HENRI LEBESGUE

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Henri Léon Lebesgue was born in 1875 and died in 1941. He studied at the Ecole Normale Supérieure, and his first post appears to have been that of maître des conférences at Rennes, which he held until 1906. He then moved to Poitiers, where he was described first as chargé de cours à la faculté des Sciences and later as professor. In or about 1912 he was called to Paris as maître des conférences, and he afterwards became professor at the Collège de France. He was elected to the Académie des Sciences in 1922. He was made an honorary member of the London Mathematical Society in 1924, and a foreign member of the Royal Society in 1930.

Towards the close of the last century, the development of mathematical analysis may be said broadly to have reached the stage at which a piece of work wherein only continuous functions were encountered could be carried through. For example, the Riemann integral solved the problem of finding a function having a given derivative if the derivative was continuous. Again, everything was known about the length of a curve and its expression as an integral if it had a continuously turning tangent. Artificial and unaesthetic restrictions had repeatedly to be made to keep discontinuous functions out. Jordan, in the preface to his *Cours d'Analyse* (2nd edition, 1893), wrote: "Certains points présentent encore quelque obscurité. Pour en citer un exemple, nous n'avons pu arriver à définir d'une manière satisfaisante l'aire d'une surface gauche, que dans le cas où la surface a un plan tangent, variant suivant une loi continue".

In 1898 vital steps were taken by Baire and by Borel. The *Thèse* of Baire was a systematic and penetrating discussion of discontinuous functions. And Borel in the small compass of four pages of his tract, *Leçons sur la théorie des fonctions,* propounded his theory of measure, fundamentally more powerful than Jordan's theory of content in being additive for an enumerable infinity of sets. All was ready for the rapid advances of the next decade which made the theory of real functions into a satisfying whole; in this transformation the leading part was played by Lebesgue.

The earliest paper (3) that I can find of Lebesgue's appeared in 1898 and dealt with Weierstrass's theorem on approximation to continuous functions by polynomials. He gave what is probably now the best-known proof, by expanding $\surd\{1+(x^2-1)\}$.

Between March, 1899, and April, 1901, Lebesgue published six notes in *Comptes Rendus* (**4–9**). Of these the earliest (unconnected with the rest) dealt with the extension to functions of two variables of Baire's theorem that a function is of class one if it is pointwise discontinuous on every perfect set; the Peano curve is employed to reduce the two variables to one. The other five notes deal with matter incorporated in the *Thèse* published in 1902, and they are of great interest as showing the succession of ideas that led to the formulation of the Lebesgue integral.

The first of the five notes (**5**) puts the problem of what surfaces are "applicable to a plane"—in the sense of a one-one correspondence such that any rectifiable curve of the surface is transformed into a rectifiable curve in the plane of the same length. Ossian Bonnet had proved that, on the assumption of a continuously turning tangent plane, the only such surfaces were developables. Here was a problem of which the general solution plainly required a release from the shackles of continuity. Lebesgue remarks ". . . combien la forme des surfaces physiquement applicables sur le plan, comme celles que l'on obtient en froissant une feuille de papier, diffère de la forme des surfaces développables". This note, breaking away from the conventions of classical differential geometry, is said to have "scandalisé Darboux".

In the next note (**6**) the area of a skew polygon C is defined as the lower bound of the areas of polyhedral surfaces having C as boundary; then the area of a skew curve by means of inscribed polygons approaching it; finally the area of a surface by dividing it up by a network of curves.

The next two notes (**7**, **8**) deal with surface integrals (not Lebesgue integrals!) and the problem of the surface of minimum area with a given bounding contour.

The final note (**9**) gives the definition of the Lebesgue integral of $f(x)$ by subdividing the range of variation of $f(x)$ instead of the range of x. Some properties are stated, including that of integrating every bounded derivative and—a result belonging to the train of thought of the other notes—the formula for the length of a curve $x=f(t)$, $y=\phi(t)$, $z=\psi(t)$, assuming only that f', ϕ' and ψ' are bounded.

Lebesgue's great *Thèse*, with the full account of this work, appeared in the *Annali di Matematica* of 1902 (**10**). The first chapter develops the theory of measure—in a more general form than Borel had given it—by assigning outer and inner measures to a set. In the second chapter we have the integral, defined first geometrically, for positive $f(x)$ as the two-dimensional measure of the set of points (x, y) where $0 \leqslant y \leqslant f(x)$ and x is in the range of integration; then follows the analytical definition already mentioned. All the leading properties of the integral are there, with the exception of the theorem that $F' = f$ almost everywhere, which followed in **12**. Double integrals are included and effectively the theorem now known as Fubini's. The next chapters expand the *Comptes Rendus* notes dealing with length, area and applicable surfaces, and there is a final chapter on Plateau's problem, that of finding a surface of minimum area with a given contour. It cannot be doubted that this dissertation is one of the finest which any mathematician has ever written.

