ARCHIBALD JAMES MACINTYRE

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1. The death on August 4th, 1967, at the age of 59, of Archibald James Macintyre, Research Professor of Mathematics at the University of Cincinnati, Ohio, U.S.A., after a brief illness came as a severe shock to all who knew him, whether personally or by repute. During my thirteen-year association with him at King's College, Aberdeen, Scotland, I naturally came to know him well, and to admire his many fine attributes. His good nature and kindness, his subtle but delightful sense of humour, the patience with which he explained mathematical problems to those less quick than he to see the point, the facility with which he was able to imbue his students with the feeling that mathematics was a living and growing subject and not a defunct language of invariable symbols, his encyclopaedic knowledge of mathematical research papers especially in the various branches of mathematical analysis—all these were highly impressive and go some way towards explaining not only the high esteem and popularity accorded to him, by both staff and students wherever he went, but also his power of attracting research students to work under his inspiring supervision.

Another aspect of his character that impinged itself early on his mathematical friends, especially on his research collaborators, was his high standard—he simply would not countenance publication unless, and until, the proposed article was of first-class quality as regards both content and exposition.

However, what impressed me most of all about him as a mathematician was his humility. He told me once that he did not regard himself as an original thinker, but merely as one who could sometimes push further ahead with the ideas and methods of others. All who have worked in the field of analysis will surely agree that such an opinion of his work does him less than justice, and I have no doubt that many of us would be only too happy to produce work of his quality.

Born on July 3rd, 1908, in Sheffield, Macintyre's early education took place at the Central Secondary School, Sheffield, after which in October 1926 he entered Magdalene College, Cambridge, as a Scholar, took his Mathematical Tripos, being awarded a First Class in the Mathematical Tripos (Part I) in June 1927, a First Class in the Mathematics Preliminary Examination in June 1928 (on which he was awarded the Davidson prize for Mathematics), and in June 1929 he became a Wrangler with Distinction in Schedule B, in the Mathematical Tripos (Part II). His mathematical contemporaries at Cambridge included Professor H. Davenport (Trinity), Professor S. Verblunsky, Dr. J. Cossar and Dr. D. W. Babbage (all of Magdalene), and his Tutor and Director of Studies at Magdalene was A. S Ramsey, the author of well-known mathematical text-books and the father of the present Archbishop

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of Canterbury and of the brilliant mathematical logician Frank Ramsey, who died at an early age.

From 1929 to 1930, A. J. Macintyre worked as a Research Student on integral functions under Dr. (now Sir) Edward Collingwood at Cambridge, and for one year from October 1930 he taught Applied Mathematics and Theoretical Physics in Swansea University College as an Assistant Lecturer attached to the Mathematics Department, whose Head was then the late Professor A. R. Richardson. The staff also included R. Wilson, later to become Professor (now Emeritus Professor) of Mathematics at the same college, with whom A. J. Macintyre struck up a firm friendship which was to lead to a significant output of research, some joint, on a variety of problems arising in classical analysis.

In October 1931 Macintyre was appointed as an Assistant Lecturer, and promoted to Lecturer in 1935, in the Mathematics Department at the University of Sheffield. Naturally he had continued with his research work since leaving Cambridge, and in 1933 was awarded a Cambridge Ph.D. for his thesis entitled " Some Properties of Integral and Meromorphic Functions ".

Macintyre's next post was as Lecturer in Mathematics at King's College, Aberdeen, his appointment being made from October 1st, 1936, to a Department headed by Professor (now Principal) E. M. Wright, co-author with G. H. Hardy of a well-known work on the theory of numbers. In 1946 Macintyre was elected a Fellow of the Royal Society of Edinburgh, and when Senior Lectureships were introduced into the University of Aberdeen in 1959 he was at once promoted to such a post. At the time he was on leave of absence from Aberdeen, as a Visiting Research Professor in Mathematics at the University of Cincinnati, Ohio, U.S.A., and on September 30th, 1959, he resigned his Aberdeen post in order to return to a permanent post as Research Professor in Mathematics at Cincinnati. In the same university he was appointed in 1963 as Charles Phelps Taft Professor of Mathematics—a post he held until his death on August 4th, 1967. He also earned the rare distinction of being elected a Fellow of the Graduate School. During his last year he was given leave to spend some time organizing research projects at the University of California on the campus at Davis.

From a research point of view Macintyre's interests were by no means confined to pure mathematics. He was also deeply interested in aerodynamics, fluid mechanics and related fields. He believed, for example, that aeroplanes should have their control surfaces—rudder and elevator—located at the front and not at the back. He conducted with great enthusiasm his experiments with model planes, and corresponded widely with experts on these matters, steadfastly refusing to accept their views unless soundly backed by evidence, at the same time being himself able to answer their objections to his revolutionary ideas. He was awarded a Caird Senior Scholarship in Aeronautics for 1943–44, and a D.S.I.R. grant from 1946–49 for special research on the Lanchester Vortex. Difficulty in obtaining suitable apparatus at that time, as well as an increasing number of students desirous of researching in pure mathematics under his friendly and perceptive eye, prevented him from pushing ahead in these fields, although his interest was maintained throughout his life.

