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OBITUARY

Douglas Geoffrey Northcott, FRS, 1916–2005

Douglas Northcott, who held the Town Trust Chair of Pure Mathematics at the University of Sheffield for 30 years, was an influential figure in the development of commutative algebra in the twentieth century: he received national and international recognition for his work on the algebra underlying the 'pre-Grothendieck' era of algebraic geometry; he was involved in many developments, that nowadays are considered central to modern commutative algebra, in the use of homological algebra as a tool in the study of commutative Noetherian rings; his seven text books have helped many research workers, both beginning and established; and the Northcott–Rees theory of reductions and integral closures of ideals, now more than 50 years old, is still mentioned at most top-level international conferences on commutative algebra.

1. *Life and career*

1.1. *Family background and education*

Douglas Northcott was born Douglas Geoffrey *Robertson*, the son of Geoffrey Douglas Spence Robertson, who was an electrical engineer, and Clara Freda Behl. Geoffrey Robertson was killed in an accident soon after the young Douglas was born; Douglas was about 2 years old when his mother married Arthur Hugh Kynaston Northcott, and he grew up not knowing of his mother's remarriage. It was only in his teens that Douglas learnt that Arthur Northcott was, in fact, his stepfather; Douglas changed his surname by Deed Poll in 1935, and he always felt himself to be part of the Northcott family and referred to Arthur Northcott as his father.

At the time when Douglas Northcott first went to school, his family lived in a very poor area off Gray's Inn Road in London. Until he was about $10\frac{1}{2}$ years of age he attended Laystall Street School, and this gave him an excellent grounding. He was recognized as a good pupil and received individual attention from some of the masters. He was shown how to solve simple simultaneous linear equations and introduced to elementary Euclidean geometry. This almost certainly laid the seeds of his love of mathematics.

In 1927, an opportunity arose for Northcott to be nominated for a 'presentation vacancy' at Christ's Hospital, and he gained admission. The school, which has a fine record in the production of outstanding mathematicians, recognized the young Northcott's mathematical aptitude at an early stage, and he benefited from the teaching of the Senior Mathematics Master, C. J. A. Trimble. The list of distinguished mathematicians who have been elected Fellows of the Royal Society and who were educated at Christ's Hospital includes William Burnside, Philip Hall, and Christopher Zeeman, in addition to Douglas Northcott. (In fact, Douglas Northcott became the Royal Society's representative on the Council of Almoners of Christ's Hospital in 1976, and he found this re-establishment of links with the school most interesting.)

In 1935, Northcott was awarded an open Bayliss Scholarship in Mathematics and entered St. John's College, Cambridge; he was a Wrangler in 1937 and obtained a Distinction in Part III of the Mathematical Tripos in 1938. During Part III, he attended lectures by G. H. Hardy, the Sadleirian Professor of Pure Mathematics responsible for many major contributions to mathematical analysis and number theory. Northcott really enjoyed Hardy's lectures on 'Fourier series' and on 'Divergent series', and through them came to know Hardy quite well. He asked Hardy if he would accept him as a research student. At that time Hardy was already supervising about six research students, but after a little hesitation he agreed to take Northcott; the year 1938–1939 was the first year of this supervision and during that year Northcott published his first paper [**1**]. (The numbering is as in the list of Northcott's publications given at the end of this obituary.) He applied for and was awarded a Commonwealth Fund Fellowship to enable him to study the theory of Banach spaces at the University of Princeton, but on the day that he was due to sail to the US, Britain declared war on Germany.

1.2. *The war years*

Douglas Northcott reported to the Cambridge University Joint Recruiting Board in September 1939 and received a certificate recommending that 'he be held in a pool to be employed in technical services, and that in the meantime he should continue with his mathematical research'. However, Northcott felt unable to proceed in this way, and he joined the Royal Artillery as a volunteer in November 1939. Hardy was very upset when he learnt what Northcott had done and offered to write to his Commanding Officer about his special qualifications. Although Northcott did not accept Hardy's offer, he was touched by his concern.

