WILLIAM PROCTOR MILNE

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(The biographical introduction to this notice is by H. S. Ruse; the appraisal of Milne's mathematical work and the compilation of Milne's mathematical publications is due to W. L. Edge.)

William Proctor Milne, who, at the time of his death, was Emeritus Professor of Mathematics in the University of Leeds, was born in Longside, Aberdeenshire, on 22 May, 1881, and died on 3 September, 1967. His wife Mary (née Burnett) had died in 1953, but he was survived by his son (Burnett) and his daughter (Elizabeth).

Having received his early education at Peterhead Academy and Aberdeen Grammar School, he had a brilliant career at the University of Aberdeen, where he was awarded several prizes, a gold medal and the Fullerton and Ferguson Scholarships. He graduated with first-class honours in Mathematics and Natural Philosophy and then went to Clare College, Cambridge. He became Fourth Wrangler in 1906, and obtained a first class in Part II of the Mathematical Tripos. He was mathematics master at Clifton College, Bristol, from 1907 until 1919, and in the latter year was appointed Professor of Mathematics at the University of Leeds, a post held by him until he retired in 1946. He went to Leeds as the sole member of the Department of Mathematics, receiving during his first term only the help of a graduate in engineering, but he was joined in December, 1919, by C. W. Gilham, who in due course became a Senior Lecturer in Mathematics; and in the following month by S. Brodetsky, who, after being Reader in Applied Mathematics, was made Professor of Applied Mathematics in 1924. The Milne–Brodetsky professorial partnership became famous in academic circles, not only because of their reputation as mathematicians, but also because they were both men of outstanding personality. Among other well-known mathematicians who were members of the staff at various times during Milne's headship of the mathematics department were W. E. H. Berwick, R. M. Gabriel, A. E. Ingham, Glenny Smeal, R. Stoneley and H. D. Ursell.

Milne was a man of lively disposition who on social occasions would often be the centre of a hilarious group of people. He showed great zest in all his widespread activities and much force of character in achieving his objectives. Of his many services to the University of Leeds and to university education generally, perhaps the greatest was that of acting as Pro-Vice-Chancellor during the second world war, when, in the absence abroad of the Vice-Chancellor, he gave compelling leadership to the committee concerned with the university's Post-War Development Plan. He was an enthusiastic supporter of the Mathematical Association, believing it to provide an essential link between mathematics teachers in schools, colleges and universities: in 1921 he persuaded the Association to hold in Leeds the first annual general meeting ever to take place outside London, and he was instrumental in founding the Yorkshire branch of the Association.

He had many interests, of which one needs special mention: he was fascinated by languages, and especially by Gaelic. The promotion of Celtic studies was indeed an interest of his barely second to that of mathematics. After his retirement he

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completed a task upon which he had been engaged for some years, unknown to anyone outside his family, namely the writing of a novel *Eppie Elrick* (*An Aberdeenshire tale of the* '15).† In it all the dialogue is in Scots, and it was intended among other things to record the dialect, as it existed in his youth, of his native Buchan district of Aberdeenshire. The book was such a success in Aberdeenshire that a reprint was required a few months after its publication.

In his early days as a schoolmaster Milne wrote, or collaborated in, five mathematical text-books‡ notable in their day for the clarity of their style, all of them running into several editions or reprints.

His great distinction as a man, as a mathematician and as a university statesman was recognized by the award of the honorary degree of LL.D. by the University of Aberdeen at the time of his retirement in 1946, and by the University of Leeds in 1955.

Among§ those who taught Milne in his undergraduate days was J. H. Grace: a colourful personality whose lectures, whether or not diversified by an occasional irregularity, are deemed to have been among the most brilliant and inspiring of that era. Grace was recognised on all sides as an authority on invariants, but he was also a geometer of striking originality and fertile imagination, and contact with him was a potent factor in determining the line that Milne would follow. And so, through Grace's help, a young schoolmaster at Clifton College was soon writing alongside H. F. Baker and G. H. Hardy in our *Proceedings*.

These early papers of Milne date from 1910 and handle problems posed by the apolarity of a plane cubic Γ and triads of its own points. Such problems suggest themselves because [**8**], if Δ_1 and Δ_2 are triangles in a plane, the locus of a point such that the two triads of lines joining it to their vertices are (considered as binary cubics) apolar is a cubic curve Γ circumscribing both Δ_1 and Δ_2 and apolar to both triads of vertices. These triads are called [**8**; p. 238] *coapolar* triads for Γ. There are, it is true, certain pairs of triangles affording apolar triads of lines centred at any point whatever in the plane; but these are identified on p. 311 of [**21**].