Lebesgue lectured on integration at the Collège de France in the year 1902–3, and his lectures were collected into a Borel tract (**1**)—the first to be written by anyone other than Borel himself. The problem of integration regarded as the search for a primitive function is the key-note of the book. The historical setting of the problem is fully displayed; there is a chapter on the ideas of Cauchy and Dirichlet, and two on the Riemann integral; then one on functions of bounded variation and two on primitive functions. Only in the last chapter of the book does Lebesgue lay down six conditions which it is desirable that the integral should satisfy. The first five are elementary, including for example, the additive property. The sixth condition is

If the sequence $f_n(x)$ increases to the limit $f(x)$, the integral of $f_n(x)$ tends to the integral of $f(x)$.

Lebesgue shows that these conditions lead to the theory of measure and measurable functions and the analytical and geometrical definitions of the integral. This last chapter, though the most exciting, is the least satisfying in the book; some of the work is given only in outline, and the outline is in parts misleading. Lebesgue slipped into inaccuracy more often than one would have expected of so fine a mathematician and so lucid a writer; his errors are always on the side of supposing some argument to be simpler than it really is.

Of the branches of analysis in which the use of an integral more powerful than Riemann's offered a rich reward, none was so promising as the theory of trigonometrical series, and to this Lebesgue turned. His first substantial paper (**11**) contained three striking theorems: (1) that a trigonometrical series representing a bounded function is a Fourier series (and an exceptional reducible set is allowed), (2) that the n-th Fourier coefficient tends to zero (the "Riemann-Lebesgue theorem"), and (3) that a Fourier series is integrable term by term.

Lebesgue gave a closer analysis of convergence problems of Fourier series in another paper (**18**). He proved that, if

$$\phi(t) = f(x+t)+f(x-t)-2f(x),$$

the Fourier series of $f(x)$ converges at x if (i) $\int_0^t |\phi(u)|\,du$ has derivative 0 for $t=0$, and (ii) $\int_\delta^a \frac{|\phi(t+\delta)-\phi(t)|}{t}\,dt$ tends to 0 with δ. This test, as developed later by Pollard and Gergen (**56**), includes all the ordinary criteria for convergence. In the same paper Lebesgue also established summability $(C1)$ wherever $|\phi(t)|$ is the derivative of its indefinite integral for $t=0$, and he showed that this holds except in a set of measure zero.

In 1904–5 Lebesgue lectured at the Collège de France on trigonometrical series and he published his lectures in another of the Borel tracts (**2**). This gives an admirable survey of the subject. There are introductory sections on the relevant parts of analysis (such as measure, integral, bounded variation) and on the historical development of the theory of trigonometrical series. There follow chapters containing the latest results about convergence and summability (mentioned above) and examples of divergent Fourier series. The last chapter expounds the Cantor-Riemann theory. Among other topics which find a place are applications to the Poisson integral and the Dirichlet problem, summation formulae (for example, Gauss's sums) and isoperimetric problems.

In (**24**) Lebesgue gave an exhaustive discussion of the question of the convergence, as n tends to infinity, of $\int_0^l f(t)\,\phi(t-x, n)\,dt$ to $f(x)$. For various classes of $f(x)$ (*e.g.* functions of integrable square or functions of bounded variation) he obtained necessary and sufficient conditions to be satisfied by $\phi(t, n)$. Hobson, working on similar lines with the rather more general integral $\int_0^l f(t)\,\phi(t, x, n)\,dt$, proved only the sufficiency of his conditions. Lebesgue gave a number of applications to well-known representations of functions by integrals, such as those of Fourier and Dirichlet, Poisson, Fejér, and Weierstrass.

A further important paper (**27**) deals with the Fourier series of functions satisfying a Lipschitz condition, with an evaluation of the order of magnitude of the remainder term. It is proved also that the Riemann-Lebesgue lemma is a best possible result for continuous functions, and some discussion is given of the "Lebesgue constants" defined by

$$\rho_n = \frac{1}{2\pi}\int_0^{2\pi} \left| \frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t} \right| dt.$$

It is known that $\rho_n \sim (4/\pi^2)\log n$; the fact that ρ_n tends to infinity with n lies at the root of the difficulties of the convergence-theory of Fourier series.

Any analyst of power comparable with Lebesgue's whom one calls to mind has attacked problems both of real and of complex function theory. It would not be true to say that Lebesgue never put $\int f(z)\,dz$ into print, but when he did it was in a minor key. The nearest approach to the ideas of complex function theory which is found in his significant work is in a group of papers on potential theory. The only paper of any length (**22**) is one in which Dirichlet's problem for the plane is carefully analysed by the methods of the calculus of variations, on the lines of Hilbert and Poincaré. Short notes published later contained important contributions. In (**30**) the idea of the "barrier" was introduced. In (**32**) an example was given of a three-dimensional region, bounded by a surface of revolution, for which the Dirichlet problem is impossible. In (**41**) conditions are given about the form of the boundary and the behaviour of super-harmonic functions which, in a sense, are final.