During Northcott's first overseas posting in India, he contracted the first of a number of illnesses. His complaint was never diagnosed, but his condition was so serious that his parents received a cable to say that he was dangerously ill. While recovering in hospital he began to think about mathematics again, and in particular about a Tauberian theorem connected with Hilbert's double series. The calculations and manipulations were rather involved, but curiously, and perhaps due to the effects of the illness, Northcott was able to do them all in his head. Later, in 1947, the results of this mental agility were published as the second of Northcott's research papers [**2**].

Northcott's regiment was posted to Ipoh in North Malaya. Before the Japanese invasion he contracted malaria. He rejoined the regiment to take part in a rearguard action from Port Swettenham down the Malay peninsular; they were on the beach near the Raffles Hotel when Singapore capitulated.

Northcott was fortunate to escape death on several occasions: some colleagues in his regiment were killed by an air attack, and it was only by chance that he was in a different place; an RAMC doctor operated on him in a make-shift hospital for what turned out to be peritonitis; he endured appalling conditions on board ship while being sent, as part of a contingent of prisoners, to work on an industrial project in Japan; and he somehow survived a camp where nearly all inhabitants had dysentery and beriberi, and were infested with vermin. He believed that the atomic bombs saved his life. It was only well into retirement that he talked much about those years.

In spite of all this there were times, while a prisoner of war, when Northcott was able to think about mathematics; indeed thinking about mathematics probably helped him to survive his war experiences. Sometimes he tried to reconstruct proofs of results which he had learnt as a student; at others he attempted to build up a theory of integration for functions with values in a Banach space. He recorded his results about this theory in a notebook which he kept in his gas-mask case. On one occasion his gas-mask was stolen and he never saw it again, and so he had to start again. His second notebook survived the war and, in due course, provided material for his PhD thesis and his fellowship dissertation.

1.3. *Return to academic life*

At the end of the war, Northcott was able to pick up the threads of academic life, although Hardy had by then retired, and was in fact a very sick man. Hardy disliked shaking hands with people, and would normally keep his hands firmly in his pockets when a general round of hand-shaking seemed imminent. However, in Northcott's case he made an exception, saying 'I suppose this is one time when I ought to shake hands'.

The formal position was that Northcott had completed only one year (1938–1939) as a research student, and so he had to spend a further two before he could submit for a PhD degree. Frank Smithies became his supervisor in place of the now-retired Hardy. Northcott spent those two years attending lectures given by J. E. Littlewood and by reading some of the papers on Banach spaces which had appeared during the war. Also, he developed the material in his army notebook into a dissertation which he planned to submit for a college fellowship, and prepared his second paper [2], in which some applications of Banach space theory apparently went a little way towards shaking Hardy's scepticism about the value of abstract methods in mathematics.

Northcott was allowed to take up the Commonwealth Fund Fellowship in 1946; when he sailed for Princeton (actually on the maiden voyage of the Queen Elizabeth as a passenger liner), he still had the intention of studying Banach spaces. He left his fellowship dissertation with his parents who undertook to post it to St. John's College on the appropriate date.

1.4. *Princeton 1946–1948*

In 1939, many of the mathematicians at Princeton had been actively interested in the comparatively new subject of Banach spaces, but when Northcott arrived in Princeton in 1946 this was no longer the case. He apprehensively attended a seminar, run jointly by Emil Artin and Claude Chevalley, with the title 'Valuation Theory'. To Northcott's surprise and delight he could not only follow the lecture, but he found it extremely enjoyable. At this time, he knew almost nothing of modern algebra and he caused something of a sensation by interrupting the (postgraduate) speaker to ask what was meant by the 'characteristic of a field'. Artin stepped in and explained the concept, and then the lecture proceeded. This was the first of many kindnesses shown to Northcott by Artin, who, during the subsequent months, explained many fundamental algebraic ideas to Northcott, so much so that Northcott became a dedicated algebraist. Much later, Northcott was very proud to be an invited speaker (along with Bartel van der Waerden, Henri Cartan, and Wolfgang Krull) at a memorial meeting in Hamburg in memory of Emil Artin.

