# ALBERT EDWARD INGHAM

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Albert Edward Ingham was born at Northampton on April 3rd, 1900, and educated at Northampton and King Edward VI's Grammar School, Stafford. His father, also Albert Edward, was a boot-machine operator and designed the " veldtschoen ", for which his firm awarded him a very modest honorarium. (The son wore boots until he was sixty.) There was an elder brother, Christopher, in the family and three younger sisters.

It is recorded that, aged three, the boy showed his aptitude for numbers and angles by learning to tell the time. He had an ear for music but had no training. Christopher had piano lessons, but when his mother, hearing the piano played in the next room, assumed that he was doing his musical homework, she found that it was A.E. who was teaching himself. He won every prize that a brilliant schoolboy could win and an open scholarship at Trinity College, Cambridge, in December 1917, going into residence in January 1919 after a few months in the Army.

As an undergraduate he was handsome, with black hair and deep blue eyes, slow-moving, seldom speaking unless spoken to, friendly if sparing of smiles. He gave an impression of rock-like strength and integrity. He gained the highest honours in the Mathematical Tripos, a Smith's Prize and an 1851 Senior Exhibition. In 1922, at his first attempt, he was elected to a Prize Fellowship at Trinity for a dissertation in which, according to him, he proved two lemmas. A friend who brought him news of his election recalls that Ingham said " Oh ", and went on working, perhaps at a third lemma.

In the early 1920s young men who had escaped or survived the war found a market, favourable to both buyers and sellers, in appointments at Cambridge or Oxford and other Universities. G. H. Hardy had a large say in the placing of pure mathematicians. Many years later, a letter of his written about 1923 was shown to me. I cannot remember just what he wrote, but his assessment of the field was emphatic: that though Ingham might be less adroit than some of his contemporaries in putting Vice-Chancellors at their ease in interviews, the depth and maturity of his Trinity dissertation marked him out as a leader of his generation in power and promise. In the event, Ingham enjoyed four years (1922–6) of research without any commitments to teach. He spent some months in Göttingen. In 1926 he was appointed Reader in the University of Leeds. In 1930, on the sudden and untimely death of F. P. Ramsey, he returned to Cambridge as fellow and director of studies at King's College, with a University lectureship. He was elected a fellow of the Royal Society in 1945 and was appointed University Reader in Mathematical Analysis in 1953.

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† This notice is also to appear in Biographical Memoirs of Fellows of the Royal Society.

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He was a member of the London Mathematical Society from 1922 and served on the Council for the ten years 1927–32 and 1940–45.

In 1957 he retired from regular College teaching after twenty-seven years of devoted labour. To a classical colleague his sufferings with a weak pupil seemed like those of the Spartan boy with the fox gnawing at his vitals. Pupils, strong and less strong alike, have testified to their gratitude for his gentle patience and the standards of perfection towards which, to their lasting gain, he encouraged them.

Ingham's lectures won higher praise from undergraduates and young graduates than those of any other teacher in the faculty. They were superbly organised, no detail was slurred over and yet the over-riding effect was one of simplicity with the main ideas and theorems in a high light. More than one lecturer dates his resolve to become a professional mathematician from hearing as a freshman Ingham's unfolding of analysis. Boys who had shown at school a preference for applied mathematics told their masters after a term or two that their eyes had been opened to analysis, which had become their favourite subject.

Not only in research and teaching but in all he did, Ingham was the embodiment of meticulous accuracy. Nothing slipshod came from his hand, his tongue or his pen. Throughout his life his colleagues were concerned at the time and energy which he spent on unrewarding chores, to his own exasperation at the slackness of others and to the detriment of his research. The duty of taking his share in the refereeing of papers was accepted with wry resignation. Unless an author had standards approaching Ingham's, his paper was likely to be taken to pieces, partly rewritten, and reassembled. Some of his friends ask whether, if he could have been more summary with secondary activities, might not some of the intact problems of number theory have been resolved by his magnificent analytical power? Others demur to the thought, holding that his conscientiousness was an integral part of his intellectual honesty.

From 1932 Ingham had the devoted support of his wife Jane (Rose Marie), daughter of Canon A. D. Tupper-Carey. They were ideally complementary, the one deliberate in all his actions, the other of surpassing quickness in thought, movement and speech. They had two sons, Michael and Stephen. Friends and pupils were always welcome to join the happy family circle at 14 Millington Road. It was characteristic that the best room in the house was turned into "the workshop" where Ingham's fine handiwork was a model for the next generation. A casual caller at the home would be likely to find young children of friends and neighbours in conversation or in play with the mathematician. The simplicity of their lives was in tune with his own and their company seemed to be for him the happiest relief from the depths of pure mathematics.