Lebesgue's handling of problems of potential theory is marked by his careful treatment of the topological questions involved. He made incursions into pure topology at a time when that subject was opening out, largely through the work of Brouwer. Brouwer published in 1911 a proof of the invariance of dimension-number, and immediately following his paper is one by Lebesgue (**28**) suggesting another proof. Lebesgue's argument rests on a result which was already in Jordan's *Cours d'Analyse*:

If each point of an n-dimensional domain D belongs to at least one of the closed sets $E_1, \ldots, E_p$, and, if these sets are small enough, then there are points common to at least $n+1$ of them.

This is closely related to the fact that a curve filling an n-dimensional domain has multiple points of order at least $n+1$ and not necessarily greater.

Lebesgue's argument for the invariance theorem was incomplete, as was pointed out by Brouwer (**53**). He gave another proof, akin to one of Brouwer's, in a note dealing also with the extension to higher dimensions of the Jordan curve theorem (**29**).

With the publication in 1904 of the *Leçons sur l'intégration* the Lebesgue integral became available as the standard integral for work in analysis that aimed at finality. It remains to trace the further developments of the theory and some closely related questions. Lebesgue wrote a number of notes (**19**) on points not adequately treated in the *Leçons*, some in answer to criticisms made by Beppo Levi (**54**), and a memoir (**26**) in which stress was laid on the aspect of the indefinite integral as a function of sets $F(E)$, and in which the differentiation of this function by means of regular families of sets was investigated. This paper laid the foundations of later work on functions of sets by Radon, de la Vallée Poussin, Carathéodory and others.

In a long paper (**17**) Lebesgue discussed measurable sets and functions of Baire's classes. He defined a "fonction représentable analytiquement" as one which may be constructed by an enumerable sequence of algebraic or limiting operations, and proved that the class of such functions is co-extensive with that of functions measurable (B). He gave examples of functions belonging to any assigned Baire class and showed that it was possible to *name* (nommer) a function not belonging to any Baire class.

The different standpoints taken by Borel and Lebesgue on questions of *constructive definitions* and *effective calculation* led to a controversy which the *Enzyklopädie* calls "teilweise heftig". Borel took the view that a number was of no interest to him if he did not know how to calculate by a finite number of operations a rational number within a given approximation to it. This led him to define the integral of a bounded function by means of the limits of the integrals of approximating polynomials (**51**). He asserted that any apparently wider scope which Lebesgue's methods might have was illusory. Borel gave also a treatment of the integral of an unbounded function by enclosing the singularities in small intervals—a development of Cauchy's ideas; such a treatment caters for functions like $(1/x)\sin(1/x)$ at $x = 0$, which do not possess absolutely convergent integrals. Borel contended that his simple and direct methods were as good as Lebesgue's for bounded functions and better for unbounded functions.

Lebesgue's reply (**34**) is a good example of his clear and cogent style. After an introductory chapter, he dealt first with Borel's process for unbounded functions and argued that it was not sufficiently definite and could be made to assign any arbitrary value to an integral. In the following chapter he argued that Borel's methods for bounded functions were less natural and less illuminating than his own. After some remarks on logical questions, Lebesgue devoted half a dozen pages to questions of priority. Borel had written of measure and integration as being indissolubly linked. Lebesgue paid generous tribute to Borel for taking the essential step of defining measure, but put on record that the first definition of the integral which it made possible was his own.

Lebesgue's work on the structure of functions and sets has now been carried further, notably by Lusin (**55**) and by the Polish school.

In the last twenty years of his life Lebesgue wrote a number of papers, but his great work had been done. His interests, as we can infer them from his writings, widened and came to include, for instance, geometry of a rather elementary kind (**42**, **48**, **50**). In another paper (**37**), which appeals both to the geometer and to the analyst, Lebesgue started from a problem considered by Bricard:—

Find the smallest value of R such that a circle of radius R will cover any plane set of points of diameter D (*i.e.* D is the upper bound of the distance between any two points of the set).

He discussed this and analogous questions concerning oval curves.

Several expository and other articles on integration, area and the like came from his pen. One of the most interesting is (**45**), containing a simple construction for the Denjoy integral (total) of an exact derivative. More important than any of his articles is the second edition of the *Leçons sur l'integration*, which appeared in 1928. The earlier historical part is little changed; the Lebesgue integral is treated rather more fully than in the first edition, and topics such as the decomposition of an absolutely additive function of sets into an absolutely continuous portion, a "fonction des sauts" and a singular function find a place. The main additions are two well-written long chapters, one on the Denjoy integral and one on the Stieltjes integral. The latter is developed from its simple beginnings in the hands of Stieltjes through the possible extensions, including finally a Denjoy-Stieltjes process.

It is not difficult to assess the importance of Lebesgue's work as an analyst. There are books and memoirs in which his is the name most frequently mentioned, and there must have been many lectures in which his was the name most frequently spoken. With the passage of time and the acceptance of his ideas as right and inevitable, the need of this explicit reference will decrease, but it is a true indication of his place among mathematicians. His work lay almost entirely in one field—the theory of real functions; in that field he is supreme.

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