PATRICK BRENDAN KENNEDY

W. K. HAYMAN

Patrick Brendan Kennedy, or Paddy as he was called by his friends, was born on 20 July, 1929, at Clarecastle, County Clare, the third of five children.

His father was a master carpenter by trade but had joined the police in 1923 after the end of the civil war. He was unusual in deciding on his own trade instead of inheriting the family farm, which was his right as the eldest son. Patrick's maternal grandfather and many of his relations on his mother's side were blacksmiths near Castlemaine. His mother's family were quiet-spoken and rather introverted people and pledged teetotallers.

In 1937 after one year's stay in Ballylongford, Paddy's father succeeded in getting himself transferred to Cork so that the children could attend the secondary school there and obtain a proper education. Paddy joined North Monastery, the secondary school at Cork in 1941 and his career there culminated in the award of the Honan scholarship in 1946, many of whose holders end up as Professors. Paddy joined University College, Cork, to study Science having been dissuaded by the president from doing Medicine as had been his original intention.

Having got his degree in 1949 and being promised a demonstratorship of £300 p.a., Paddy immediately thought of helping his family, and proceeded to help his elder brother Tadhg through college. Two years later Paddy obtained his M.Sc. from the National University. His examiner, V. C. A. Ferraro, was very impressed by his work in analysis and suggested that Paddy should join me in Exeter, where I was at that time a lecturer, as my first research student. The following two years were very happy for us both. We were young and enthusiastic and had long conversations about functions walking in Rougemont gardens and other places around Exeter.

I had recently returned from America full of the ideas of Maurice Heins and put Paddy onto a question raised by Heins. This he solved in a thoroughly elegant manner. But the teaching was both ways. Halberstam and I and some of our other colleagues benefited from a course on probability and measure theory given by Paddy. Paddy was a wonderful conversationalist and tremendous good company. I have known few greater joys in life than sitting opposite him each smoking a cigarette and chatting, oblivious of time.

After two years Paddy's work with me was essentially complete and he started on his University career as an assistant lecturer at Aberystwyth where he was married to Pamela Fishwick in March 1954. After the end

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of the session Paddy escaped back to Cork under threat of being called up in the British Army, not an encouraging prospect for a patriotic Irishman. He became lecturer and in 1956 Professor at Cork and was elected a fellow of the Royal Irish Academy in 1962. In 1963 he returned to England as first Professor of mathematics at York, where he died tragically on the night of 8/9 June, 1966, leaving his widow and three children.

No record of Paddy's life is complete without mentioning his chess. He was Irish champion in 1949, being the youngest person ever to hold that title, and the only one to win it at the first attempt and without losing a match. His games in the championship were characterized by a combination of logic and lack of nerves. He played whatever the situation demanded, whether that was a wild or very risky-looking attack or a precise manoeuvre. He seemed to lose interest in the game somewhat after the championships, possibly from a feeling of anticlimax. Had he worked at it he might have been one of the top half dozen players ever to come out of Ireland. What is less well known is that Paddy was also a very promising pianist. He had lessons in 1950 from Mrs. Tilly Fleischmann and learnt very quickly to play Chopin. She said of him at the time that he was the only pupil she had had, who never played a wrong note. He was also very interested in listening to music, chiefly Beethoven, and his general conversation suggested the arts man rather than the scientist.

Throughout his career as a university teacher Paddy threw himself with great gusto into academic politics. He saw things in black and white colours and had no hestiation in telling everybody that it would be iniquitous to do anything but A even when it was fairly clear that course B would in fact be adopted. Coming from a very young Professor such an attitude sometimes caused resentment. It also caused him a great deal of disappointment when some of the things he hoped for did not come off. He could not be above the battle. If he was in something, whether it was a committee or a problem in mathematics he was in it up to the neck. In this way he got a good deal done both mathematically and in building up the departments at both Cork and York, but at high cost to his nerves. He was extremely conscientious and his final tragic breakdown can only be attributed to excessive and quite unnecessary worry about his students and his work.