He was friendly and hospitable and would be lavish of his time to help a pupil or a colleague. With his instinctive kindness he would notice if someone at a party (perhaps a visitor to his College) was lonely and would draw him into a circle. He called almost everyone by his surname. This was in part for better definition—John was more likely to be ambiguous than Williamson—but it was also formal courtesy. One cannot conceive of him thinking of a Provost of King's as Noel or Edmund, nor would he address a young faculty secretary as Jill or Kay or Anne.

The simplicity of his life has already been mentioned. It did not occur to him to want a car or a radio, let alone a television set. For forty years he used the Sunbeam bicycle that he had won at school as a prize. He was an expert photographer; he developed his own colour films and did everything from first principles. He had a reading knowledge of Russian as well as of the more usual languages. He was a good cricketer, as was his father—who would have been of minor county class if he had been able to give the time. He was always eager for a friendly match between the High Table and the College Staff or the Choir School. Like Hardy and Littlewood, he was a devotee of cricket-watching at Fenner's.

In a College conspicuous for the loyalty of its members, he was one of the most loyal. The King's Record relates that "in college meetings he predictably put the conservative view, sometimes with a melancholy irony that could not fail to amuse. Impatient of any nonsense or fuss, he had complete integrity and was not afraid of being in a minority of one". The link of the College with its Choir School had a particular interest for him. He had the great happiness of seeing his elder son Michael elected a Fellow of King's in 1961 and join the staff of the University Observatory at Oxford.

There were unexpected departures from the order and system which governed nearly all his doings. The large table on which he worked was a chaos of books, manuscripts, letters, notices, and a minute or two of search was often needed to bring to light what he wanted. For many years he and his wife spent their summer holiday walking (with rucksacks) among mountains. They asserted that no plans were made in advance; they would go to Victoria Station (not to an airport!) and decide there where to buy a ticket to. It was on such a holiday that he died. On September 6th, 1967, on a high path near Chamonix, his heart failed. Fortunately other walkers were within sight. He died almost without pain, conscious that everything that could be done for him was being done.

The 1920s were exciting years for a young analyst, and one with Ingham's ability would have been encouraged to plunge at once into deep water. Hardy left for Oxford in 1920, Littlewood stayed in Cambridge. Their collaboration suffered no interruption. High among their interests at that prolific period were (1) the zeta function and the analytic theory of numbers, (2) Tauberian theorems and the like. It is significant that the two "lemmas" of Ingham's fellowship dissertation were properties of the zeta function and that the first paper which he presented to the London Mathematical Society arose out of Littlewood's Tauberian theorem. These two interests were to cover nearly all of his life's work.

Research was not then organized under a "supervisor" as it has been in later years and, though it is likely that Ingham needed little direct help, he must have had boundless stimulus from Littlewood. Littlewood was happier solving problems than writing up the results for publication and he would toss a manuscript across to a pupil, giving him ideas which he could develop. In particular there was a famous Bohr–Littlewood manuscript on the zeta-function which was intended to be a Cambridge tract but never was. The accessibility of Littlewood and a timely sentence from him, perhaps "Work at a hard problem; you may not solve it but you'll solve another one", would be enough to cheer Ingham on a sticky wicket.

It is now time to review in detail Ingham's contributions to knowledge. His one book, the Cambridge tract on *The Distribution of Prime Numbers*, was published in 1932. It is, and will remain, a classic. The roots of both this tract and Titchmarsh's on the Zeta Function were in the Bohr–Littlewood manuscript. Titchmarsh in 1951 expanded his tract into a substantial book (Oxford). When Ingham's tract went out of print he was urged to make a similar expansion or, at least, a revision. Such a rewriting would have meant, with his standards, more toil than he could face. The tract was ultimately reprinted, with minor changes, by Stechert-Hafner (New York) in 1964.

The papers comprise two main groups. Professor Davenport has written the following analysis of those on the theory of numbers, which form the larger group. The references are to the bibliography at the end of the notice.

*The Riemann zeta function and the theory of numbers*

The papers on the zeta function and the distribution of primes are **4**, **7**, **9**, **10**, **15**, **18**, **20**, **23**.

Other papers having their origin in problems of the analytic theory of numbers are **21**, **22**, **24**, **25**, **29**.

