EDMUND LANDAU

G. H. HARDY *and* H. HEILBRONN*.

Edmund Landau, an honorary member of this Society since 1924, was born in Berlin on 14 February 1877, and died there on 19 February 1938. He was the son of Professor Leopold Landau, a well-known gynaecologist. After passing through the "French Gymnasium" in Berlin, he entered the University of Berlin as a student of mathematics, and remained there, apart from two short intervals in Munich and Paris, until 1909. His favourite teacher was Frobenius, who lectured on algebra and the theory of numbers. Landau worked through these lectures very thoroughly, and used his notes of them throughout his life. He took his doctor's degree in 1899†, and obtained the "venia legendi", or right to give lectures, in 1901.

In 1909 Landau succeeded Minkowski as ordinary professor in Göttingen. The University of Göttingen was then in its mathematical prime; Klein and Hilbert were Landau's colleagues, and young mathematicians came to Göttingen for inspiration from every country. After the war the University recovered its position quickly, so that Landau had always ample opportunities of training able pupils, and often had a decisive influence on their careers.

In 1933 the political situation forced him to resign his chair. He retired to Berlin, but still lectured occasionally outside Germany. In 1935 he came to Cambridge as Rouse Ball Lecturer, and gave the lectures which he developed later into a Cambridge Tract‡. He continued to take an active interest in mathematics, and his last lectures were in Brussels in November 1937, only a few months before his sudden death.

Landau's first and most abiding interest was the analytic theory of numbers, and in particular the theory of the distribution of primes and prime ideals. In his "doctor-dissertation" (**1**) he gave a new proof of the identity

$$\sum_{1}^{\infty} \frac{\mu(n)}{n} = 0 \tag{1}$$

(conjectured by Euler and first proved by von Mangoldt), and this was

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* We have to thank Dr. Alfred Brauer for information concerning the facts of Landau's career.

† His thesis was **1** in the list of papers on p. 310.

‡ G in the list of books on p. 307.

the first of a long series of papers on the zeta-function and the theory of primes.

The "prime number theorem"

$$\pi(x) \sim \frac{x}{\log x} \tag{2}$$

was first proved by Hadamard and de la Vallée-Poussin in 1896. De la Vallée-Poussin went further, and proved that

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O\{xe^{-A\sqrt{(\log x)}}\} = \operatorname{li} x + O\{xe^{-A\sqrt{(\log x)}}\}, \tag{3}$$

for a certain positive A. The proofs of both Hadamard and de la Vallée-Poussin depended upon Hadamard's theory of integral functions, and in particular on the fact that $\zeta(s)$, apart from a simple pole at $s = 1$, is regular all over the complex plane.

In 1903 Landau found (in **2**) a new proof of the prime number theorem which does not depend upon the general theory of Hadamard. For this proof we need know only that $\zeta(s)$ can be continued "a little way over" the line $\sigma = 1$; we do not need the functional equation, the Weierstrass product, and the other machinery used in the earlier proofs. On the other hand we do not obtain quite so precise a formula as (3).

This discovery of Landau's was very important, since it permitted a decisive step in the theory of the prime ideals of an algebraic field κ. This theory depends upon the properties of the Dedekind zeta-function

$$\zeta_\kappa(s) = \Sigma \frac{1}{(N\mathfrak{a})^s}, \tag{4}$$

where $\mathfrak{a}$ runs through the integer ideals of κ (except 0), and $N\mathfrak{a}$ is the norm of $\mathfrak{a}$. It was not proved until much later (by Hecke in 1917) that $\zeta_\kappa(s)$ can be continued all over the plane, but Landau had no difficulty in showing that it has properties like those of $\zeta(s)$ used in his proof of the prime number theorem. He thus obtained the "prime ideal theorem": if $\pi_\kappa(x)$ is the number of prime ideals of κ whose norm is less than x, then

$$\pi_\kappa(x) \sim \frac{x}{\log x}. \tag{5}$$

Later*, using Hecke's discoveries, he proved the formula for $\pi_\kappa(x)$ corresponding to (3).

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* C, § 20.

The logic of prime number theory has developed a good deal since 1903 and even since 1917, and Landau kept fully in touch with all these developments. Thus his paper **12** contains the shortest and most direct proof of the prime number theorem known today (a proof based on the ideas of Wiener). He was also intensely interested in the logical relations between different propositions in the theory. Thus he first proved (in **6** and **10**) that (1) and (2) are "equivalent", that each can be deduced from the other by "elementary" reasoning, although there is no "elementary" proof of either*. It was Landau who first enabled experts to classify the theorems of prime number theory according to their "depths".