In Princeton, Artin directed Northcott's attention to André Weil's famous paper ⟨**20**⟩, and this stimulated Northcott to begin his contributions, some of which are described in § 2.1 below, to the algebra underlying what might be referred to as the 'pre-Grothendieck' era of algebraic geometry. His first papers in this area, [**3**, **4**], were described by Weil (at the beginning of ⟨**22**⟩) as containing 'some interesting new theorems'. During the years that followed, Northcott's expertise in this area developed rapidly.

1.5. *Cambridge 1948–1952*

While in the US, Northcott heard that his application for a Research Fellowship at St. John's College, Cambridge, had been successful, and after 21 months in the US, he returned to England in 1948. The fellowship was initially for a three-year period, but it was subsequently renewed for a further three years. During 1949–1951 he held an Assistant Lectureship at the University of Cambridge, and in 1951 he was made a full Lecturer.

Northcott organized a very successful working seminar in Cambridge on Weil's book ⟨**21**⟩. David Rees, a participant in the seminar, was inspired by it to become a commutative algebraist. In addition, Northcott had become familiar with very recent work of Zariski and Chevalley on geometry. In particular, he had learnt a great deal about issues related to completions of local and semi-local rings from Chevalley's paper ⟨**7**⟩. He also became familiar with the work of F. S. Macaulay as contained in the latter's Cambridge Tract ⟨**14**⟩ published in 1916.

In 1949, Douglas Northcott married Rose Hilda Austin, a charming and vivacious person; their first daughter, Anne Patricia, was born in 1950, and their second, Pamela Rose, in 1952. Among their friends in Cambridge at this time were Frank and Nora Smithies, and David Rees and Joan Cushen (later Mrs David Rees).

The need for more living space than was provided by the Northcotts' small college flat in Cambridge resulted in Douglas's investigation of posts elsewhere, and he was appointed to the Town Trust Chair of Pure Mathematics at the University of Sheffield in 1952.

1.6. *The first years at Sheffield*

A glance at Northcott's list of publications shows that his first 10 years at Sheffield were extraordinarily productive: during that time, he published, on average, more than four papers a year, and also his first two books appeared. He had been encouraged to write the first of these, the Cambridge tract on *Ideal theory* [**15**], in Cambridge by William Hodge. This, on its publication in 1953, was a great success; its five chapters, on (I) The Primary Decomposition, (II) Residue Rings and Rings of Quotients, (III) Some Fundamental Properties of Noetherian Rings, (IV) The Algebraic Theory of Local Rings, and (V) The Analytic Theory of Local Rings, stimulated interest in commutative algebra. Wolfgang Krull, who had made fundamental contributions to the subject (Krull's Principal Ideal Theorem, Krull's Intersection Theorem, Krull dimension, . . .), was impressed by Northcott's book, and encouraged his students to read the new tract. Krull and Northcott became friends, and Northcott was an invited speaker at the meeting which marked Krull's retirement.

1.7. *Sheffield in the 1960s and 1970s*

When Douglas Northcott first joined the staff of the University of Sheffield, there was a single Department of Mathematics; he was the only professor, and the administrative head. Subsequently the department split into four departments, of Pure Mathematics, Applied & Computational Mathematics, Probability & Statistics, and Computer Science. Northcott remained head of a department for 30 years, until his retirement in 1982. Rose's unstinting support helped him to cope with the demands of the headship (and other significant administrative roles, such as that of Dean of the Faculty of Pure Science (in 1958–1961) and Vice-President of the London Mathematical Society (in 1968–1969)); she even looked after babies of members of staff in times of crisis. Even though Douglas served as head for 30 years, he was, with Rose's support, able to make time for writing seven books and about seventy research papers following his PhD. However, the majority of those papers had been written by the early 1960s, and it is probably fair to say that he concentrated more on his writing of books during the last 20 years of his professional life.