The first in time was **7**, the main results of which were communicated to the London Mathematical Society at its meeting on April 26th, 1923 (see *Proceedings* (2), 22). This paper is concerned principally with the asymptotic behaviour of

$$J_2(\sigma, T) = \int_1^T |\zeta(\sigma+it)|^2\, dt \tag{1}$$

and

$$J_4(\sigma, T) = \int_1^T |\zeta(\sigma+it)|^4\, dt \tag{2}$$

as $T \to \infty$, though for technical reasons it is found desirable to consider a more general form of the first. The most interesting and delicate case is when $\sigma = \frac{1}{2}$. Here the results proved are:

$$J_2(\tfrac{1}{2}, T) = T\log T + cT + O(T^{\frac{1}{2}}\log T), \tag{3}$$

where $c$ is a certain numerical constant, and

$$J_4(\tfrac{1}{2}, T) = \frac{1}{2\pi^2}\, T\log^4 T + O(T\log^3 T). \tag{4}$$

Littlewood had obtained a less precise estimate for $J_2$ from the approximate functional equation for $\zeta(s)$. To establish (3) Ingham had to carry out a refined analysis of the remainder term in that functional equation.

The proof of (4) is more straightforward, but is based on the approximate functional equation for $\zeta^2(s)$, which had been placed at Ingham's disposal by Hardy and Littlewood, though it was not published by them until later.†

In the course of the proof of (4), Ingham found it necessary to estimate the sum

$$\sum_{m=1}^{n} d(m)\, d(m+k),$$

where $d(m)$ denotes the number of divisors of $m$. This no doubt suggested to him the more detailed investigation of this sum, and of the analogous sum

$$\sum_{m=1}^{n-1} d(m)\, d(n-m),$$

which is made in **4**. His results on these two questions were substantially improved upon later by Estermann.‡ For references to more recent work see an article by Linnik.§

Paper **9** is of a quite different character from the two just discussed; it may be described as concerned with a qualitative rather than a quantitative question. Until this paper appeared, there was essentially only one proof that $\zeta(1+it) \neq 0$, namely that given in somewhat different forms by Hadamard and by de la Vallée Poussin in 1896. The same proof covers the non-vanishing of Dirichlet's $L$-functions at $1+it$, except in the case when the character is real (and non-principal) and $t = 0$. For this latter result there were essentially two proofs, one arithmetical and due to Dirichlet, the other analytical and due to de la Vallée Poussin. Neither of them is similar to the proof for $\zeta(s)$. In **9**, Ingham proved a general theorem which covers both cases:

*Let*

$$g(s) = \prod_{p} (1-\varepsilon_p p^{-s})^{-1},$$

*where $\varepsilon_p$ is a number (real or complex) of absolute value* 1 *or* 0, *and the product is over all primes. Suppose that $g(s)$, which is obviously regular for $\sigma > 1$, can be continued along the real axis as far as the point $s = \frac{1}{2}$, this point included. Then $g(1) \neq 0$.* In the application one takes $\varepsilon_p = \chi(p)\, p^{-ia}$, where $\chi$ is a character (which can be 1) and $a$ is a real number (which can be 0).

† *Proc. London Math. Soc.* (2), 29 (1929), 81–97 or Hardy's *Collected Papers* II (Oxford, 1967), 213–229.

‡ *Journal für Math.*, 164 (1931), 173–182.

§ *Proc. Internat. Congress of Mathematicians* Edinburgh 1958 (Cambridge, 1960), 313–321.

In **10,** Ingham considered the existence of

$$\lim_{T\to\infty} \frac{1}{T} \int_1^T |\zeta(\sigma+it)|^\lambda \, dt,$$

where $\lambda$ is a positive real number. With the help of general convexity theorems, including Gabriel's two-variable convexity theorem†, he proved that the limit exists if $\sigma > \frac{1}{2}$ and $0 < \lambda \leqslant 4$. A simpler but more special proof was given later by Davenport.‡