Landau's second big discovery was in an entirely different direction. Picard's theorem states that an integral function, not a constant, assumes all values with at most one exception. Picard deduced his theorem in 1879 from the properties of the "modular function", and it was not until 1896 that Borel found the first elementary proof.

In 1904 Landau, studying Borel's proof, made a most important, and then very unexpected, extension (**3**). If a_0 and a_1 are given, $a_1 \neq 0$, and

$$f(x) = a_0 + a_1 x + \dots$$

is regular at the origin, then there is a number $\Omega = \Omega(a_0, a_1) > 0$, depending on a_0 and a_1 only, such that $f(x)$, if regular in the circle $|x| < \Omega$, must assume one of the values 0 and 1 somewhere in the circle. It is obvious that this theorem includes Picard's theorem.

A few weeks later Schottky developed Landau's theorem further. Suppose that $a_0 \neq 0$, $a_0 \neq 1$, and $0 < \vartheta < 1$. Then there is a number $\Phi = \Phi(a_0, \vartheta)$ with the following property: if $f(x)$ is regular, and never 0 or 1, for $|x| < 1$, and $f(0) = a_0$, then

$$|f(x)| < \Phi(a_0, \vartheta)$$

for $|x| < \vartheta$. Landau's theorem is a simple corollary†.

Schottky's theorem was imperfect in one important respect, since his function Φ was unbounded near $a_0 = 0$ and $a_0 = 1$. Landau, in **13**,

* The proof that (2) implies (1) is given in A, § 156, but that of the converse implication is later.

† See B (ed. ii), p. 103.

removed this imperfection. Suppose that $a > 0$, $0 < \vartheta < 1$. Then there is a number $\Psi = \Psi(a, \vartheta)$ with the property: if $f(x)$ is regular, and never 0 or 1, for $|x| < 1$, and $|f(0)| \leqslant a$, then

$$|f(x)| < \Psi(a, \vartheta)$$

for $|x| < \vartheta$. In **13** he makes an important application to the theory of $\zeta(s)$ and $\zeta_\kappa(s)$.

These theorems have inspired a great amount of later work. Carathéodory, for example, found the "best" Ω in terms of the modular function. The elementary proofs have also been transformed by the discovery of "Bloch's theorem", and are developed in this way in the second edition of Landau's *Ergebnisse* (B).

This theorem and the prime ideal theorem were probably the most striking of Landau's original discoveries. We state a few more with the minimum of comment.

(1) Every large positive integer is a sum of at most 8 positive integral cubes (**9**).

This 8 is the only number which has resisted the later analytic attacks on Waring's problem.

(2) Every positive definite polynomial with rational coefficients can be represented as a sum of 8 squares of polynomials with rational coefficients. In particular, every positive definite quadratic with rational coefficients is a sum of 5 squares of rational linear functions; and 5 is the best possible number (**5**).

Mordell has since proved the corresponding theorem for quadratics with *integral* coefficients.

(3) If $f(x) \sim x$ when $x \to \infty$, and $xf'(x)$ increases with x, then $f'(x) \to 1$ (**8**, 218).

This theorem (which contains the kernel of a differencing process used by de la Vallée-Poussin and others in the analytic theory of numbers) is, perhaps, the first genuine example of a "O-Tauberian" theorem.

(4) If $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ is regular, and $|f(x)| < 1$, for $|x| < 1$, then

$$|a_0 + a_1 + \ldots + a_n| \leqslant 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 + \ldots + \left(\frac{1 \cdot 3 \ldots 2n-1}{2 \cdot 4 \ldots 2n}\right)^2.$$

There is equality, for every n, with an appropriate $f(x)$ depending on n (**11**).

(5) A Dirichlet's series $\Sigma a_n e^{-\lambda_n s}$, with non-negative coefficients, has a singularity at the real point of its line of convergence (**4**).

This had been proved before for power-series by Vivanti and Pringsheim, but their method of proof cannot be extended to the general case.