His writing is characterized by careful attention to detail; his books were principally aimed at graduate students, but their clarity, detailed discussion of difficult points and reliable accuracy means that they also serve as informative and reassuring references for experienced researchers. While Northcott was writing the books, he would present seminars at Sheffield containing some of the material. Thus, for example, he gave Sheffield seminars on 'Invariants and resolutions' in 1973–1974 prior to the appearance of *Finite free resolutions* [**79**] in 1976, and seminars on 'Affine sets and affine groups' in 1976–1977 while he was preparing [**81**] for publication in 1980. He said that he wrote books to improve his own understanding of their subjects; for example, *Affine sets and affine groups* was motivated by a desire for a greater understanding of the Hochster–Roberts theorem ⟨**12**⟩, that the ring of invariants of a linearly reductive affine linear algebraic group over a field K acting rationally on a regular Noetherian K-algebra is Cohen–Macaulay.

Northcott's seminars, like his books and papers, were always carefully prepared, with meticulous attention to detail. His audience normally consisted of five or so members of staff, usually including P. Vámos, T. B. Cruddis, his former research students A. J. Douglas and D. W. Sharpe, and also, from 1971, the present author, together with two or three research students. Over the years, members of Northcott's staff benefited greatly from his seminars, as well as from the resulting books; for example, the covers of my own copies of *Ideal theory* [**15**], *An introduction to homological algebra* [**46**] and *Lessons on rings, modules and multiplicities* [**65**] are all hanging off through much heavy use. Northcott's own favourite among his books was *Lessons* ... [**65**]. He was delighted when his two Cambridge tracts, *Ideal theory* [**15**] and *Finite free resolutions* [**79**], were reprinted in 2004.

Other notable mathematical events during the later years of Northcott's professional life included a visit to Sheffield by David Eisenbud in the early 1970s, when Northcott showed uncharacteristic excitement about, and enthusiasm for, Eisenbud's joint results with Buchsbaum ⟨**4**–**6**⟩ about exactness of finite complexes of free modules; an LMS–EPSRC Durham Symposium on 'Commutative algebra' in 1981, organized jointly by Northcott and myself, which attracted many distinguished commutative algebraists and at which Hochster gave a masterly series of lectures on the homological conjectures in commutative algebra (see ⟨**18**⟩); and a London Mathematical Society meeting in Sheffield in 1982 in honour of Northcott as his retirement approached.

The retirement of Northcott in 1982 was followed soon afterwards by the retirement of David Rees, and it was subsequently always a source of sadness to me that the remaining 'commutative algebra base' at UK universities was not large enough to support another Durham Symposium on commutative algebra.

2. *Mathematical work*

Although Douglas Northcott began his mathematical research career with work in mathematical analysis, and although his first two papers were in that area, it was his work in commutative algebra, and particularly his work on geometric local rings and the algebra underlying the algebraic geometry of Zariski and Weil, that led to his international recognition. There is a dichotomy, or even a trichotomy, in Northcott's approach to commutative algebra. When he was concerned with applications to geometry, he mainly worked with a *geometric ring over a field* k, that is, a ring obtained from the polynomial ring $k[X_1, \ldots, X_n]$ in n indeterminates by a sequence of operations each consisting of passage to a homomorphic image or formation of a ring of fractions. The second part of the dichotomy concerned Northcott's consideration of how the results of the geometric case are affected when he allowed the rings under consideration to be arbitrary (commutative) Noetherian rings. The final part (of the trichotomy) arose when he dropped the 'Noetherian' hypothesis, and considered behaviour over arbitrary commutative rings.

2.1. *Geometric rings over a field*

On his return to Cambridge in 1948, Northcott vigorously pursued his new-found interest in the local algebra underlying algebraic geometry. Before the end of 1951, he completed five papers [**7**–**10**, **12**] in the area of geometric rings over a field. We here consider this series of papers in some detail, because the last of them, [**12**], resulted in the award to Northcott of the London Mathematical Society's 1953 Junior Berwick Prize.

In the following detailed discussion of these papers, we shall denote the completion of a local or semi-local ring Q by $\widehat{Q}$; thus, the completion of a local ring Q_1^* will be denoted by $\widehat{Q_1^*}$. Additionally, the residue field of a local ring Q will be denoted by κ_Q. Note that, in the case when Q is a geometric local ring over a field k, then κ_Q is a finitely generated extension field of k.