Macintyre's family life was a happy one and the family had a host of friends. He had married Sheila Scott of Edinburgh, whose father, a schoolmaster, later went with them to America, on December 27th, 1940, and who, a Girtonian and herself a Wrangler in Mathematics, had taught at St. Leonard's School (St. Andrews) and at Stowe School. She also was a good mathematician and after her marriage not only lectured in mathematics at the Universities of Aberdeen and Cincinnati but also published a number of research papers on mathematical analysis, in particular on the convergence of Abel's and other series, and on the Whittaker constant.

The Macintyres had three children, one of whom, Douglas Scott, born in August 1946, died suddenly of enteritis in March 1949, while Allister William, born on February 8th, 1944, is now a Computer Programmer and Susan Elizabeth, born in March 1950, was married on July 10th, 1967, to Mr. J. Gaines of Cincinnati. Mrs. Macintyre, who, like her husband, had always been popular with both staff and students, died in America on March 21st, 1960, at the comparatively early age of 50, three years before her father, who had settled in America with them.

2. *Introduction*

Since it is clearly not possible for me to describe the whole of Macintyre's mathematical output of research papers in the space available, I have selected topics on two bases—(i) those in which he made the more significant advances, (ii) those in which exceptionally neat methods were used—(i) and (ii) being naturally not mutually exclusive classes.

As a classical analyst, Macintyre considered a diversity of problems, throughout many of which, however, runs a strong thread, viz., his keen interest in overconvergence. Amongst his 43 papers, often grown from seed sown in his Ph.D. thesis, one finds such topics as asymptotic paths, the flat regions of meromorphic functions, interpolation problems based on the Laplace transform and other formulae for regular functions, Tauberian theorems in connection with certain canonical products, and numerous problems, many published jointly with R. Wilson, in the theory of the singularities of $f(z) = \sum_{n=0}^{\infty} c_n z^n$ on the circle of convergence.

In what follows, references are given to two lists which appear at the end of this essay. The first is a list of Macintyre's publications, references to which are preceded by the letter M.

3. *Asymptotic paths*

An early example of Macintyre's elegant methods is his proof [**M3**], by means of a theorem due to Grötzsch [**12**; 367–9, Lemma 1] on conformal transformations, of the following theorem of Ahlfors [**1**(*a*), **1**(*b*)], which generalised Denjoy's result†

† As well as some given by Collingwood and Valiron in [**9**].

[10] that an integral function of finite order ρ cannot have more than 2ρ asymptotic paths: If $f(z)$ is an integral function bounded on n continuous curves which divide the plane into n unbounded simply connected domains in each of which $f(z)$ is unbounded, then $\varliminf_{r \to \infty} r^{-\frac{1}{2}n} \log M(r) = 0$, $\left(M(r) = \max_{|z|=r} |f(z)|\right)$.

4. *Flat regions*

4.1. A significant part of Macintyre's work concerns the so-called flat regions of integral and meromorphic functions $f(z)$. J. M. Whittaker, who published [25(*a*), 25(*b*)] important theorems on the subject, had defined R as a flat region if

$$H^{-1} < \{\log |f(z_1)|\}/\{\log |f(z_2)|\} < H \quad \forall z_1, z_2 \in R \qquad (H > 1 \text{ constant}). \tag{4.1}$$

Macintyre obtained more general and/or sharper results of this nature in three ways.

(i) A sharp form of Schottky's theorem, due to Ostrowski [18(*a*), 85, Th. V*], shows that, if $p(z)$ ($\neq 0, 1$) is regular in $|z| < s$, then it is flat in $|z| < \theta s$ ($\theta \leqslant \frac{1}{2}$) if $|p(0)|$ is large enough. In [M5] Macintyre gave a new proof of a sharper version of Whittaker's results, for functions meromorphic and of finite order in the plane, with deficient poles, by dissecting the complex plane into curvilinear quadrilaterals and showing that Ostrowski's theorem could be applied over a suitable selection of these quadrilaterals.

(ii) From the Nevanlinna formula (4.2) below, Macintyre [M9] and Macintyre and Wilson [M15] obtained the inequalities†

$$\log |f(z)| > -KT(kr), \quad \left|\frac{d^q}{dz^q} \log f(z)\right| < Kr^{-q}\{T(kr)\}^q \qquad (4.1a, 4.1b)$$

valid over $|z| < r$, $z \notin N$, with $k > 1$, q (an integer) $\geqslant 1$, where $f(z)$ is meromorphic over $|z| \leqslant kr$, T denotes the Nevanlinna characteristic and N a neighbourhood of the zeros and poles of $f(z)$. The formula [8(*g*), 9; 17(*a*), 4]

$$\left.\begin{aligned} \log f(z) = iC + \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\phi})| \, g(Re^{i\phi}, z) \, d\phi \\ + \sum_{|a_m| < R} h(z, R, a_m) - \sum_{|b_n| < R} h(z, R, b_n) \end{aligned}\right\} \tag{4.2}$$

where $g = (Re^{i\phi}+z)/(Re^{i\phi}-z)$ and $h(z, R, c_s) = \log\{R(z-c_s)/(R^2 - \bar{c}_s z)\}$, shows that the essential problem was to find good bounds for $|p(z)|$ where $p(z) = \prod_{s=1}^{M} (z-c_s)$,

† For integral functions of finite order Boutroux [5] had given complicated proofs of several cases of (4.1) and M. L. Cartwright [8(*g*), 72–77] gives a proof of (4.1*a*) starting with the Hadamard product. References and further similar theorems are given in [8(*a*); 8(*d*), p. 162; 8(*e*); 8(*g*), p. 92].

and for $\left|\frac{d^q \log p(z)}{dz^q}\right|$, with N as small as possible† in comparison with $C = |z| \leqslant r$.