Kennedy's mathematical work was concerned largely with asymptotics. In this field we first establish that under certain hypotheses a function of a continuous or integral variable has at most (or at least) a certain order of magnitude. Secondly we prove that this result is best possible, in the sense that no stronger estimate of this type is true in general. Finally, we may in favourable circumstances be able to investigate further those functions for which the extremal growth is obtained. Kennedy was extremely successful in all the three fields outlined above. He was also most ingenious in constructing examples to show that his own and other people's results were best possible, and these examples are frequently quite simple although far from obvious. The main fields in which Kennedy worked are functions of a complex variable, Fourier series with gaps, and Tauberian theorems and we shall consider his work in each of these in turn.

1. *Theory of functions of a complex variable* [**1**, **3**, **4**, **11**, **13**, **14**, **15**, **17**, **19**].

Kennedy's powers were already fully displayed in two papers [**1**, **4**] which constituted his thesis. Let $u(z)$ be a function subharmonic in the plane. Further let $\gamma_1, \gamma_2, \ldots, \gamma_{n+1}=\gamma_1$ be n curves going from $z=0$ to ∞ and having no common points and suppose that $u(z)$ is bounded above on each of the curves γ_k but not in any of the complementary domains D_k determined by these curves. Set

$$\sigma_k(r)=\sup u(z) \text{ for } z \text{ in } D_k \text{ and } |z|=r,$$

and

$$\sigma(r)=\max \sigma_k(r), \quad \text{for } 1 \leqslant k \leqslant n.$$

Then it was known that

$$\alpha=\varliminf_{r\to\infty}\frac{\sigma(r)}{r^{k/2}}>0. \tag{1}$$

The special case when $u(z)=\log|f(z)|$ and $f(z)$ has n distinct asymptotic values a_k along the curves γ_k represents Ahlfors' celebrated solution of the Denjoy conjecture. The general result is due to Heins. Heins investigated the case when $\alpha<+\infty$ in (1) and proved that in this case, we have as $r\to\infty$

$$\log\sigma(r)\sim\frac{n}{2}\log r \tag{2}$$

and

$$\sum_{k=1}^{n}\log\sigma_k(r)=\frac{n^2}{2}\log r+O(1). \tag{3}$$

Heins also conjectured that (2) could be sharpened to

$$\log\sigma(r)=\frac{n}{2}\log r+\lambda+o(1),$$

a result which he had previously proved in the case $n=1$. This conjecture was disproved by Kennedy [**1**], who showed that the best possible results are for every $n\geqslant 2$

$$\log\sigma(r)=\frac{n}{2}\log r+o(\log r)^{\frac{1}{2}}, \tag{2'}$$

and

$$\sum_{k=1}^{n}\log\sigma_k(r)=\frac{n^2}{2}\log r+\lambda+o(1). \tag{3'}$$

In order to prove his positive theorems Kennedy developed a refinement of Ahlfors' distortion theorem, which allowed him to replace $O(1)$ by $\lambda+o(1)$. The result has since been extended by Eke and has many applications. Equally interesting are the counter-examples [**4**]. Here Kennedy first constructed sub-harmonic functions with the required properties and then, by an ingenious modification of a previous technique of Kjellberg, he showed how to approximate these subharmonic functions by functions of the type $\log|f(z)|$, where $f(z)$ is entire, so that Kennedy's final examples are actually of integral functions. Kennedy also needed and proved for subharmonic functions of finite order an analogue of the Hadamard representation of integral functions of finite order. The approximation technique in particular has proved important in other connections and has been refined by Dr. Al-Katifi.

Further Kennedy showed [**1**] that if $\theta(r)$ is the argument of a point of intersection of γ_k, with $|z|=r$ and if $\alpha<+\infty$ in [**1**], then

$$\theta(r)=o(\log r)^{\frac{1}{2}} \tag{4}$$

and that this result is best possible [**4**]. However, in his proof of (4) there was a gap which was filled a little later by Clunie.