Recall that a local integral domain Q is said to be *analytically unramified* if its completion is reduced, that is, has no non-zero nilpotent element, or, equivalently, if the zero ideal in $\widehat{Q}$ is the intersection of (finitely many) prime ideals, while Q is said to be *analytically irreducible* if $\widehat{Q}$ is an integral domain.

Zariski had proved that a geometric local integral domain Q (over a field k) is analytically unramified. In [**7**], Northcott established an elementary consequence of this work of Zariski, namely that the number of minimal prime ideals of $\widehat{Q}$ is equal to the number of maximal ideals of the integral closure of Q. This complements nicely Zariski's result that if Q above is actually integrally closed, then it is analytically irreducible.

Zariski had also proved earlier, by use of the structure theorems for the completions of geometric local integral domains, that a regular geometric local integral domain Q (over a field k) is a unique factorization domain. In [**8**], Northcott gave a short proof of this result which avoided use of completions. (It must be remembered that the period under discussion here was well before M. Auslander and D. A. Buchsbaum completed the proof ⟨**1**⟩ that every regular local ring is a unique factorization domain.)

For our discussion of [**9**], we introduce some notation.

Notation 2.1.1. We consider an analytically irreducible geometric local integral domain Q (over a field k), with field of fractions F and maximal ideal $\mathfrak{m}$; we also consider a finite extension field F^* of F of degree $[F^* : F] =: n$. We denote by Q^* the integral closure of Q in F^*; note that Q^* is a semi-local ring which is finitely generated as a Q-module. We denote the maximal ideals of Q^* by $\mathfrak{n}_1, \ldots, \mathfrak{n}_h$.

For each $i = 1, \ldots, h$, the localization $(Q^*)_{\mathfrak{n}_i}$ of Q^* at $\mathfrak{n}_i$ can be identified in a natural way with a subring of F^*; the local integral domains $(Q^*)_{\mathfrak{n}_i}$ $(i = 1, \ldots, h)$ are referred to as the *extensions of Q in F^**. Note that

$$Q^* = \bigcap_{i=1}^{h} (Q^*)_{\mathfrak{n}_i}.$$

As the $(Q^*)_{\mathfrak{n}_i}$ $(i = 1, \ldots, h)$ are integrally closed geometric local integral domains over k, they are analytically irreducible. We denote the fields of fractions of the completions $\widehat{Q}$ and $\widehat{(Q^*)_{\mathfrak{n}_i}}$ $(i = 1, \ldots, h)$ by K and K_i^* $(i = 1, \ldots, h)$, respectively.

The completion $\widehat{Q^*}$ of the semi-local ring Q^* need not be an integral domain, but it is reduced; its full ring of fractions K^* is therefore isomorphic to a direct product of fields; in fact, it follows from ⟨**7**, Proposition 8⟩ that $K^* \cong \prod_{i=1}^{h} K_i^*$.

Most of the main new results of [**9**] can be summarized in the following theorem.

THEOREM 2.1.2 (Northcott [**9**, Theorems 4–8]). *We use the notation of* 2.1.1.

(i) *For each $i = 1, \ldots, h$, the topological closure of Q in $\widehat{(Q^*)_{\mathfrak{n}_i}}$ can be identified with the completion $\widehat{Q}$ of the local ring Q, and $\widehat{(Q^*)_{\mathfrak{n}_i}}$ is a finitely generated $\widehat{Q}$-module. Consequently, the degree $[K_i^* : K]$ is finite; this degree, denoted by n_i, is defined to be the* local degree of $(Q^*)_{\mathfrak{n}_i}$ over Q.