Paper **15** is the first which relates directly to the distribution of the primes. Let $\pi(x)$ denote the number of primes not exceeding $x$, and let

$$\operatorname{li} x = \int_0^x \frac{du}{\log u}$$

(where the Cauchy principal value is taken at $u = 1$). By the Prime Number Theorem, $\pi(x)$ is asymptotically equal to $\operatorname{li} x$ as $x \to \infty$. Although numerical evidence suggested that $\pi(x)$ might be always less than $\operatorname{li} x$, a famous theorem of Littlewood (1914) states that the difference $\pi(x) - \operatorname{li} x$ in fact changes sign infinitely often. An account of Littlewood's proof was given by Ingham in Chapter 5 of his Tract. There is no difficulty if one supposes that the Riemann Hypothesis is false. But if it is true, the proof requires one to show that the oscillating terms in the "explicit formula" for $\pi(x) - \operatorname{li} x$ sometimes add up to a large positive amount. Littlewood's proof was very delicate and complicated, and in the present paper Ingham gave a simpler and more direct proof. He further showed that, on the assumption of the Riemann Hypothesis, there exists a constant $A > 1$ such that every interval $(x, Ax)$, with $x$ sufficiently large, contains both integers $n$ for which $\pi(n) > \operatorname{li} n$ and integers $n'$ for which $\pi(n') < \operatorname{li} n'$. For references to later work on these questions, see Ingham's comments in Hardy's *Collected Papers* II, 98–99. Some of the ideas underlying **15** emerge with greater generality and clarity in **23**.

Of all Ingham's papers, probably **18** is the one which is known to the widest circle of mathematicians, since it concerns a problem of general interest, namely the magnitude of $p_{n+1} - p_n$, where $p_n$ denotes the $n$th prime. Hoheisel was the first to prove, in 1930, that there is some constant $\theta < 1$ such that

$$p_{n+1} - p_n < p_n^{\theta}$$

for all sufficiently large $n$. His value of $\theta$ was only slightly less than 1. Ingham proved that the result holds with $\theta = \frac{5}{8}$, and indeed with a slightly smaller value of $\theta$ depending on whatever estimate may be proved for $|\zeta(\frac{1}{2}+it)|$ as $t \to \infty$. The proof depends on a new upper bound for $N(\alpha, T)$, the number of zeros of $\zeta(s)$ in the rectangle $\alpha \leqslant \sigma \leqslant 1$, $|t| \leqslant T$. Two such upper bounds are established in the paper;

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† *J. London Math. Soc.*, 2 (1927), 112–117.

‡ *J. London Math. Soc.*, 10 (1935), 136–138.

the first of these was improved later by Ingham in **20**, and will be found as Theorem 9.19B in Titchmarsh's *The theory of the Riemann zeta-function* (Oxford, 1951). But it is the other which is more effective when $\alpha$ is near to 1, and which dominates the problem of $p_{n+1}-p_n$. Here no further progress has since been made.

The last paper of the group, **23**, is one of fundamental importance. It has already had many consequences, and probably its potentialities are still not exhausted. It relates to two conjectures, one put forward by Mertens in 1897, the other by Pólya in 1919. Let $\lambda(n)$ denote $+1$ or $-1$ according as $n$ has an even or odd number of prime factors, repeated factors being counted with their multiplicities. Pólya's conjecture was that

$$L(x) = \sum_{n \leqslant x} \lambda(n) \leqslant 0$$

for all $x \geqslant 2$. Numerical evidence confirms this up to $x = 250{,}000$, but the theory of the $\zeta$-function, and the analogy with Littlewood's theorem on $\pi(x) - \operatorname{li} x$, suggested that probably the conjecture was false.

By very ingenious reasoning, Ingham showed (in effect) that the conjecture would be disproved if one could find particular values of $T$ and $u$ for which

$$A^*(T, u) > 0.$$

Here

$$A^*(T, u) = \frac{1}{\zeta(\frac{1}{2})} + 2\mathscr{R} \sum_{0<\gamma_n<T} \left(1 - \frac{\gamma_n}{T}\right) \alpha_n e^{i\gamma_n u},$$

where $\frac{1}{2}+i\gamma_n$ is the typical zero of $\zeta(s)$, and

$$\alpha_n = \frac{\zeta(1+2i\gamma_n)}{(\frac{1}{2}+i\gamma_n)\,\zeta'(\frac{1}{2}+i\gamma_n)}.$$

(It should be noted that in disproving Pólya's conjecture one is entitled to assume the Riemann hypothesis and further to assume that all the zeros of $\zeta(s)$ are simple.)

Thus the way was opened to the possibility of disproving the conjecture by a computation which used only a finite number of zeros of $\zeta(s)$. Note that $\zeta(\frac{1}{2})$ is negative, so that in order to succeed it is necessary to make the exponential sum assume a positive value which is large enough to outweigh this negative constant term. Thus the problem is not purely computational; having chosen a value for $T$, one must restrict the search for a possible $u$ to some carefully selected set of values which will tend to produce a positive sum. Success was achieved by the late C. B. Haselgrove†, a former research student of Ingham who was then director of the Computing Laboratory at Manchester University. He found that $A^*(T, u) > 0$ when $T = 1000$ and $u = 831{\cdot}847$. This disproved Pólya's conjecture, but did not in itself yield an explicit value of $x$ for which $L(x) > 0$. Later R. S. Lehman‡ proved that

$$L(906180359) = +1.$$

† *Mathematika*, 5 (1958), 141–145.
‡ *Math. Comp.*, 14 (1960), 311–320.