For $|p(z)|$ Cartan's lemma [**4**, 46–47; **8**(g), 75–77] was already to hand. For the other Macintyre and Fuchs had just obtained [**M12**] what was needed. I note in passing that (a) if N is measured areally in comparison with C, results sharper than (4.1b) are found, and (b) (4.1b) had to be weakened when $q = 1$ if the relative measure is linear, *unless*‡ as a result of ingenious work by Macintyre and Fuchs [**M12**], *z is restricted to lie on a radius* $\arg z = a$ *constant.*

Inequalities analogous to (4.1ab) for functions meromorphic in an angle were similarly proved in **M15**, and flat region results followed in both cases by integrating (4.1b) when $q = 1$.

(iii) In a third approach to "flat region" problems Macintyre investigates [**M8**] the extent of the regions in which an integral function $f(z)$ remains large in the neighbourhood of a point $z = re^{i\theta}$ at which $|f(z)|$ takes its maximum value $M(r)$. Complicated work by Wiman on comparison of power series had resulted in theorems which led Macintyre to conclude that $\phi(T) = e^{-NT} f(ze^T)/f(z)$ is practically constant for such z, with the choice $N = zf'(z)/f(z)$, and he uses $\phi(T)$ to obtain not only "flat region" results§ [**M8**], but also [**M7**] the simplest proof then known, and new generalisations, of Bloch's theorem¶.

5. *Laplace's transformation and integral functions*

This is the title of an important paper [**M10**] which not only gave (I) new proofs and extensions of theorems of M. L. Cartwright [**8**(f)] and other interpolation results, but also was to lead to (II) new results in the theory of singularities on the circle of convergence of power series. Here I deal only with (I). In §6 below appears an account of (II).

M. L. Cartwright had used the Lagrange interpolation formula and the Phragmén-Lindelöf principle to prove that, if $F(z) = \sum_0^\infty a_n z^n$ is an integral function satisfying $|F(z)| < Me^{\tau|z|}$ with $\tau < \pi$, then (i) $|F(x)| < K(\tau)A$ for all real x and an explicit|| $K(\tau)$, if $|F(n)| < A$ $(n = 0, 1, 2, \ldots)$, and (ii) $F(x) \to 0$ as $x \to +\infty$ if $F(n) \to 0$ as $n \to \infty$ $(n = 1, 2, \ldots)$. Following Pólya's well-known article [**19**(a)] on gaps

† N consists of "small" circles centred at the c_s, of course.

‡ Vitally important re §6 below.

§ Which considerably sharpened the best results [24(b)] then known in this direction.

¶ [**11**; 266, 523].

|| In **M34** Liu and Macintyre improve upon previous estimates of $K(\tau)$ by using S. Bernstein's theorem [**4**, 206] that $|F'(x)| \leqslant M\tau$ when $|F(x)| \leqslant M$ for all real x. In **M33** Macintyre and Shah use Nevanlinna theory to obtain an analogue of Bernstein's theorem for *meromorphic* functions satisfying similar relations.

and singularities of power series, Macintyre [M10] used the Borel-Laplace transform

$$F(z) = \frac{1}{2\pi i}\int_C e^{z\zeta} f(\zeta)\, d\zeta \tag{5.1}$$

with a suitable contour C and $f(\zeta) = \sum_0^\infty a_n\, n!\, \zeta^{-n-1}$, and showed how to replace $f(\zeta)$ by a second associated function

$$\psi(\zeta) = \pm\tfrac{1}{2}F(0) \pm \sum_{n=1}^{\infty} F(\pm n)\, e^{\mp n\zeta} (\pm \mathscr{R}(\zeta) > \tau)$$

so that (5.1) is in fact an interpolation formula. Term by term integration gave

$$F(z) = \lim_{\delta\to 0} \frac{1}{\pi} \sum_{-\infty}^{\infty} \frac{\sin \omega(z-n)}{(z-n)} F(n)\, e^{-\delta|n|}$$

if $\overline{\lim}_{n\to\infty} | F(\pm n)|^{1/n} \leqslant 1$, $\tau < \omega < 2\pi - \tau$, from which a whole family of interpolation results, as well as (i) and (ii) above, followed.

The method is applied also to obtain analogous results† for any order ρ, as well as for functions regular only in an angle, so giving a new proof of Cartwright's analogues to (i) and (ii) above [**8**(*c*); **8**(*f*), Th. 10].

6. *The coefficient theory of the Taylor series*‡

6.1. It was at Swansea in October 1930 that Macintyre first met Wilson and started an association which was to last throughout his life. Dienes, then at Birkbeck College, London, had only recently left Swansea, where fruitful discussions with Wilson about the manuscript for his book [**11**] on the Taylor series, which was to appear in 1931, had already inculcated in Wilson a strong interest in the subject, which was soon to inspire Macintyre also. The first joint paper [**M2**] by Macintyre and Wilson, however, was concerned with interpolatory function theory, dealing with the order of a meromorphic function interpolated at a sequence of complex points z_s, thus generalising a result of Mursi and Winn [**16**] for the case of an integral function with the z_s real.

