THOMAS MURRAY MACROBERT

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Thomas Murray MacRobert, who died at his home in Glasgow on 1st November, 1962, was born on 4th April, 1884, in Dreghorn, Ayrshire, the son of the Rev. Thomas MacRobert, M.A., and Isabella Edgeley Fisher. His father was for fifty-seven years minister of the Dreghorn congregation, at that time in the Evangelical Union and later in the Congregational Union. The Rev. Thomas MacRobert was prominent in the counsels of the Congregational Union and served as its President. A Liberal in politics, he was a friend of Keir Hardie, who brought his infant daughter to be baptised at Dreghorn.

In 1901 MacRobert entered Glasgow University. His original intention was to follow his father in the Congregational ministry; he used to say that he gave up the idea because he considered that he would have made a poor preacher. After a successful undergraduate career (not entirely devoted to study, as his prowess in a Rectorial fight shows) he graduated in 1905, Master of Arts with First Class Honours and Bachelor of Science with Special Distinction, both degrees in Mathematics and Natural Philosophy. Anxious to widen his mathematical knowledge, he determined to go to Cambridge and sat the Scholarship examination at Trinity College, obtaining a Major Scholarship. He was a Wrangler in Part I of the Mathematical Tripos in 1908 and was placed in the First Class in Part II in 1910. MacRobert enjoyed his time at Cambridge. He spoke in the Union, supporting the policies of the Liberal Government, for which he had a great admiration; he even considered seriously making his career in politics.

In October 1910 MacRobert joined the Mathematics Department at Glasgow University as Assistant to Professor Gibson, and it was here that he was to spend the whole of his teaching life. In 1913 he was appointed Lecturer and in 1927 he succeeded Professor Gibson as Professor of Mathematics, retiring from the Chair in 1954. Glasgow University conferred on him in 1917 the higher degree of D.Sc. for his work on functions of a complex variable, and in 1955, after his retirement, the honorary degree of LL.D.

The affection and respect in which he was held by colleagues and former students was amply demonstrated when, after his retirement, they commissioned Mr. Norman Hepple (now R.A.) to paint his portrait. This fine

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† This notice is based on a fuller account of Professor MacRobert's life and work by Dr. R. P. Gillespie and Professor A. Erdélyi, which has appeared in the *Proceedings of the Glasgow Mathematical Association*, 6 (1963), 57–64.

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picture, conveying something of his character and personality, hangs in the Mathematics Classroom at Glasgow University.

In 1914 MacRobert married Violet McIlreaith, who predeceased her husband by five years. They are survived by one daughter and two sons.

Not only Glasgow students received MacRobert's willing help; many mathematicians overseas, whose researches followed the lines of his work, benefited greatly from his advice, and he gave them generous help, particularly in the preparation of the manuscripts of their papers for publication in British journals. He was one of the original founders of the Glasgow Mathematical Association, of which he was twice President and latterly Honorary President. It was entirely due to his initiative that the Association, with the support of the University Court of the University of Glasgow, embarked in 1951 on the publication of its *Proceedings*, and he served on the editorial committee until his death.

Professor MacRobert's publications span almost half a century, his first paper having appeared in 1916, while the last one was in the press at the time of his death. All in all he wrote three books, revised or prepared for publication three works by other authors, and published some seventy research papers and notes. The list of his publications shows one unusual feature. The second item in chronological order (preceded only by a two-page note in 1916) is a substantial book which went through five editions (the fifth edition appearing in the last year of its author's life) and numerous reprintings and is still in use, 45 years after its first appearance.

The book in question is *Functions of a complex variable*. The theoretical part of this book is on a comparatively modest level, mirroring presumably the level current in undergraduate instruction before the First World War: one is conscious that the book precedes by a decade and a half or so the more advanced texts by Titchmarsh and Copson. This part of the book changed little in subsequent editions. The main strength of the work lies in the presentation of special functions, such as the gamma function, functions of the hypergeometric type, and elliptic functions. This is the part of analysis that occupied the most important place in MacRobert's research. Here he was an acknowledged master and not only improved the presentation in each edition but also added new results obtained by himself, his associates, and others. Results of the research which he carried out in the course of a long and diligent life appear in the appendices and the "miscellaneous examples". In this respect Appendix V and the third group of miscellaneous examples are especially noteworthy in that they contain an excellent presentation of the E-function.

Since the first publication of *Functions of a complex variable* a large number of text-books on complex variable theory and on special functions have appeared, among them some distinguished and famous works. The very fact that through almost half a century MacRobert's book maintained a place of its own beside the later books testifies to its quality.