(6) If $N(\sigma_0, T)$ is the number of zeros of $\zeta(s)$ in the domain $\sigma \geqslant \sigma_0 > \frac{1}{2}$, $|t| \leqslant T$, then

$$N(\sigma_0, T) = o(T).$$

This was proved, first with O and then as stated, by Bohr and Landau in their joint papers **14** and **15**. It is known that $N(\frac{1}{2}, T)$ is of order $T \log T$, so that most of the zeros of $\zeta(s)$ lie very near $\sigma = \frac{1}{2}$. This was the first successful attempt to show that the Riemann hypothesis is at any rate "approximately" true. Carlson proved later that $o(T)$ may be replaced by $O(T^{a})$, where $0 < a < 1$, and Titchmarsh and Ingham have since improved Carlson's value of a.

Landau also published a very large number of new, shorter, and simpler proofs of known theorems. We mention only his well-known proof of Weierstrass's approximation Theorem (**7**). This depends upon the singular integral

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \{1-(u-x)^2\}^n f(u)\, du,$$

and was perhaps the first proof in which the approximating functions are "visibly" polynomials. It is reproduced, in a more general form, in Hobson's book (vol. ii, ed. 2, 459–461).

Landau wrote over 250 papers, but it is possible that he will be remembered first for his books, of which he wrote seven.

A. *Handbuch der Lehre von der Verteilung der Primzahlen* (Leipzig and Berlin, Teubner, 1909: 2 vols., 961 pp.).

B. *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie* (Berlin, Springer, 1916; second edition, 1929: 122 pp.).

C. *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und Ideale* (Leipzig and Berlin, Teubner, 1918; second edition, 1927: 147 pp.).

D. *Vorlesungen über Zahlentheorie* (Leipzig, Hirzel, 1927: 3 vols., 1009 pp.).

E. *Grundlagen der Analysis* (Leipzig, Akademische Verlagsgesellschaft, 1930: 134 pp.).

F. *Einführung in die Differentialrechnung und Integralrechnung* (Groningen, Noordhoff, 1934: 368 pp.).

G. *Über einige neuere Fortschritte der additiven Zahlentheorie* (Cambridge Tracts in Mathematics, No. 35, 1937: 94 pp.).

Of these books, E and F are elementary, and we say nothing about them, interesting and individual as they are. All the rest are works of first-rate importance and high distinction.

Landau was the complete master of a most individual style, which it is easy to caricature (as some of his pupils sometimes did in an amusing way*), but whose merits are rare indeed. It has two variations, the "old Landau style", best illustrated by the *Handbuch*, which sweeps on majestically without regard to space, and the "new Landau style" of his post-war days, in which there is an incessant striving for compression. Each of these styles is a model of its kind. There are no mistakes—for Landau took endless trouble, and was one of the most accurate thinkers of his day—no ambiguities, and no omissions; the reader has no skeletons to fill, but is given every detail of every proof. He may, indeed, sometimes wish that a little more had been left to his imagination, since half the truth is often easier to picture vividly than the whole of it, and the very completeness of Landau's presentation sometimes makes it difficult to grasp the "main idea". But Landau would not, or could not, think or write vaguely, and a reader has to read as precisely and conscientiously as Landau wrote. If he will do so, and if he will then compare Landau's discussion of a theorem with those of other writers, he will be astonished to find how often Landau has given him the shortest, the simplest, and in the long run the most illuminating proof.

The *Handbuch* was probably the most *important* book he wrote. In it the analytic theory of numbers is presented for the first time, not as a collection of a few beautiful scattered theorems, but as a systematic science. The book transformed the subject, hitherto the hunting ground of a few

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* For example in a mock *Festschrift* written on the occasion of his declining an invitation to leave Göttingen for another university.

adventurous heroes, into one of the most fruitful fields of research of the last thirty years. Almost everything in it has been superseded, and that is the greatest tribute to the book.

Landau would not publish a second edition of the *Handbuch* (which must necessarily have been a new book), but preferred to incorporate the results of later researches in his *Vorlesungen*, which is no doubt his *greatest* book. This remarkable work is complete in itself; he does not assume (as he had done in the *Handbuch*) even a little knowledge of number-theory or algebra. It stretches from the very beginning to the limits of knowledge, in 1927, of the "additive", "analytic", and "geometric" theories. Thus part 6 (vol. i, pp. 235–269) carries the solution of Waring's problem to where it stood before Vinogradov's recent work. Part 12 (vol. iii, pp. 201–328) contains practically everything then known about "Fermat's last theorem", and the rest of the book is conceived on the same scale. In spite of this enormous programme, Landau never deviates an inch from his ideal of absolute completeness. For example, he never refers to his *Algebraische Zahlen*, but proves from the beginning everything he needs.