If

$$\overline{\lim_{r\to\infty}}\frac{\theta(r)}{\log r}=\mu,$$

Ahlfors had shown that

$$\overline{\lim_{r\to\infty}}\frac{\log\sigma(r)}{\log r}\geqslant\frac{n}{2}(1+\mu^2),$$

and Kennedy's method allowed him to construct examples to show that this inequality also is sharp.

In another paper with a somewhat similar outlook Kennedy and I considered p-valent functions of maximal growth. For such functions the maximum growth is attained along a ray $\arg z=\theta_0$ and I had previously shown that if $\delta>0$ and $\delta\leqslant|\theta-\theta_0|\leqslant\pi$ then

$$\log|f(re^{i\theta})|=O\left(\log\frac{1}{1-r}\right)^{\frac{1}{2}}.$$

We showed in [**11**] that O can be replaced by o here and that the result is then best possible, even for univalent functions. Here again the examples have had further applications.

Let C be the class of functions

$$f(z)=\sum_1^\infty a_n z^n$$

regular in $|z|<1$ and such that the image of $|z|<1$ has finite area

$$A=\pi\sum_1^\infty n|a_n|^2.$$

A sub-class of C is the class B of bounded univalent functions. Let

$$l(r,f) = \int_0^r |f'(t)|\,dt,$$

$$\lambda(r,f) = \sum_1^\infty |a_n| r^n,$$

$$M(r,f) = \max_{|z|=r} |f(z)|.$$

Evidently

$$M(r,f) \leqslant l(r,f) \leqslant \lambda(r,f).$$

A simple application of Schwarz's inequality was used by Rosenblatt in 1916 to show that for $f(z) \in C$

$$\lambda(r,f) = o\left(\log \frac{1}{1-r}\right)^{\frac{1}{2}}.$$

Kennedy and Twomey prove in [**19**] the stronger inequality

$$\int_0^1 \frac{\lambda(r,f)^2\,dr}{(1-r)[\log(1/1-r)]^2} \leqslant 4A/\pi.$$

for this class. They also prove that this result is best possible in the following sense. Given $\Phi(r)$, such that $\Phi(r)^2/\log[1/(1-r)]$ decreases to zero as $r \to 1$ and that

$$\int_0^1 \frac{\Phi(r)^2\,dr}{(1-r)[\log(1/1-r)]^2} < +\infty,$$

then there exists a function $f \in B$ and a univalent function $f_0 \in C$, such that for all sufficiently small $1-r$

$$\lambda(r,f) \geqslant l(r,f) > \Phi(r) \text{ and } M(r,f_0) > \Phi(r).$$

A weaker counter example had previously been given by Kennedy in [**3**]. The authors of [**19**] also prove analogous results for the sums $|a_1| + |a_2| + \ldots + |a_n|$.

In the remaining group of papers [**13, 14, 15, 17**] Kennedy considered the growth of $T(r,f')$ when $T(r,f) = O(1)$, as $r \to 1$, where $T(r,f)$ denotes the Nevanlinna characteristic of a function $f(z)$ meromorphic in $|z| < 1$. He proved [**17**] that in this case.

$$\int_0^1 \exp 2T(r,f')\,dr < +\infty,$$

and that this result cannot be greatly sharpened.