His next book appeared ten years later, in 1927. Its title, *Spherical harmonics*, does not describe its contents adequately, and the sub-title, "An elementary treatise on harmonic functions with applications", indicates what the book was originally intended to be rather than what it actually became. Apparently MacRobert set out to write an elementary presentation of Legendre functions (without using analytic function theory) with applications to potential theory. The plan grew under his hands, and he wound up with a book on the solution of the classical boundary value problems of mathematical physics by Fourier series, Legendre functions, and Bessel functions. This book, too, has proved sufficiently popular to merit a second edition in 1947.

Approximately another ten years later (in 1937 and 1938) appeared the third, and the last, of MacRobert's books, *Trigonometry* (in four parts), written in collaboration with W. Arthur (then Lecturer in MacRobert's department). The scope of this work is much wider than is indicated by the titles of its four parts. In particular, the third part, entitled *Advanced trigonometry*, is really a text-book on infinite series and products, including topics such as double series and uniform convergence. Like all MacRobert's books, it contains a remarkably rich collection of examples.

Besides writing original works, MacRobert found time to revise or see through publication books by others. He collaborated with Gray on the second edition of Gray and Mathews's *Bessel functions*, which was published in 1922. The appearance in the same year of Watson's monumental work on the same subject caused this revised edition to have somewhat less impact than it might have had in different circumstances; but the book (which up to that time was the most important book on its subject) has merit, and is still widely used, especially by applied mathematicians. In 1926 appeared the revised edition of Bromwich's *Infinite series*. The revision of this important work embodies considerable improvements, both of a mathematical nature and in a more systematic presentation, and this revised edition proved of great and lasting influence in its field. Finally, he prepared for publication *Advanced calculus*, by his predecessor in the Glasgow Chair, G. A. Gibson.

It has already been indicated that MacRobert's publication record shows a trend opposite to that of many other mathematicians. All of his books were written in the first half of his career, while both the number and importance of his research papers increased during the second half. He continued his research activities after his retirement from the Glasgow Chair and was producing interesting papers up to the time of his death.

The turning point in his research activities is, beyond any manner of doubt, the discovery of the E-function in the middle thirties. Up to that point MacRobert had no clearly delineated field of research but applied his analytical skill to a variety of problems connected with special functions, especially functions related to hypergeometric functions, and related topics.

Most of the special functions in question can be expressed in terms of the generalized hypergeometric series ${}_pF_q$ where

$$\frac{\Gamma(a_1)\ldots\Gamma(a_p)}{\Gamma(c_1)\ldots\Gamma(c_q)}\,{}_pF_q\left(a_1,\ldots,a_p;\,c_1,\ldots,c_q;\,-\frac{1}{z}\right)$$
$$=\sum_{n=0}^{\infty}\frac{\Gamma(a_1+n)\ldots\Gamma(a_p+n)}{n!\,\Gamma(c_1+n)\ldots\Gamma(c_q+n)}\left(-\frac{1}{z}\right)^n. \tag{1}$$

This series converges for each z if $p \leqslant q$, it converges for $|z|>1$ if $p=q+1$, and (unless it terminates) it diverges for each z when $p>q+1$. Nevertheless, cases have been known for a long time in which such a divergent series represents certain functions of the hypergeometric type asymptotically as $z\to\infty$. Presumably in an attempt to construct proofs by induction (on p and q) of certain results involving (convergent) generalized hypergeometric series, MacRobert discovered a multiple integral which possesses (1), with $p>q+1$, as its asymptotic expansion. Defining

$$E(a_1,\ldots,a_p;\,c_1,\ldots,c_q;\,z)$$

by (1) when $p\leqslant q+1$ and by his multiple integral when $p\geqslant q+1$, he found that the two definitions are equivalent when both hold (*i.e.*, when $p=q+1$ and $|z|>1$), and that (1) when divergent provides the asymptotic expansion of E in a certain sector. Moreover, a great many identities involving the E-function hold irrespective of the relative size of p and q, and many known formulae involving functions of the hypergeometric type can be expressed elegantly in terms of E.

These discoveries convinced MacRobert that in the E-function he had found an appropriate extension of generalised hypergeometric functions, and from 1938 onwards he devoted most of his research efforts to the investigation of this function. In the course of the years he built up an impressive body of results, including, in particular, a formidable number of integrals with E-functions. These results not only serve to unify and express conveniently known facts involving special functions of the hypergeometric type but also lead to many new formulae. A considerable number of research workers are now engaged in research in this field; some of them are MacRobert's pupils, while others learned about the subject from his publications.

Publications

BOOKS

Functions of a complex variable, Macmillan. 1st Ed. 1917, 1925; 2nd Ed. 1933, 1938; 3rd Ed. 1945, 1950; 4th Ed. 1954; 5th Ed. 1962.

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