6.2. For the discussion of the coefficient theory I assume that $|z| = 1$ is the circle of convergence of $f(z) = \sum_{n=0}^{\infty} c_n z^n$ and, for the first set of results, that there is only one singularity S, viz., an isolated essential point at $z = 1$. Wilson had conjectured that $\lim_{n'\to\infty} c_{n'+1}/c_{n'} = 1$ where $\{n'\}$ is a sequence of n of density 1. This was proved [**26**] when S is of finite exponential order by J. M. Whittaker and Wilson,

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† Using $f(\zeta) = \sum_0^\infty a_n \Gamma(\sigma+n\sigma)\, \zeta^{-n-1}$, $\sigma = 1/\rho$. Compare [2]. Results in 18(*b*) are more simply obtained, and are extended.

‡ [3] gives a comprehensive account of developments in this theory up to 1955. §6 here is a summary of a detailed survey kindly prepared for me by Professor Wilson.

using some techniques from a paper of Whittaker's [25(*a*)] on the flat regions (see §4 above) of integral functions.

It was Macintyre who perceived that the connection between the two papers was not only formal but also fundamental. In simple terms, where an integral function is "large" it is also "flat". More precisely, if $G(z)$ is the function approximately interpolating the c_n, so that $c_n = G(n)+c_n{}^*$, where $\overline{\lim}_{n\to\infty} |c_n{}^*|^{1/n} < 1$, and $G(z)$ is an integral function of at most order 1, minimum type [19(*b*); 744], then along any radius

$$\log|G(z)| > -\gamma|z|, \quad |G'(z)/G(z)| < \varepsilon \quad (|z| = r) \qquad (6.1\text{a}, 6.1\text{b})$$

for a set of r of linear density 1, where γ and ε are any positive numbers†.

From (6.1) it easily follows when the isolated essential singularity S at $z = 1$ is unrestricted that

$$|c_n| > e^{-\gamma n}, \quad |c_{n+1}/c_n - 1| < \varepsilon \qquad (6.2\text{a}, 6.2\text{b})$$

for a common sequence of n of density 1.

Inequalities of the form (6.2a) had already been given by Pólya, valid for a sequence of n of upper density, maximal density, or density‡ equal to unity, according to the nature of more specialised S which he had defined.§ The theorems of **M15**, discussed in §4 above, enabled Macintyre and Wilson [**M14**] to obtain analogous results concerning the sequence for which (6.2b) holds¶ for each S.

M14 also contains a definition, due to Macintyre, of the *virtually isolated singularity* and a proof that (6.2a, 6.2b) hold in this case for a sequence of n of density 1. An error, however, not easily ascertainable [see **M31**; 3] occurred in the authors' proof. It was to be nearly twenty years before Macintyre, in a brilliant paper [**M31**] based on an application of Bourion's theory of over-convergence applied to the special properties of the singularity concerned, demonstrated the correctness of the theorem stated.

6.3. So far attention has been confined to the case of a single singularity on the circle of convergence. In [**M13**] Macintyre and Wilson showed that, if there are k singularities, the density (or lower density) of the small coefficients, i.e., the c_n which satisfy $|c_n| < e^{-\gamma n}$ ($\gamma > 0$), is zero unless all the singularities are of the same kind and placed at the vertices of a regular polygon, in which case|| the density is $(k-1)/k$. More complete results are given when $k = 2$; when $k > 2$, a different method, founded on arguments in a Whittaker–Wilson paper [26]—**M14** had not

† (6.1*ab*) are of course special cases of (4.1*ab*). It is here that footnote ‡ of §4 applies.

‡ See [3, 13] or [19 (*a*)] for definitions of these densities.

§ Almost isolated (a.i.); easily approachable or approachable; and a.i. of finite exponential order, respectively.

¶ See [3, 66–78].

|| The problem was further explored by Ponting [20]. See also [3, 81–2].

then been written—restricted the discussion to isolated essential points of finite exponential order.

6.4. Two further interesting joint papers **[M17, M25]** by Macintyre and Wilson deserve mention in connection with the coefficient theory. Both are based on **M10**. In **M17** a conjecture of Wilson's is proved and developed, concerning the close relationship between the orders, types and directions of strongest growth of $\psi(z)$, $F(z)$, $G(z)$, $z^{-1}f(z^{-1})$, where $\psi(z) = \sum_0^\infty F(n)z^n = G[1/(1-z)]$ with $G(z)$ an integral function and $f(z)$ the Laplace transform of $F(z)$. Wilson considerably sharpened some of the results later.†

In **M25** novel Laplace transform techniques and operational methods, together with Wright's theory of the generalised Bessel function **[28]**, were used to determine the asymptotic form of the c_n arising from singularities like $\exp P[1/(1-z)]$, where $P(z)$ is a polynomial, and from related singularities. In the introduction Macintyre sketched an elegant theory which provided the basis required.

6.5. A comprehensive research tract proposed by Macintyre on the coefficient theory, containing the above results together with later work on the isolable singularity of Pólya **[19(*b*)]** and on Hadamard multiplication of singularities (extending **19**(*b*), Th. VIII), had been in preparation by Macintyre and Wilson for some time, but, although several chapters were finished, the rest of the tract was only in draft form at the time of Macintyre's death.