(ii) *We have $n = n_1 + \ldots + n_h$.*

(iii) *An element of F is integral over Q if and only if it is integral over $\widehat{Q}$.*

(iv) *An element of F^* is integral over Q if and only if it is integral over $\widehat{(Q^*)_{\mathfrak{n}_i}}$ for all $i = 1, \ldots, h$.*

It should be noted that, in order to prove part (iii) of Theorem 2.1.2, Northcott used a result from [**10**] which showed that, for elements $\theta_1, \ldots, \theta_n \in F$, each prime ideal $\mathfrak{P}$ of $Q[\theta_1, \ldots, \theta_n]$ for which $\mathfrak{P} \cap Q = \mathfrak{m}$ is the contraction to $Q[\theta_1, \ldots, \theta_n]$ of a prime ideal of $\widehat{Q}[\theta_1, \ldots, \theta_n]$.

Paper [**12**] is the most important of the above-mentioned five papers. It depends on the results of [**9**], and so we again employ the notation introduced in 2.1.1, but we here make the additional assumption that Q is integrally closed. (Recall Zariski's result, mentioned above, that an integrally closed geometric local integral domain (over a field k) is automatically analytically irreducible.) An interesting point about [**12**] is that some of the results Northcott obtained in that paper, such as Proposition 2, indicate that $\widehat{Q}$ behaves very much as if it were integrally closed. Northcott wrote [**12**] without knowing that $\widehat{Q}$ is integrally closed; however, by the galley proof stage, a paper ⟨**23**⟩ of Zariski containing a proof that $\widehat{Q}$ is integrally closed had appeared, and Northcott added 'in proof' an additional theorem (described below) that depended on Zariski's result.

THEOREM 2.1.3 (Northcott [**12**, Theorems 9–11]). *Let the situation and notation be as in* 2.1.2. *In addition, let F' be a finite extension field of F^*, and note that, then, the set of extensions of Q in F' is precisely the union of the sets of extensions of the h rings $(Q^*)_{\mathfrak{n}_i}$ $(i = 1, \ldots, h)$ in F'; note also that the extensions of Q in F^* are precisely the intersections with F^* of the extensions of Q in F'.*

(i) *Let Q' be a local geometric subring of F' (over the field k) having F' as its field of fractions. Denote the maximal ideal of Q' by $\mathfrak{m}'$. Then Q' is an extension of Q in F' if the following four conditions are satisfied:*

(a) *$Q \subseteq Q'$;*
(b) *$Q'\mathfrak{m}$ is $\mathfrak{m}'$-primary;*
(c) *$\kappa_{Q'}$ is a finite extension field of κ_Q; and*
(d) *Q' is integrally closed.*

(ii) *Now suppose that Q is a regular local ring and let Q' be an extension of Q in F' that is also regular. Then the local degree of Q' over Q is equal to $[\kappa_{Q'} : \kappa_Q]\lambda$, where λ denotes the length of the Q'-module $Q'/Q'\mathfrak{m}$.*

(iii) *Furthermore, if all the extensions of Q in F' are regular, then the integral closure of Q in F' can be generated, as a Q-module, by $[F' : F]$ elements.*

Northcott pointed out that an important special case of [**12**, Theorem 9] can be translated without difficulty into Zariski's Main Theorem on birational correspondences; he illustrated his theory by detailed consideration of Bezout's Theorem.

The additional theorem that Northcott added to [**12**] at proof stage, after Zariski's result that $\widehat{Q}$ is integrally closed had been published, concerned unramified extensions of Q in F'. In the situation, and with the notation, of Theorem 2.1.3 above, we say that an extension Q' of Q in F', having field of fractions F' and maximal ideal $\mathfrak{m}'$, is *unramified* if $\kappa_{Q'}$ is a separable extension of κ_Q and the local degree of Q' over Q is equal to $[\kappa_{Q'} : \kappa_Q]$; Northcott's result 'added in proof' showed that, if Q' is an extension of Q in F' such that $\kappa_{Q'}$ is a separable extension of κ_Q, then Q' is an unramified extension of Q if and only if $Q'\mathfrak{m} = \mathfrak{m}'$.