Suppose that $f(z)$ is regular and bounded in $|z| < 1$ and let $2\pi\mu_f$ be the measure of the set of singular points of $f(z)$ on $|z| = 1$. Then if

$$\lambda_f = \limsup_{r \to 1} T(r,f')\left(\log \frac{1}{1-r}\right)^{-1},$$

Kennedy proved [**14**] that $\lambda_f \leqslant \mu_f \leqslant 1$ and that for any fixed μ such that $0 < \mu \leqslant 1$ we can have $\lambda_f = \mu_f = \mu$. Further in [**15**] by a similar method he constructed an example in which

$$N\left(r, \frac{1}{f'}\right) \sim \left(\log \frac{1}{1-r}\right)^{-1}, \text{ as } r \to 1.$$

This solved a problem raised at the Cornell function theory conference in 1961. In [**13**] he constructed some examples in the case $\mu_f = 0$, so that $f(z)$ is regular almost everywhere on $|z| = 1$, and nevertheless

$$T(r, f') \to \infty, \text{ as } r \to 1. \tag{5}$$

In this connection it should be noted that Kennedy was the first to construct examples of functions $f(z)$ regular and bounded in $|z| < 1$ and satisfying (5), which have regular points on $|z| = 1$. These results neatly complement an earlier theorem of mine, though I was fortunate enough to stop Kennedy from calling his paper ' Counterexample to a theorem of Hayman.'

2. *Fourier series* [6, 7, 9, 10, 12, 16]

The second main direction of Kennedy's work lay in the theory of series, particularly Fourier series with gaps. Suppose that $f(x) \in L(0, 2\pi)$ and has a Fourier series of the form

$$\sum_{k=1}^{\infty} (a_{n_k} \cos n_k x + b_{n_k} \sin n_k x), \tag{6}$$

where n_k is a suitably rapidly growing sequence of positive integers. Then it is natural to ask the following question. When do certain continuity or other conditions for $f(x)$ on a set or interval E have the same effect as if these conditions were satisfied on the whole interval $[0, 2\pi]$?

The weakest gap hypothesis considered by Kennedy is

$$n_{k+1} - n_k \to \infty. \tag{7}$$

With this hypothesis and if E is an interval he proves [**6**] the following:

(i) If $f(x)$ has bounded variation on E, then $a_n, b_n = O(n^{-1})$.

(ii) If $f \in \text{Lip}\ \alpha$ on E, then $a_n, b_n = O(n^{-\alpha})$.

Further if $\alpha > \frac{1}{2}$, and if $f(x)$ also satisfies (i) then

$$\Sigma(|a_n| + |b_n|) < +\infty. \tag{8}$$

In [**7**] Kennedy shows that in these conclusions (7) cannot be replaced by even a very weak average condition. Given η, such that $0 < \eta < \pi$ and α, such that $0 < \alpha < 1$ he constructs functions for which

$$n_k \geqslant e^{k(\pi - \eta)/22}, \quad k = 1, 2, \ldots$$

and which satisfy (ii) with E the interval $[-\eta, \eta]$ and for which (8) is false and $|a_n|$, $|b_n|$, tend to zero arbitrarily slowly.

In [**10**] and [**16**] he obtains results of a similar nature by weakening the conditions on E and strengthening (7). For any set E we say that $f \in \mathrm{Lip}\, \alpha(E)$ if

$$f(x+h)-f(x)=O(h^{\alpha}) \tag{9}$$

uniformly for x on E as $h \to 0$ through unrestricted real values.

Some of these results have since been superseded by a theorem of Kahane and Izumi (*J. d'Analyse Math.* 14 (1965), 235–46) who prove that if $f \in \mathrm{Lip}\, \alpha(E)$, where E is a single point and

$$\varliminf_{k\to\infty} \frac{n_{k+1}}{n_k} > 1, \tag{10}$$

then

$$a_n,\, b_n = O(n^{-\alpha}).$$

However, Kennedy also proves in [**10**] that if $f \in \mathrm{Lip}\, \alpha(E)$, where the closure of E has positive measure, and if

$$\frac{n_{k+1}-n_k}{n_k^{\beta} \log n_k} \to \infty,$$

where $0<\beta<1$, then $a_n, b_n = O(n^{-\alpha\beta})$, as $n \to \infty$. Here $\alpha\beta$ cannot be replaced by any larger constant. If, further,

$$\alpha > \tfrac{1}{2}(\beta^{-1}-1), \text{ then } \Sigma(|a_n|+|b_n|)<\infty.$$

Much of the above work of Kennedy uses ideas in an earlier paper of Noble and an inequality for L^2-means due to Paley and Wiener.