Milne's first endeavour [**8**] to characterise coapolar triads on Γ exploits the fact that the projection of any apolar triad from a point O of Γ onto Γ itself is also an apolar triad; he gives an interesting proof of this in a later paper [**29**; p. 380]. He is thus able, by choosing O suitably, to use a collinear apolar triad. And, in passing, one may note his idea of projecting coapolar triads into two that make up the intersection of Γ with a conic [**23**]: the use of elliptic functions shows at once that 36 centres of projection are available on Γ. But in [**21**] Milne shows directly that, given an apolar triad ABC and a point P on Γ, there are two triads including P which are coapolar with ABC. Neither of these triads is, in general, collinear; if one is, the three tangents of Γ concur at, say, J and so the line, being part of the polar conic of J, belongs [**4**; p. 9] to the Cayleyan envelope of Γ.

Those triads coapolar with a given one fall into closed sets of six, each set being a determinantal system: i.e. the six non-collinear triads among the nine intersections of two groups of three lines. This phenomenon occurs [**8**, p. 236] at the very outset

† Published by Scrogie, Peterhead, 1956.

‡ I to V in the list of publications given below.

§ From this point everything that follows is by W. L. Edge.

in the attenuated form with both groups of lines concurrent; the complete version occurs in [**21**; p. 313], and its full symmetry emerges in [**27**]. Here Milne shows that, if Δ_1 and Δ_2 are coplanar triangles, the web of cubics through their six vertices includes a pencil $\wp$ apolar both to Δ_1 and Δ_2. Moreover, the remaining base points of $\wp$ form a triangle Δ_3, and the three triangles are symmetrically related. This discovery was anticipated by Morley in 1904.

In [**33**], still influenced by J. H. Grace, Milne acts on a suggestion of Richmond's and regards the web of cubics circumscribing Δ_1 and Δ_2 as mapping, in Clebsch's manner, the plane sections of a cubic surface F. Let the six lines of either half of a double-six S on F be partitioned as two triples: those nine lines on F that do not belong to S but which meet one line of each triple are the intersections of a pair of Steiner trihedra whose vertices are, say, H and K. Milne proves that if Δ_1 and Δ_2 map the two triples then Δ_3 maps the intersections of F with HK, and that the pencil $\wp$ maps the sections of F by the planes through HK.

In [**38**, **39**, **43**, **46**], Milne is concerned with a cubic surface F and the plane quartic which is the apparent contour of F from a point on F itself. The plane equation of F is, as Salmon noted, a linear combination of $\sigma^3 = 0$ and $\tau^2 = 0$, where σ, τ correspond to Aronhold's invariants S, T of a ternary cubic. The four tangents to a non-singular plane cubic Γ from a point P of itself have the same cross-ratios wherever P is on Γ; these four tangents are equianharmonic when $S = 0$, harmonic when $T = 0$. When Γ acquires a cusp both S and T vanish: the tangent planes to F at the points of its parabolic curve H form the developable $\sigma = \tau = 0$. H is bitangent to all the 27 lines on F, and Milne is able to show, by algebra in [**38**] and by his favourite " methods of synthetic geometry " in [**43**], that the 24 tangent planes to F at the 24 contacts of H with the 12 lines of a double-six all touch a quadric. The 36 quadrics so related to F were called " Milne quadrics " by A. L. Dixon; they are contact quadrics of $\sigma = 0$. Several properties of plane quartics that had just been found by Jolliffe with the help of what Milne stigmatises as " rather intricate analysis " appear in [**39**] as corollaries of Milne's synthetic geometry.

Mention of the plane quartic serves to introduce Milne's main investigations: work which engrossed him and to which he applied his finest talent; the scrutiny of the contact primes (as we now call Milne's hyperplanes) and contact quadrics of the canonical curves of genera 4 and 5. The plane quartic, of genus 3, has 28 bitangents and 63 systems of contact conics. A non-hyperelliptic canonical curve C^p of genus p, of order $2p-2$ in projective space $[p-1]$, has $2^{p-1}(2^p-1)$ contact primes and $2^{2p}-1$ systems of contact quadrics—i.e. of quadrics which touch C^p at $2p-2$ points and do not contain it wholly: thus C^4 has 120 tritangent planes and 255 systems of contact quadrics. The geometry of the 28 bitangents had engaged Steiner, Hesse, Cayley, Aronhold and Salmon; the elegance and symmetry of that figure merited the attention even of these eminent mathematicians. It is but natural to hope that the contact primes of C^p will possess properties at least as fascinating, tempered though the hope may be by the expectation that the greater elaboration of the figure may place well-nigh insuperable obstacles in the way of its unravelment. But Milne was undeterred and took the field with [**35**], a study of C^4 which, as it is the intersection of a unique quadric Γ_2 and a cubic surface, he calls a " quadricubic " curve.