The second conjecture of the title, due to Mertens, has still not been disproved. It is probable that Ingham's method is adequate for the purpose, but it seems as yet that the amount of computation that would be needed to give a reasonable chance of success would be prohibitive.

*Paper* **21.** The two classical lattice point problems in question are the circle problem and Dirichlet's divisor problem. The circle problem is concerned with the number $R(x)$ of integer pairs $u$, $v$ satisfying

$$u^2+v^2 \leqslant x.$$

If we put

$$R(x) = \pi x + P(x),$$

it is known that $|P(x)|$ behaves like $x^{\frac{1}{4}}$ in mean square, and it was proved by Hardy that there is a positive constant $A$ such that

$$P(x) < -Ax^{\frac{1}{4}} \log^{\frac{1}{4}} x$$

for some arbitrarily large values of $x$. Any inequality in the opposite direction presents much greater difficulty, since its proof demands the use of Kronecker's theorem on Diophantine approximation in place of Dirichlet's theorem. In the paper under discussion, Ingham proves that for *any* $C$, however large,

$$P(x) > Cx^{\frac{1}{4}}$$

for some arbitrarily large values of $x$. He indicates at the end of the paper the possibility of getting a more quantitative result. Work on these lines was carried out by K. S. Gangadharan†, a research student of Ingham, who proved that

$$P(x) > Ax^{\frac{1}{4}}(\log\log x)^{\frac{1}{4}}(\log\log\log x)^{\frac{1}{4}}.$$

A further improvement has been effected recently by Katái and Corradi.

There is a similar situation with Dirichlet's divisor problem, except that now it is the negative values of the error term that present the greater difficulty.

*Paper* **22.** In a famous paper, Hardy and Ramanujan‡ gave an asymptotic expansion for $p(n)$, the number of unrestricted partitions of $n$. This was the first instance of the use of the "circle method", afterwards developed in different directions by Hardy and by Hardy and Littlewood.

As Hardy and Ramanujan remarked, a Tauberian argument, based on the behaviour of the generating function

$$\sum p(n)\, x^n = \Pi(1-x^n)^{-1}$$

as $x \to 1$ from the left through real values, can be used to prove that

$$\log p(n) \sim \pi \sqrt{\left(\frac{2n}{3}\right)},$$

† *Proc. Cambridge Philos. Soc.*, 57 (1961), 699–721.

‡ *Proc. London Math. Soc.* (2), 17 (1918), 75–115, or Hardy, *Collected Papers* I, 306–339.

but appears to be incapable of giving the first term of the asymptotic expansion, i.e.

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\left(\frac{2n}{3}\right)}\right).$$

In the present paper Ingham shows that this can be proved by a more elaborate Tauberian argument, using the behaviour of the generating function as $x \to 1$ in an angle inside the unit circle.

The advantage of the Tauberian method is that it does not make use of the more profound properties of the generating function (derived from the theory of the elliptic modular functions), and is therefore applicable to a wider range of partition problems. Theorem 2 of the present paper gives an asymptotic formula for the number of partitions of $n$ into given parts $\lambda_1, \lambda_2, \ldots$, under conditions of some generality.

For later work on this question, see a paper of Roth and Szekeres.†

*Paper* **24**. This paper is concerned primarily with what Hardy called "Ingham summability" and discussed in Appendix IV of his *Divergent series* (Oxford 1949).

A series is said to be summable (I), with sum $s$, if

$$\sum_{n \leqslant x} \frac{n}{x}\left[\frac{x}{n}\right] a_n \to s$$

as $x \to \infty$, where $[t]$ denotes the largest integer not exceeding $t$. The method is not "regular", that is, a convergent series is not always summable by the method.

In Theorem 2 of the paper, Ingham proves the satisfying result that the usual one-sided Tauberian condition

$$a_n > -K/n,$$

where $K$ is a positive constant, suffices to ensure that a series summable (I) shall be convergent, with the same sum. This theorem is deduced from the more general Theorem 1, which is concerned with the following question. Suppose we know that a positive function $f(x)$, defined for $x \geqslant 1$, satisfies

$$\sum_{n \leqslant x} f(x/n) = ax \log x + bx + o(x),$$

where $a$, $b$ are constants. Under what conditions can we infer that $f(x) \sim ax$ as $x \to \infty$? It is proved that it suffices if $f(x)$ is increasing in the wide sense. The proof employs Wiener's Tauberian theory, but a further essential ingredient is the non-vanishing of $\zeta(s)$ on $\mathscr{R}s = 1$.