The above definition of $\mathrm{Lip}\, \alpha(E)$ raises the question of whether a function can belong to $\mathrm{Lip}\, \alpha(E)$ for some sets E without belonging to $\mathrm{Lip}\, \alpha$ on any interval. This question is affirmatively settled in [**10**]. In [**12**] Kennedy proves more strongly that if $\omega(t)$ is any continuous positive function in (0, 1) such that $\omega(t) \to 0$ as $t \to 0$ and E is any set in [0, 1] then there exists f defined on [0, 1] such that $f(x)$ is discontinuous at every interior point of the complement of E, while

$$|f(x+h)-f(x)| \leqslant \omega(h), \quad x \in E \text{ and } x+h \in [0, 1].$$

Zygmund had shown in 1930 that if (10) holds and $a_{n_k}, b_{n_k} \to 0$ as $k \to \infty$, then the trigonometric series (6) converges on a set which is everywhere dense and has everywhere the power of the continuum. In [**9**] Kennedy shows that (10) cannot be relaxed in this. In fact if $\phi(t) \to 0$, as $t \to \infty$ then there exists a sequence n_k, and a_{n_k}, b_{n_k}, which tend to zero as $n \to \infty$, such that

$$\frac{n_{k+1}}{n_k} > 1 + \phi(n_k)$$

and the series (6) diverges everywhere.

3. *Tauberian theorems* [2, 5, 8, 18, 20]

Kennedy's remaining papers deal with results of Tauberian type. Heywood had proved that if $c_n \geqslant 0$ for all sufficiently large n and if

$$\sum_0^\infty c_n = 0, \tag{11}$$

and

$$f(x) = \sum_0^\infty c_n x^n,$$

then

$$(1-x)^{-\gamma} f(x) \in L(0, 1)$$

if and only if

$$\Sigma c_n \log n \text{ converges} \quad (\gamma = 1) \tag{12}$$

or

$$\Sigma c_n n^{\gamma-1} \text{ converges} \quad (\gamma < 1). \tag{13}$$

In [2] Kennedy considers arbitrary c_n, sets

$$S_n = \sum_{r=0}^n c_r,$$

and shows that any two of the conditions (12),

$$S_n \log n \to \text{a finite limit } \alpha, \tag{14}$$

and

$$\int_0^{1-\epsilon} (1-x)^{-1} \sum_0^\infty c_n x^n \to \text{a finite limit, as } \epsilon \to 0,$$

implies the third. Since (11) and (12) trivially imply (14) with $\alpha = 0$, Heywood's result follows in the case $\gamma = 1$.

In [8] the author keeps Heywood's assumption that $c_n \geqslant 0$ finally but replaces $(1-x)^{-\gamma}$ by a more general function $\phi(x)$ and obtains analogous results.

Let x_n be a uniformly distributed sequence in the interval [0, 1]. Then it follows from Hardy's Tauberian theorem, that since x_n is $[C, 1]$ summable but not convergent

$$\overline{\lim_{n\to\infty}}\, n\,|n_{n+1} - x_n| = +\infty.$$

In the opposite direction Kennedy [5] showed that given any function $\phi(n)$ which tends to infinity with n there exists a uniformly distributed sequence x_n, such that

$$n(x_{n+1} - x_n) = O\{\phi(n)\}$$

as $n \to \infty$.

In the same general direction Kennedy and Szüsz prove that in Littlewood's Tauberian theorem the condition $a_n = O(n^{-1})$ cannot be sharpened even for increasing functions $f(x)$. Given $\phi(n)$ as above they construct a power series

$$f(x) = \sum_0^\infty a_n x^n$$

which is increasing and bounded in [0, 1] and diverges at $x = 1$ while $|a_n| < \phi(n)/n$ for all values of n.