Clebsch and E. Pascal had found properties of the contact quadrics of C^4, but by transcendental methods involving abelian functions, not by " the processes of synthetic geometry". Milne [**35**; p. 374] asserts that "by this application of synthetic methods many new properties are revealed, and the whole configuration is considerably simplified". However similar this may sound to the voice of a salesman crying his wares, it is the literal truth. It is ironical that Milne's main discovery, though not his masterly argumentation, was anticipated by P. Roth in 1911. Among the cubic surfaces through C^4, 255 are four-nodal. The enveloping cone to such a surface G from a point V of itself consists of two quadric cones, each a contact quadric of C^4. As V varies on G the hexads of contacts with C^4 are all those of the contact quadrics of one system. Moreover, both Milne and Roth identify 28 points V for which one of the pair of quadric cones enveloping G touches C^4 at the contacts of a pair of tritangent planes. But Roth's arguments, though far from wholly transcendental, are not purely synthetic either. Milne's reasoning in [**35**], and again in [**36**], where C^4 is specialised so that Γ_2 is a cone, is dazzling in its resource and dexterity; any geometer would take pride in having composed it.

C^5 is the base curve of a net of quadrics in [**4**], and the Jacobian curve ϑ of this net is in birational correspondence with a plane quintic q. Thus Milne, in studying C^5, is led to study q and is able to partition its 2015 contact conics as 1023+992; each of the 1023 is linked with one of the 1023 systems of contact quadrics of C^5 while the 992 consist of 496 pairs, each pair being linked with one of the 496 quadritangent solids of C^5. Milne [**42**; p. 133] had found an analogy in [**4**] of his (and Roth's) discovery in [**3**]: each of the 1023 systems of contact quadrics of C^5 is allied to a five-nodal quartic primal whose nodes are on ϑ and which has C^5 for a nodal curve. He discusses this in [**44**], the whole text, unusually for him, being phrased in terms of the geometry of four dimensions, and there is a detailed account of the contact conics of q in [**51**] where special attention is devoted to the *triadic sets* of three such conics: of three conics, that is, whose 15 contacts with q all lie on the same cubic curve.

By this time Milne's papers were being flanked in our *Proceedings* by those of Baker's Cambridge pupils; Milne was an eager student of the works of these comparative youngsters and, as he testifies in [**48**, **49** and **50**], was thereby instigated to make some contributions himself. But he remained to the end an ardent devotee of the canonical curves. In [**53**] he projects C^4 from a point of itself into a binodal plane quintic: if Cayley's "absolute" conic envelope is taken to be the pair of nodes this quintic is bicircular, and Milne obtains properties of its 120 tritangent circles and of its contact conics. It is perhaps permissible, after the lapse of time, to disclose that the present writer refereed [**54**] in 1937; the paper still glows with the bracing zest and clarity that graced its birth 30 years ago.

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3. "The focal and bi-tangent properties of bi-circular quartics," *Proc. Edinburgh Math. Soc.* (1), 29 (1910), 90–99.

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5. "An easy geometric representation of the sextic covariant of a binary quartic," *Proc. Edinburgh Math. Soc.* (1), 30 (1911), 65–66.

6. " Further investigations of circular cubics and bi-circular quartics," *Proc. Edinburgh Math. Soc.* (1), 30 (1911), 75–88.
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9. "A symmetrical method of apolarly generating cubic curves," *Proc. London Math. Soc.* (2), 10 (1912), 207–210.
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25. " The graphical treatment of power series," *Math. Gaz.*, 9 (1917–19), 198–208.
26. " Mathematics and the pivotal industries," *Math. Gaz.*, 9 (1917–19), 312–316.
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52. " Three tritangent planes of the quadricubic curve," *Proc. London Math. Soc.* (2), 41 (1936), 454–461.
53. " The bicircular quintic curve," *Proc. London Math. Soc.* (2), 44 (1938), 130–139.
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