Further results about summability (I) are given at the end of the paper; from these it emerges in particular that (I) implies $(C, \delta)$ and is implied by $(C, -\delta)$, for any $\delta > 0$.

† *Quart. J. Math.* (2), 5 (1954), 241–259.

*Paper* **25**. In spite of its title, this is a sequel to **24**, and like that paper is connected indirectly (but essentially) with the prime number theorem.

Suppose $\phi(t)$ is defined for $0 < t \leqslant 1$, and integrable in the sense of Riemann in $0 < \delta \leqslant t \leqslant 1$ for each $\delta$. Let

$$S(h) = h \sum_{n \leqslant 1/h} \phi(nh), \quad I(h) = \int_h^1 \phi(t)\,dt.$$

It was proved by Wintner (and it follows also from one of the results of **24**) that if $S(h)$ has a finite limit as $h \to 0$, then $I(h)$ has the same limit. The proof depends in a surprising way on the absolute convergence of

$$\int_1^\infty \frac{g(v)}{v}\,dv, \text{ where } g(v) = \sum_{n \leqslant v} \frac{\mu(n)}{n}$$

and $\mu(n)$ is the Möbius function, arising in this context as the coefficient of $n^{-s}$ in the Dirichlet series for $1/\zeta(s)$.

In **25** Ingham shows that it is more illuminating to generalize the equally spaced ordinates $nh$ in the Riemann sum to $l_n h$, where $d_n = l_n - l_{n-1} = o(l_n)$. The necessary and sufficient condition that the corresponding $I(h)$ has the same limit as the sum $S(h)$ involves "Möbius functions" which appear as the coefficients in the Dirichlet series for $1/Z(s)$, with $Z(s) = \sum d_n \lambda_n^{-s}$ $(l_{n-1} \leqslant \lambda_n \leqslant l_n)$.

*Paper* **29**. This paper, with its combination of delicate ideas from number theory and analysis, could hardly have been written by any single mathematician, or by any two mathematicians other than Erdös and Ingham.

The general question considered has some affinity with that of **24**. But instead of the hypothesis

$$\sum_{n \leqslant x} f\left(\frac{x}{n}\right) = ax \log x + bx + o(x),$$

there is the hypothesis that

$$f(x) + \sum_{a_n \leqslant x} f\left(\frac{x}{a_n}\right) = (1+A)\,x + o(x),$$

where $1 < a_1 \leqslant a_2 \leqslant \ldots$ is a sequence of integers, and $\sum 1/a_n$ converges with sum $A$. The question is, whether this implies that

$$f(x) \sim x$$

as $x \to \infty$. It is difficult to summarize the conclusions which depend very much on whether $A < 1$ or $A = 1$ or $A > 1$. Generally speaking it may be said that the elementary methods are successful only if $A \leqslant 1$, unless special relations exist among the numbers $a_n$.

*Papers on the theory of series and Tauberian theorems*

Professor Bosanquet has analysed this work, which is contained in the papers numbered **1**, **12**, **13**, **14**, **17**, **26**, **28**, **30**.

**1**(*b*). This refers to a manuscript written in 1924 as part of Ingham's report to the Royal Commissioners for the Exhibition of 1851, from whom he held a Senior Studentship. The story of it is characteristic of Ingham's perfectionism. He sent a copy to Hardy, who (on his own initiative) sent it to the London Mathematical Society. It was accepted for publication, subject to some revision, and in particular to the toning down of some destructive criticism of other writers, permissible in a confidential report but unsuited to a journal.

In a pioneer paper on Tauberian theorems for Riesz's typical means (*Proc. London Math. Soc.* (2), 12 (1913), 174–180) Hardy had made two mistakes, (i) by leaving a gap in the inductive proof of the $O$-theorem, and (ii) by neglecting a term in the $O_L$-theorem that would be significant if the indices increased erratically. Neither omission was trivial.

In 1918 Ananda-Rau filled the gap in (i). Ingham now gave an example to show that the result in (ii) was incorrect without an extra condition. This was shown independently by Ananda-Rau (1930). Ingham gave also a number of positive results.