As mentioned earlier, the London Mathematical Society awarded Northcott its 1953 Junior Berwick Prize for paper [**12**]. During the remainder of the 1950s, Northcott's reputation as an expert in the algebra underlying the 'pre-Grothendieck' era of algebraic geometry continued to grow: he gave an invited lecture at the Algebraic Geometry Symposium at the 1954 International Congress of Mathematicians in Amsterdam on 'Specialization methods in algebraic geometry', and, when he was elected to Fellowship of the Royal Society in 1961, he was cited as 'the foremost authority in [the UK] on the modern methods introduced by Zariski and Weil into the study of abstract algebraic geometry'.

2.2. *General commutative Noetherian rings and local rings*

Already in the early 1950s, some of Northcott's work was concerned with general (commutative Noetherian) local rings. Paper [**11**] is a brief note that presents a short proof of Krull's result that, if $\mathfrak{a}$ is a proper ideal of a local ring R, and $r \in R$, then $r \in \bigcap_{n=1}^{\infty} \mathfrak{a}^n$ if and only if there exists $a \in \mathfrak{a}$ such that $r = ar$. This has the consequence that, if $0 = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_t$ is a minimal primary decomposition of the zero ideal of R, then

$$\bigcap_{n=1}^{\infty} \mathfrak{a}^n = \bigcap_{\substack{i=1 \\ (1+\mathfrak{a}) \cap \sqrt{\mathfrak{q}_i} = \emptyset}}^{t} \mathfrak{q}_i$$

provides a minimal primary decomposition of $\bigcap_{n=1}^{\infty} \mathfrak{a}^n$.

Paper [**13**] is a straightforward and clear account of the theory of the Hilbert–Samuel polynomial in the geometric case, written just after Samuel's thesis ⟨**17**⟩ had become available.

In [**16**], Northcott gave an alternative proof of Cohen's Theorem ⟨**8**, Theorem 21⟩ that a proper ideal $\mathfrak{a}$ of a regular local ring that can be generated by $\operatorname{ht} \mathfrak{a}$ elements must be unmixed. This presaged Northcott's later interest in the foundations of the theory of Cohen–Macaulay rings.

For many commutative algebraists around the world, the names of 'Northcott' and 'Rees' fit together very naturally, on account of their joint paper [**18**], which, citation indices suggest, is by some distance Douglas Northcott's most cited research paper. In this paper, Northcott and Rees introduced *reductions* of ideals. This concept, and the related concept of *integral closure,* have had a major influence on research in commutative algebra in the more than 50 years since they were introduced; indeed, there have been frequent mentions of them at top-level international conferences in commutative algebra in the early years of the 21st century, and Irena Swanson's and Craig Huneke's book ⟨**19**⟩ *Integral closures of ideals, rings and modules* has recently appeared in the London Mathematical Society's Lecture Note Series.

Let R be a commutative Noetherian ring (with identity), and let $\mathfrak{b}$ and $\mathfrak{a}$ be proper ideals of R. The ideal $\mathfrak{b}$ is said to be a *reduction of* $\mathfrak{a}$ precisely when $\mathfrak{b} \subseteq \mathfrak{a}$ and there exists $s \in \mathbb{N}_0$ (the set of non-negative integers) such that $\mathfrak{b}\mathfrak{a}^s = \mathfrak{a}^{s+1}$; then the least such s is called the *reduction number of* $\mathfrak{a}$ *with respect to* $\mathfrak{b}$. One can view such a $\mathfrak{b}$ as an approximation to $\mathfrak{a}$ which nevertheless retains some of the properties of $\mathfrak{a}$: for example, a prime ideal $\mathfrak{p}$ of R is a minimal prime ideal of $\mathfrak{a}$ if and only if it is a minimal prime ideal of $\mathfrak{b}$, and when that is the case, the multiplicity of $\mathfrak{a}$ corresponding to $\mathfrak{p}$ is equal to the multiplicity of $\mathfrak{b}$ corresponding to $\mathfrak{p}$.