7. *Tauberian theorems*

7.1. It was in this field that Macintyre suggested that I should start my research for the Ph.D. degree. The initial problem, first discussed by Valiron **[24**(*a*), 237**]** and later by Titchmarsh **[23]**, both of whom gave real variable solutions of considerable length, was to obtain the relation $n(r) \sim Kr^\rho$ $(0 < K < \infty,\ 0 < \rho < 1,\ r \to \infty)$ for the number of zeros in $|z| = r$ of an integral function $f(z)$ of genus 0, all of whose zeros are negative, given that, with $z = x > 0$,

$$\log f(z) \sim K\pi z^\rho \operatorname{cosec} \pi\rho \quad (|z| \to \infty). \tag{7.1}$$

Under Macintyre's hand was evolved a proof **[6]** based on function theory methods,‡ which is not only much simpler but also allows reductions in the hypotheses. Using Montel's limit theorem we show§ that (7.1) holds uniformly in the sector S: $|\arg z| \leqslant \pi-\delta$, and obtain an elementary Tauberian problem by taking the imaginary parts for $\delta(>0)$ small.

Trivial modifications give the corresponding result when $f(z)$ is of genus p, with $p < \rho < p+1$, and Valiron's theorem is thus completely proved.

† e.g. **[27]**. See also **[3, 18]**.

‡ Heins **[14]** gave a similar proof about the same time.

§ **[4**, 57**]** gives details of the first step but the method thereafter differs from ours **[6]**.

The main relaxations which we found possible in the hypotheses are of two kinds (A), (B). (A) The path to infinity in (7.1) may be taken, as Montel's theorem allows, to be *any* continuous path P to infinity† in S, and P may itself be replaced by a discrete sequence (z_n) such that $|z_n| \to \infty$ and $z_{n+1}/z_n \to 1$ or even (i) $|z_{n+1}/z_n| \to 1$. The proof under the assumption (i) follows from **M27**, where we obtained, from a lemma due to J. M. Whittaker [**25**(*c*); 57], new proofs of Vitali's convergence theorem, Blaschké's theorem [**15**; 181] and Montel's theorem, and, from a lemma due to Hall [**13**], a Montel analogue in which the z_n tend to the boundary of the region concerned. (B) (7.1) may be replaced [**M22**] by its real parts for $\arg z = \alpha$ (constant, $|\alpha| \leqslant \pi - \delta$) with certain side conditions, and with $z = z_n$ satisfying (i)‡. The result was obtained for *any* non-integral order ρ.

For $\rho < \frac{1}{2}$ Titchmarsh [**4**, 59; **23**, 195] had obtained the result also when $\alpha = \pi$, i.e., the path in (7.1) runs along the line of zeros of $f(z)$. We gave a new proof valid for all non-integral $\rho < \frac{3}{2}$.

7.2. In our remaining paper [**M21**] concerning these canonical products, we showed that Valiron's theorem is a limiting case of an oscillation theorem, i.e., that $0 < l \leqslant \overline{\lim_{x\to\infty}}\, x^{-\rho} \log f(x) \leqslant L < \infty$ implies

$$0 < \phi(l, L) \leqslant \overline{\lim_{n\to\infty}}\, n^{-1/\rho} a_n \leqslant \Phi(l, L) < \infty, \quad \text{(the } -a_n \text{ are the zeros of } f(z)\text{)},$$

where Φ/ϕ is arbitrarily near unity when L/l is sufficiently near unity. The proof follows as before, with the use, suggested by Macintyre, of Nevanlinna's two constant theorem§ in place of Montel's theorem.

8. *The two Macintyres*

8.1. As far as I can discover, only one paper [**M24**] was prepared jointly by Macintyre and his wife, Sheila Scott Macintyre. Sheila, herself a classical analyst, had numerous research publications to her credit, especially on the Whittaker constant (defined in **4**; 173) and on interpolation series of various kinds for integral functions. The object of the Macintyre's paper is to investigate conditions under which the Abel series $\sum = \sum_{0}^{\infty} z(z-n)^{n-1} F^{(n)}(n)/n!$ (i) converges, (ii) is asymptotically equivalent to $F(z)$ in $\mathscr{R}(z) > 0$, when $F(z)$ is regular for $|\arg z| \leqslant \frac{1}{2}\pi$, where $|F(re^{i\theta})| < Kr^{-\gamma} e^{rb(\theta)}$ with $K > 0$, γ real, and $b(\theta)$ the supporting function of a certain convex set. (i) follows at once from the Cauchy integral for $F^{(n)}(n)$, but (ii) required the use of the inverse Laplace transform (5.1) with a suitable C, to which Schmidli's method [**21**] of expanding $e^{z\zeta}$ in powers of ζe^{ζ}, followed by term by term integration, is applied.

† In fact, from [**8**(*b*)], a set of interpolation points of positive linear density suffices.

‡ [**8**(*c*)] contains *inter alia* a discussion of the case in which $\rho = 1$.

§ [**17**(*b*), 41].

8.2. Macintyre (alone) used this transform again in **M28**. Here $F(z)$ is an integral function of exponential type, $f(\zeta)$ is replaced by $\sum_0^\infty v_n(\zeta)\, T_n(F)$, and an expansion for $F(z)$ is obtained on integration. Buck [7] had obtained a similar expansion by writing $e^{z\zeta}$ as $\sum_0^\infty u_n(z)\, g_n(\zeta)$ and integrating. While Buck's method is the more powerful in dealing with Abel's series, Macintyre's applies to some series, e.g., the Lidstone and Whittaker two-point and (iii) the analogous Poritsky and Gontcharoff n-point series, which do not appear in Buck's analysis.