Ingham later explained that he never revised the paper for publication, partly because he " had the uneasy feeling that the proofs of the constructive part were not in the best form, and that in any case a good deal of it was probably well enough known to Hardy and Littlewood ".

A full account of these Tauberian results is now in Hardy, *Divergent Series*, Chapters 6 and 7. The notes at the ends of those chapters are relevant.

**12.** This paper is concerned with a remark of Wiener, quoted by Hardy, that a function and its (complex) Fourier transform cannot both be very small at infinity without being null. For example, if $f(x)$ and $F(x)$ are both $O(e^{-\frac{1}{2}x^2})$, then $f(x) = F(x) = Ce^{-\frac{1}{2}x^2}$; and if, further, one function is $o(e^{-\frac{1}{2}x^2})$, then both are null. Ingham remarks that if $f(x)$ vanishes outside $(-A, A)$, then $F(x)$ cannot be $O(e^{-\epsilon x})$ without being null. He goes on to show that, if $\chi(y)$ is any positive decreasing function tending to zero, then there is a function vanishing outside $(-A, A)$, but not null, such that $F(x) = O(e^{-x\chi(x)})$, if and only if $\int^{\infty} \frac{\chi(y)}{y}\,dy$ converges. This gives information about the class of kernels $K(x)$ defining transformations $\frac{1}{T}\int_{-T}^{T} K(t/T)\,g(t)\,dt$, suitable for use in applications of Wiener's method in Tauberian theorems.

**13**, **14** and **26**. These papers are concerned with two analogous groups of Tauberian theorems for a general Dirichlet series $f(s) = \sum a_n e^{-\lambda_n s} = \sum a_n l_n^{-s}$. A typical pair is: ($\lambda$-type) Schnee's extension of Tauber's theorem, where the convergence of $\sum a_n$ follows from $\sum_1^n \lambda_r a_r = o(\lambda_n)$ and $f(s) \to A$ as $s \to 0+$, and

($l$-type) M. Riesz's extension of Fatou's theorem, where the convergence of $\sum a_n$ follows from $\sum_1^n l_r a_r = o(l_n)$ and $f(s)$ regular at $s = 0$. In some $l$-theorems, $f(s)$ satisfies a condition near a whole line $\mathscr{R}(s) = a$. For example, in Ikehara's theorem, $f(s)$ is regular on $\mathscr{R}(s) = a$ except for a pole at $s = a$. Heilbronn and Landau extended Ikehara's theorem, by only assuming a property of $f(s)$ near a segment $\sigma = a$, $-T \leqslant t \leqslant T$, and obtaining a result involving bounds depending on $T$, which becomes Ikehara's theorem in the limit as $T \to \infty$.

In **13**, Ingham re-examines Heilbronn and Landau's results and obtains other $l$-theorems with the Heilbronn-Landau refinement. His method is to use a transformation, defined by a suitable kernel satisfying Wiener's conditions. He also formulates an $l$-version of the high indices theorem, which he refines successively in **14** and **26**. In the $\lambda$-version, where $\lambda_n/\lambda_{n-1} \geqslant 1+d$ $(d > 0)$, the appropriate Tauberian condition, $a_n = O(1)$ or $o(1)$, is omitted, and the main step is to prove that it is a consequence of the other hypothesis. In the $l$-version, where $l_n/l_{n-1} \geqslant e^\gamma$ $(\gamma > 0)$, i.e. $\lambda_n - \lambda_{n-1} \geqslant \gamma$, the function satisfies a condition on a segment $\sigma = 0$, $-T \leqslant t \leqslant T$. In the weakest form, stated in **13**, this is assumed for all large $T$, and is used to deduce that $a_k = o(1)$. In **14**, Ingham, with the hypothesis

$$T = \frac{\pi+\varepsilon}{\gamma} > \frac{\pi}{\gamma},$$

sharpens his earlier analysis to obtain the inequalities

$$\sum |a_n|^2 \leqslant \frac{C(\varepsilon)}{2T} \int_{-T}^{T} |f(t)|^2\, dt,$$

$$|a_n| \leqslant \frac{C(\varepsilon)}{2T} \int_{-T}^{T} |f(t)|\, dt.$$

In **26** he succeeds in making the final refinement of the latter inequality to $\varepsilon = 0$ and $C(0) = 2$ (the best possible). An elegant treatment of the last inequality was later given by Mordell (*Illinois J. Math.*, 1 (1957), 214). L. Schwartz, *Etude des sommes d'exponentielles* (Hermann 1959), Chapter 3, §3, relates Ingham's results to those of others.