The richness of content of the book, and the power of condensation it shows, are astonishing. Thus the classical theorems about decompositions into two, three, and four squares are proved in twenty-eight pages (vol. i, pp. 97–125). And Landau can find room (vol. i, pp. 153–171) for four different evaluations of Gauss's sums.

The *Vorlesungen* is not only Landau's finest book but also, in spite of the great difficulty and complexity of some of the subject matter, the most agreeably written. The style here is the rather informal style of his lectures, which he was persuaded by his friends to leave unchanged.

The *Algebraische Zahlen* gives a short and self-contained account (pp. 1–54) of the theory of algebraic numbers and ideals, intended as an introduction to the proofs of the prime ideal theorem and its refinements which occupy the remainder of the book. He does not go so deeply into the algebraic theory as, for example, Hecke, being content with what is required for his applications.

The *Ergebnisse* is probably Landau's most *beautiful* book. It contains a collection of elegant, significant, and entertaining theorems of modern function theory: Hadamard's and Fabry's "gap theorems", Fatou's theorem, the most striking "Tauberian" theorems, Bloch's theorem, the Picard-Landau group of theorems, and the fundamental theorems concerning "schlicht" functions. It is one of the most attractive little volumes in recent mathematical literature, and the most effective answer to any one who suggests that Landau's mathematics was dull.

Finally, his last work, the Cambridge Tract originating from his Rouse Ball lecture, gives an account of Vinogradov's "Waring" and Schnirelmann's "Goldbach" theorems, and of a group of half solved "elementary" problems of additive number theory which open a new field of research for young and unprejudiced mathematicians. There is a review of this tract by Mr. Ingham in the current volume of the *Mathematical Gazette*, to which we should have little to add.

Landau was certainly one of the hardest workers of our times. His working day often began at 7 a.m. and continued, with short intervals, until midnight. He loved lecturing, more perhaps even than he realized himself; and a lecture from Landau was a very serious thing, since he expected his students to work in the spirit in which he worked himself, and would never tolerate the tiniest rough end or the slightest compromise with the truth. His enforced retirement must have been a terrible blow to him; it was quite pathetic to see his delight when he found himself again in front of a blackboard in Cambridge, and his sorrow when his opportunity came to an end.

No one was ever more passionately devoted to mathematics than Landau, and there was something rather surprisingly impersonal, in a man of such strong personality, in his devotion. Everybody prefers to do things himself, and Landau was no exception; but most of us are at bottom a little jealous of progress by others, while Landau seemed singularly free from such unworthy emotions. He would insist on his own rights, even a little pedantically, but he would insist in the same spirit and with the same rigour on the rights of others.

This was all part of his passion for order in the world of mathematics. He could not stand untidiness in his chosen territory, blunders, obscurity, or vagueness, unproved assertions or half substantiated claims. If X had proved something, it was up to X to print his proof, and until that happened the something was nothing to Landau. And the man who did his job incompetently, who spoilt Landau's world, received no mercy; that was the unpardonable sin in Landau's eyes, to make a mathematical mess where there had been order before.

Landau received many honours in his lifetime. He was a member of the Academies of Berlin, Göttingen, Halle, Leningrad, and Rome; but no honour seemed to please him quite so much as his election to honorary membership of this society, and he came specially from Germany to attend our sixtieth anniversary dinner. This was natural, since there was no country where his reputation stood quite so high as in England, and none where his work has borne more fruit.

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[This list contains only papers referred to in the notice.]

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3. "Über eine Verallgemeinerung des Picardschen Satzes", *Berliner Sitzungsberichte* (1904), 1118–1133.
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5. "Über die Darstellung definiter Funktionen durch Quadrate", *Math. Annalen*, 62 (1906), 272–285.
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LOUIS NAPOLEON GEORGE FILON

G. B. JEFFERY.

Louis Napoleon George Filon was born in France on 22 November, 1875. He was the son of Augustin Filon, the French littérateur who was tutor to the Prince Imperial. His parents came to England when he was about three years old and lived at Margate. At this time his father was blind and his mother in delicate health, so that the boy was brought up in a serious atmosphere; an only child of invalid parents.

His father undertook his early education which centred mainly round the Classics. He began Latin and Greek before he was six. His own memories of this time were of regular Latin essays on Roman History