Let K be a positive integer and set $n_1 = 1$, $n_{k+1} = Kn_k + 1$, $k \geqslant 1$. Following Szüsz, Kennedy shows very simply in [**20**] that if a_k is a decreasing sequence of positive numbers such that

$$\sum_{k=1}^\infty |a_k \sin(\pi n_k x)| < \infty$$

for some x which is not an integer, then $\Sigma a_k < \infty$.

At the time of his death Kennedy was engaged on writing a short book on real analysis and a longer book on the applications of subharmonic functions to function theory.

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Papers by P. B. Kennedy

(*M.R.* refers to references in mathematical reviews.)

1. " On a conjecture of Heins ", *Proc. London Math. Soc.* (3), 5 (1955), 22–47. *M.R.*, 16, 809.
2. " Integrability theorems for power series ", *Quart. J. Math.* (Oxford) (2), 6 (1955), 316–320. *M.R.*, 19, 265.
3. " Conformal mapping of bounded domains ", *Journal London Math. Soc.*, 31 (1956), 332–336. *M.R.*, 17, 1191.
4. " A class of integral functions bounded on certain curves ", *Proc. London Math. Soc.* (3), 6 (1956), 518–547. *M.R.*, 18, 647.
5. " A note on uniformly distributed sequences ", *Quart. J. Math.* (Oxford) (2), 7 (1956), 125–127. *M.R.*, 20, 3404.
6. " Fourier series with gaps ", *Quart. J. Math.* (Oxford) (2), 7 (1956), 224–230. *M.R.*, 20, 4733.
7. " Fourier series with gaps. II ", *Quart. J. Math.* (Oxford) (2), 8 (1957), 84–88. *M.R.*, 20, 4734.
8. " General integrability theorems for power series ", *Journal London Math Soc.*, 32 (1957), 58–62. *M.R.*, 18, 896.
9. " Remark on a theorem of Zygmund ", *Journal London Math. Soc.*, 33 (1958), 71–72. *M.R.*, 20, 1882.
10. " On the coefficients in certain Fourier series ", *Journal London Math. Soc.*, 33 (1958), 196–207. *M.R.*, 20, 4735.
11. With W. K. Hayman, " On the growth of multivalent functions ", *Journal London Math. Soc.*, 33 (1958), 333–341. *M.R.*, 20, 5875.
12. " A remark on continuity conditions ", *Proc. American Math. Soc.*, 10 (1959), 203–204. *M.R.*, 21, 2713.

13. " On a theorem of Hayman concerning quasibounded functions ", *Canadian J. of Math.*, 11 (1959), 593–600. *M.R.*, 21, 5742.
14. " A property of bounded regular functions ", *Proc. Roy. Irish Acad.*, Sect. A 60 (1959), 7–14. *M.R.*, 21, 5741.
15. " A problem on bounded analytic functions ", *Proc. American Math. Soc.*, 15 (1964), 325–326. *M.R.*, 28, 5180.
16. " Note on Fourier series with Hadamard gaps ", *Journal London Math. Soc.*, 39 (1964), 115–116.
17. " On the derivative of a function of bounded characteristic ", *Quart. J. of Math.* (Oxford) (2), 15 (1964), 337–341. *M.R.*, 30, 3221.
18. With P. Szüsz, " On a bounded increasing power series ", *Proc. American Math. Soc.*, 17 (1966), 580–581.
19. With J. B. Twomey, " Some properties of bounded univalent functions and related classes of functions ", *Proc. Roy. Irish Acad.*, Sect. A 65 (1967), 43–49.
20. " On a theorem of Szüsz ", *Acta Math. Acad. Sci. Hungar.*, 17 (1966), 423–424.

Imperial College,
London, S.W.7.