**17** and **30**. In **17** Ingham introduces a new method for proving Tauberian theorems, and applies it to a proof of the high indices theorem. In the original method of Littlewood, a peak function was obtained by taking the $k$th derivative of $f(s) = \sum a_n e^{-\lambda_n s}$, with a sufficiently large $k$. Thus Littlewood obtained $(-1)^k f^{(k)}(s) = \sum a_n \lambda_n{}^k e^{-\lambda_n s}$. Ingham observes that a peak function may also be obtained by taking a $k$-th difference with increment $h$, where $h$ is sufficiently small or (as in his proof of the high indices theorem) $h$ is related to $s$. For example, we may take

$$M^{-k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}f(ks+js)=\sum a_n M^{-k}(e^{-\lambda_n s}-e^{-2\lambda_n s})^k,$$

where $M$ is the maximum of $e^{-x}-e^{-2x}$. His proof of the theorem is simpler than that of Hardy and Littlewood, in particular because the main step reduces it to an $o$-Tauberian theorem instead of an $O$-theorem. In **30**, in the volume of the *Proceedings* published in honour of Professor Littlewood's 80th birthday, Ingham returns again to the topic to give "a simplified version that has gradually evolved over a long period of lecturing on the subject". Here the emphasis is on the choice of peak functions without the use of approximation theory, and (as he points out) differences are nowhere explicitly mentioned.

**28.** This paper is concerned with extensions of Wiener's well-known theorem about the reciprocal of a function with an absolutely convergent Fourier series. Ingham gives an elementary proof of a known extension, originally obtained by functional analysis by Hewitt & Williamson and Edwards, for the case of a generalized Dirichlet series in which the exponents do not necessarily increase. He also extends this in the manner of Lévy and Zygmund.

*Ingham's other work and his influence*

Comments have still to be made on the following papers in the bibliography: **2**, **3**, **5**, **6**, **8**, **11**, **16**, **19**, **27**.

**2.** Soon after Ingham became a Fellow of Trinity, J. E. Jones (later Sir John Lennard-Jones) arrived from Manchester as a research student in mathematical physics. Jones consulted Ingham on problems about cubic crystals which were directly expressible by the generalized zeta functions of Epstein. The look on Ingham's face as he turned the handle of a Brunsviga calculator was that of the Spartan boy.

**5** and **6** contain theorems of the Phragmén-Lindelöf type about mean values

$$\frac{1}{2y}\int_{-y}^{y}|f(x+iw)|^p\,dw,$$

where $f$ is regular in a vertical strip $\alpha \leqslant x \leqslant \beta$. A necessary preliminary is a discussion of convexity theorems for

$$J(x)=\int_{-\infty}^{\infty}|f(x+iy)|^p\,dy.$$

The results are applied in **10** to the zeta function.

**3**, **11**, **16** are short notes, and **8** a longer paper, which are all good specimens of Ingham's analytical force.

**19** and **27** are two of Ingham's longer reviews. His reviews were searching and influential and were read by many mathematicians besides those immediately interested. A list of his more important reviews is appended.

In summing up we must recall the high respect in which Ingham was held by mathematicians, pure and applied alike. It was not easy to get an opinion out of him—his first defence was a disclaimer of knowledge—but anything that he ultimately said was based on all the relevant information and was impeccable. The influence of his papers was greater by far than might be suggested by their modest number or their modest style of presentation. They have been read and re-read by mathematicians throughout the world. Anyone who was fortunate enough to make a significant addition to their contents had made his mark. Among Ingham's research students were R. A. Rankin, C. Hooley and the late C. B. Haselgrove and W. B. Pennington.

A tribute from Norbert Wiener is of interest. In 1926, when they were both in Göttingen, Ingham pointed out the Tauberian aspect of the theorems at which Wiener was working. Wiener records that "It is to Ingham that I owe a scientific lead which carried me to much of my best work" (*I am a mathematician* (New York, 1956), 115).

It will be plain to the reader that the hard core of this notice is the work of Professors Davenport and Bosanquet. I invoke a prevalent phrase in saying that it is "for technical reasons" that I figure as the author, making acknowledgments to them, instead of the other way round. The Provost and Fellows of King's College, Cambridge (among whom Mr. L. P. Wilkinson is to be named) have generously allowed me to incorporate parts of a memoir in the Annual Report of their Council (November 1967). Professor Littlewood and Dr. L. A. Pars have helped me to reconstruct past events. Jane and Michael Ingham have been kind and helpful in every way.

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