The basis of this paper is the close relation† among the functions

$$f(\zeta, \alpha) = \sum_0^\infty F^{(n)}(\alpha)\, \zeta^{-n-1}$$

for different α. As we know, (iii) are determined by the values of $F^{(n)}(\alpha_p)$ for some n and p. Macintyre expresses this information in terms of the $f(\zeta, \alpha_p)$, deduces $f(\zeta)$ via a simple functional equation, and $F(z)$ via (5.1).

The ideas involved are deeply developed, the method is modified so as to obtain analogous interpolation series for integral functions of any order (thus solving explicitly a problem for which Pólya had obtained [**19**(c)] an existence theorem), and the paper ends with a proof that summability properties of certain special series display a consistency not always exhibited by the convergence properties.

9. *Gap power series*

9.1. In the early 1950's Macintyre considered [**M23**] the question as to whether the existence of asymptotic paths of integral functions $F(z)$ can or cannot be excluded by a knowledge of the sequence of integers $\{\lambda_n\}$ alone, where $F(z) = \sum_{n=0}^\infty c_n z^{\lambda_n}$ $(0 \leqslant \lambda_n \uparrow \infty)$. Pólya had proved [**19**(a); 636–9] that, if (i) $\lambda_n \sim n \log\log n$, then there exists an $F(z)$ which tends to zero as $z = x(>0) \to \infty$. Macintyre shows how to construct a function with as thin a $\{\lambda_n\}$ as possible, with the aid of which he replaces (i) by the more general $\sum_{n=0}^\infty \lambda_n^{-1} = \infty$. If, also, the sequence $\{\lambda_n/n\}$ is bounded, the $F(z)$ constructed is of finite order. As is pointed out, two converse results are suggested, viz., that asymptotic paths are excluded by (Case I) convergence of $\sum_{n=0}^\infty \lambda_n^{-1}$, (Case II) $\lambda_n/n \to \infty$ with $F(z)$ of finite order. References are given to a method which would prove Case I for radial paths, and to a proof of Case II by Pólya [**19**(a); 631, Th. IX], who here raised the question as to whether the relation $\overline{\lim}_{r\to\infty} \{\log m(r)/\log M(r)\} = 1$ holds. Macintyre [**M23**] obtains the result, which clearly excludes the possibility of asymptotic paths and so offers an alternative (somewhat simpler) proof of Case II than Pólya's, that, in this case, to each $\varepsilon > 0$,

† Compare [**19**(a), 590(b)].

there are arbitrarily large R such that $\{\log|F(z)|/\log M(R)\} > 1-\varepsilon$ for all z on a certain continuous curve enclosing the origin and confined to the annulus $R(1-\varepsilon) < |z| < R$. The proof of this theorem is an ingenious elaboration of methods due to Pólya [**19**(*a*)] and Sunyer Balaguer [**22**].

9.2. The early 1950s also saw the arrival in Aberdeen of Paul Erdös for a short stay as Research Fellow in the Department of Mathematics at King's College. One of the consequences was a joint paper [**M26**] on gap series. Pólya had shown [**19**(*a*), 640] that $\overline{\lim_{r\to\infty}}\, m(r)/M(r) = \overline{\lim_{r\to\infty}}\, \mu(r)/M(r) = 1$, $\left(\mu(r) = \max_{|z|=r} |c_n z^{\lambda_n}|\right)$, from $\underline{\lim}_{n\to\infty} \{\log(\lambda_{n+1}-\lambda_n)\}/\log\lambda_n > \frac{1}{2}$. Let $\sum(h, k)$ denote $\sum_{n=0}^{k} (\lambda_{n+h}-\lambda_n)^{-1}$ for positive integral h, k. In **M26** Pólya's results are (i) obtained under the weaker condition $\sum(1, \infty) < \infty$, (ii) modified under a variety of hypotheses on $\sum(h, k)$. The proofs depend largely upon showing that the gap series concerned are (sometimes) dominated by the maximum term, or upon constructing gap series which have this property.

9.3. In 1959 Macintyre returned [**M32**] to problems concerning gap series and, for example, by setting $c_n = G(n)$ with $G(z)$ an appropriate integral function, demonstrated that, if (a) $F(z)$ is holomorphic throughout a simply connected schlicht domain D which contains $|z| = 1$, (b) H is a closed subset of D,

$$\text{(c)}\quad \lambda_{n+1}/\lambda_n > \lambda(D, H) > 1,$$

then the series is overconvergent throughout H. The proofs, given for two cases, viz., when D is the whole plane cut from $z = 1$ to $z = \infty$, and when D is one of a family of domains bounded by certain logarithmic spirals, admit some weakening of the hypotheses.

10. *Conclusion*

In this essay I have tried to depict what I consider to be the highlights of Macintyre's mathematical output. One thread, in particular, runs throughout his work, viz., his strong interest in overconvergence, as is seen in **M1, M31, M32, M35** and in the research tract (§6 above). In connection with the coefficient theory he used Bourion's theory of overconvergence brilliantly to deal with difficult cases, viz., (i) of the isolable singularity [**19**(*b*)] (in the tract) and (ii) of the virtually isolated singularity [**M31**, defined in **M14**].

As one would expect, the germs of his earlier work are to be found in his Ph.D. thesis, while his later publications, several written in conjunction with his research students at Cincinnati, are often developments of earlier work.

It has, of course, not been possible to describe here the whole of Macintyre's output, but when one considers also the great variety of topics discussed in his remaining publications not mentioned above, I think that all will agree what a great debt mathematics owes, and will owe, to the fertility of Macintyre's mind.

