where $H_*(\pi;\ J)$ is to be understood as the homology of $C(\pi) \otimes_\pi J$. Similar results are available in cohomology and were enunciated by Eckmann, Freudenthal and Eilenberg–MacLane; here we may speak of a *ring* isomorphism $H^*(K;\ J) \cong H^*(\pi;\ J)$ if J is a ring of coefficients.

Hopf's procedure is, as we have suggested, the forerunner of homological algebra. We now interpret (12) as saying that $C(\pi)$ provides a *free resolution of* $\mathbb{Z}$ over π, so that, if J is a (left) π-module, $H_n(\pi;\ J)$ is precisely the value of the nth left derived functor of the functor $\mathbb{Z} \otimes_\pi -$, evaluated at J, in other words,

$$H_n(\pi;\ J) = \mathrm{Tor}_n^\pi(\mathbb{Z}, J). \tag{16}$$

Today homological algebra is far more than the homology and cohomology theory of groups. Starting with the general notion of derived functors of a given additive functor between abelian categories, one develops the cohomology theory of groups, Lie algebras, augmented associative algebras, and other algebraic systems as special cases. Commutative ring theory, Hopf algebra theory and various other " concrete " algebraic theories are now largely incorporated into homological algebra; on the other hand, there are also more abstract directions which homological algebra has taken, for example, the theory of derived categories and the homology theory of small categories. However, nobody would dispute that the origin of homological algebra is to be found in Hopf's study of the homology groups of aspherical complexes, based on Hurewicz' observations.

The Hopf formula (9) for $H_2\,\pi$, namely

$$H_2\,\pi = R \cap [F, F]/[F, R], \tag{17}$$

where F is a free group and $F/R \cong \pi$, may be recovered from the general procedure just described by setting $J = \mathbb{Z}$ and taking for F the free group on the elements of π. It is an interesting historical fact that the group $H_2\,\pi$, as a function of the *finite* group π, had already been considered much earlier by Schur in his study of (complex) projective representations. Essentially, Schur showed in 1904 that the obstruction to lifting a projective representation of π to an ordinary (complex) representation lay in a certain group which he called the *multiplicator* of π; and the multiplicator of π is, in the event, just $H_2\,\pi$. Though Hopf may well have been aware of Schur's work, there is no reason to suppose that the connection of the homology theory of groups which he invented with this early work became apparent until some years later.

The work of Hopf and his students during the war ranged over many parts of topology and differential geometry; a report **[42]** on this work was given by Hopf in 1946. This report was originally to have been published in 1941 in honour of Brouwer's 60th birthday, but the war prevented the appearance of this Festschrift. In the report as it did appear, Hopf included results obtained subsequent to 1941. Hopf continued to make active contributions to topology (fibre-bundle theory, relations between homotopy and homology, fields of surface elements on 4-manifolds) and to differential geometry right up to his retirement in 1965 and beyond. One should also add that he exhibited again his unique capacity for discerning the deep connections between topology and algebra in his application [47] of the theory of ends of groups to the study of non-compact spaces; this theory was utilized most effectively by Stallings in his proof that a finitely-generated group of cohomological dimension 1 must be free.

One is struck, on reading the works of Shakespeare, with the fact that they are full of quotations. So it is with the works of Heinz Hopf; his ideas are as fresh and as vigorous today as when he first expounded them. To those fortunate enough to have known him, his personality and way of life will always remain an inspiration. But even those who only know him through his published work will find therein a model of outstanding mathematical creativity reinforced by the art of lucid and elegant exposition.

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