May I offer my sincere thanks to all those who have helped me in any way to prepare this essay in its present form, in particular to the Registrars of Swansea

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List of publications by A. J. Macintyre

M1. " Un théorème sur l'ultraconvergence ", *C. R. Acad. Sci. Paris*, 199 (1934), 598–599.

M2. " On the order of interpolated integral functions and of meromorphic functions with given poles " (with R. Wilson), *Quart. J. Math. Oxford Ser.*, 5 (1934), 211–220.

M3. " On the asymptotic paths of integral functions of finite order ", *J. London Math. Soc.*, 10 (1934), 34–39.

M4. " Elementary proof of theorems of Cauchy and Mayer ", *Proc. Edinburgh Math. Soc.* (2), 4 (1934–36), 112–117.

M5. " A theorem concerning meromorphic functions of finite order ", *Proc. London Math. Soc.* (2), 39 (1935), 282–294.

M6. " Two theorems on Schlicht functions ", *J. London Math. Soc.*, 11 (1936), 7–11.

M7. " On Bloch's theorem ", *Math. Z.*, 44 (1938), 536–540.

M8. " Wiman's method and the flat regions of integral functions ", *Quart. J. Math. Oxford Ser.*, 9 (1938), 81–88.

M9. " On the minimum modulus of integral functions of finite order ", *Quart. J. Math. Oxford Ser.*, 9 (1938), 182–184.

M10. " Laplace's transformation and integral functions ", *Proc. London Math. Soc.* (2), 45 (1939), 1–20.

M11. " On a theorem concerning functions regular in an annulus ", *Recueil Math.* (*Mat. Sb.*) N.S., 5 (47) (1939), 307–308.

M12. " Inequalities for the logarithmic derivatives of a polynomial " (with W. H. J. Fuchs), *J. London Math. Soc.*, 15 (1940) 162–168.

M13. " Coefficient density and the distribution of singular points on the circle of convergence " (with R. Wilson), *Proc. London Math. Soc.* (2), 47 (1940), 60–80.

M14. " Some converses of Fabry's theorem " (with R. Wilson), *J. London Math. Soc.*, 16 (1941), 220–229.

M15. " Logarithmic derivatives and flat regions of analytic functions " (with R. Wilson), *Proc. London Math. Soc.* (2), 47 (1942), 404–435.

M16. " Some elementary inequalities in function theory " (with W. W. Rogosinski), *Edinburgh Math. Soc. Notes*, 35 (1945), 1–3.

M17. " Associated intergral functions and singular points of power series " (with R. Wilson), *J. London Math. Soc.*, 22 (1947), 298–304.

M18. " Note on the preceding paper "†, *J. London Math. Soc.*, 23 (1948), 209–211.

M19. " Euler's limit for e^x and the exponential series ", *Edinburgh Math. Soc. Notes*, 37 (1949), 26–28.

M20. " Extremum problems in the theory of analytic functions " (with W. W. Rogosinski), *Acta Math.*, 82 (1950), 275–325.

M21. " An oscillation theorem of Tauberian Type " (with N. A. Bowen), *Quart. J. Math. Oxford Ser.* (2), 1 (1950), 243–247.

M22. " Some theorems on integral functions with negative zeros " (with N. A. Bowen), *Trans. Amer. Math. Soc.*, 70 (1951), 114–126.

† " A functional inequality " (by Sheila Scott Macintyre), *J. London Math. Soc.*, 23 (1948), 202–209.

M23. " Asymptotic paths of integral functions with gap power series ", *Proc. London Math. Soc.* (3), 2 (1952), 286–296.

M24. " Theorems on the convergence and asymptotic validity of Abel's series " (with S. S. Macintyre), *Proc. Roy. Soc. Edinburgh, Sect. A*, 63 (1952), 222–231.

M25. " Operational methods and the coefficients of certain power series " (with R. Wilson), *Math. Ann.*, 127 (1954), 243–250.

M26. " Integral functions with gap power series " (with P. Erdös), *Proc. Edinburgh Math. Soc.* (2), 10 (1954), 62–70.

M27. " Interpolatory methods for theorems of Vitali and Montel type " (with N. A. Bowen), *Proc. Roy. Soc. Edinburgh, Sect. A*, 64 (1954), 71–79.

M28. " Interpolation series for integral functions of exponential type ", *Trans. Amer. Math. Soc.*, 76 (1954), 1–13.

M29. " The history and the theory of the aeroplane and the sailplane ", *Aberdeen University Rev.*, 36 (1956), 241–250.

M30. " A direct proof of Morera's theorem ", *Arch. Math.*, 8 (1957), 374–375.

M31. " An ‹ verconvergence theorem of G. Bourion and its application to the coefficients of certain power series ", *Ann. Acad. Sci. Fenn. Ser. A, I.* Mathematica No. 250/23 (1958), 3–11.

M32. " Size of gaps and region of overconvergence ", *Collect. Math.*, XI 3 (1959), 165–174.

M33. " On an extension of a theorem of Bernstein to meromorphic functions " (with S. M. Shah), *J. Math. Anal. Appl.*, 3 (1961), 351–354.

M34. " Cartwright's theorem on functions bounded at the integers " (with H. C. Liu), *Proc. Amer. Math. Soc.*, 12 (1961), 460–462.

M35. " Studies on overconvergence " (with H. G. Mushenheim), *Collect. Math.*, XV (1963), 3–36.

M36. " Upper bounds for a Bloch Constant " (with P. S. Chiang), *Proc. Amer. Math. Soc.*, 17 (1966), 26–31.

M37. " Convergence of $i^{i^{i\cdots}}$", *Proc. Amer. Math. Soc.* 17 (1966), 67.

M38. " Inequalities for functions regular and bounded in a circle " (with Cecil Craig), *Pacific J. Math.*, 20 (1967), 449–454.

M39. " A proof of the power series expansion without differentiation theory " (with Willy John Wilbur), *Proc. Amer. Math. Soc.*, 18 (1967), 419–424.

M40. " Some theorems of Bloch type " (with P. S. Chiang), *Proc. Amer. Math. Soc.*, 18 (1967), 953–954.

M41. " Integers, no three in arithmetic progression " (with P.-S. Chiang), *Mathematics magazine*, 41 (1968), 128–130.

M42. " e and e^2 as continued fractions " (with C. I. Lubin), *Amer. Math. Monthly* (2) 73 (1966), 1124, problem 5445.

M43. " Entire functions related to the Dirichlet divisor problem " (with N. A. Bowen), Proc. of Symposia in Pure Mathematics (organised by the *Amer. Math. Soc.*), XI (1968), 66–78.

References

1. L. Ahlfors, (*a*) *Comptes Rendus*, 188 (1929), 688; (*b*) *Acta Soc. Sci. Fenn. Nova Ser. A*, 1 (1929), No. 9.
2. V. Bernstein, *Mem. Acad. Ital.*, 7 (1936), 131–189.
3. L. Bieberbach, *Analytische Forsetzung* (Heft 3 in Ergebnisse der Mathematik und ihrer Grenzgebiete—Springer-Verlag, 1955).
4. R. P. Boas, *Entire functions* (Academic Press, 1954).
5. P. Boutroux, *Acta Math.*, 28 (1903), 97–224.
6. N. A. Bowen, *Quart. J. Math. Oxford Ser.*, 19 (1948), 90–100.
7. R. C. Buck, *Trans. Amer. Math. Soc.*, 64 (1948), 283–298.

8. M. L. Cartwright, (*a*) *Proc. London Math. Soc.* (2), 32 (1931), 51–71; (*b*) *Proc. Cambridge Philos. Soc.*, 31 (1935), 26–30; (*c*) *Proc. Cambridge Philos. Soc.*, 31 (1935), 347–350; (*d*) *Proc. London Math. Soc.* (2), 38 (1935), 158–179; (*e*) *Proc. London Math. Soc.* (2), 38 (1935), 503–541; (*f*) *Quart. J. Math. Oxford Ser.*, 7 (1936), 46–55; (*g*) *Integral functions* (Cambridge, 1956).

9. E. F. Collingwood and G. Valiron, *Proc. London Math. Soc.* (2), 26 (1927), 169–184.

10. A. Denjoy, *Comptes Rendus*, 145 (1907), 106.

11. P. Dienes, *The Taylor series* (Oxford, 1931).

12. H. Grötzsch, *Lpz. Berichte*, 80 (1928), 367–376.

13. T. Hall, *Ark. Mat., Ast och Fysik*, 25A, 28 (1937), 1–8.

14. M. Heins, *Ann. of Math.* (1), 49 (1948), 200–213.

15. P. Montel, *Leçons sur les familles normales de fonctions analytiques* (Paris, 1927).

16. M. Mursi and C. E. Winn, *Quart. J. Math.*, 4 (1933), 173–179.

17. R. Nevanlinna, (*a*) *Le théorème de Picard-Borel* (Paris, 1929); (*b*) *Eindeutige Analytische Funktionen* (Berlin, 1936).

18. A. Ostrowski, (*a*) *Comment Math. Helv.*, 5 (1933), 55–87; (*b*) *Math. Ann.*, 108 (1933), 718–756.

19. G. Pólya, (*a*) *Math. Z.*, 29 (1929), 549–640; (*b*) *Ann. of Math.* (2), 34 (1933), 731–777; (*c*) *Trans. Amer. Math. Soc.*, 50 (1941), 129–139.

20. F. W. Ponting, *Collect. Math.*, 10 (1958), 137–184.

21. S. Schmidli, *Thesis uber gewisse Interpolationreihen* (Zurich, 1942).

22. F. Sunyer Balaguer, *Mem. R. Acad. Cienc. Artes, Barcelona* (3), 29 (1948), 475–516.

23. E. C. Titchmarsh, *Proc. London Math. Soc.* (2), 26 (1926), 185–200.

24. G. Valiron, (*a*) *Ann. Toulouse* (3), 5 (1914), 177–257; (*b*) *Comptes Rendus*, 204 (1937), 33–35.

25. J. M. Whittaker, (*a*) *Proc. Edinburgh Math. Soc.*, 2 (1930), 111–28; (*b*) *Quart. J. Math. Oxford Ser.*, 2 (1931), 252–258; (*c*) *Interpolatory function theory* (Cambridge, 1935).

26. J. M. Whittaker and R. Wilson, *J. London Math. Soc.*, 14 (1939), 202–208.

27. R. Wilson, *Quart. J. Math. Oxford Ser.* (2), 12 (1961), 145–152.

28. E. M. Wright, *J. London Math. Soc.*, 8 (1933), 71–79.