The ideal $\mathfrak{b}$ is said to be a *minimal reduction of* $\mathfrak{a}$ if and only if $\mathfrak{b}$ is a reduction of $\mathfrak{a}$ and there is no reduction $\mathfrak{c}$ of $\mathfrak{a}$ with $\mathfrak{c} \subset \mathfrak{b}$ (the symbol '$\subset$' is reserved to denote strict inclusion). Also, an ideal which has no reduction other than itself is called a *basic ideal*.

Most of [**18**] is written under the hypothesis that R is a local ring Q with infinite residue field, and so that hypothesis will be in force until further notice; also $\mathfrak{m}$ will denote the maximal ideal of Q. Northcott and Rees defined the *analytic spread* $\ell(\mathfrak{a})$ of $\mathfrak{a}$; this turns out to be equal to the dimension of $\mathcal{G}(\mathfrak{a})/\mathfrak{m}\mathcal{G}(\mathfrak{a})$, where $\mathcal{G}(\mathfrak{a})$ denotes the associated graded ring $\bigoplus_{i \in \mathbb{N}_0} \mathfrak{a}^i/\mathfrak{a}^{i+1}$ of $\mathfrak{a}$. They proved that every reduction of $\mathfrak{a}$ requires at least $\ell(\mathfrak{a})$ generators, that a reduction of $\mathfrak{a}$ is a minimal reduction of $\mathfrak{a}$ if and only if it can be generated by $\ell(\mathfrak{a})$ elements, and that each reduction of $\mathfrak{a}$ contains a minimal reduction of $\mathfrak{a}$. Thus all minimal generating sets of all minimal reductions of $\mathfrak{a}$ have exactly $\ell(\mathfrak{a})$ elements.

They went on to show that $\ell(\mathfrak{a})$ also admits the following interpretation. Elements $v_1, \ldots, v_t \in \mathfrak{a}$ are said to be *analytically independent in* $\mathfrak{a}$ if and only if, whenever $h \in \mathbb{N}$ (the set of positive integers) and $f \in R[X_1, \ldots, X_t]$ (the ring of polynomials over R in t indeterminates) is a homogeneous polynomial of degree h such that $f(v_1, \ldots, v_t) \in \mathfrak{a}^h\mathfrak{m}$, then all the coefficients of f lie in $\mathfrak{m}$. Then if $\mathfrak{b}$ is a reduction of $\mathfrak{a}$, $\dim_{Q/\mathfrak{m}}(\mathfrak{b}/\mathfrak{m}\mathfrak{b}) =: t$ and $\{v_1, \ldots, v_t\}$ is a minimal generating set for $\mathfrak{b}$, it turns out that $\mathfrak{b}$ is a minimal reduction of $\mathfrak{a}$ if and only if $v_1, \ldots, v_t$ are analytically independent in $\mathfrak{a}$. Consequently, $\ell(\mathfrak{a})$ is equal to the largest number of elements of $\mathfrak{a}$ that are analytically independent in $\mathfrak{a}$, and

$$\operatorname{ht} \mathfrak{a} \leqslant \ell(\mathfrak{a}) \leqslant \dim_{Q/\mathfrak{m}}(\mathfrak{a}/\mathfrak{m}\mathfrak{a}).$$

Also in [**18**], Northcott and Rees established the fundamental connections between reductions and integral closures. The relevant work in [**18**] was developed for ideals in the local ring Q with infinite residue field, but that hypothesis is unnecessarily restrictive, and so we now return to the general commutative Noetherian ring R. We say that $r \in R$ is *integrally dependent on* the ideal $\mathfrak{b}$ of R if and only if there exist $n \in \mathbb{N}$ and $c_1, \ldots, c_n \in R$ with $c_i \in \mathfrak{b}^i$ for $i = 1, \ldots, n$ such that

$$r^n + c_1 r^{n-1} + \ldots + c_{n-1} r + c_n = 0.$$

(In fact, in [**18**] Northcott and Rees eschewed the terminology 'integrally dependent' that had been used by earlier authors in favour of 'analytically dependent'; however, the terminology 'integrally dependent' became standard.)

Let $\mathfrak{b} \subseteq \mathfrak{a}$ be ideals of R. Then $\mathfrak{b}$ is a reduction of $\mathfrak{a}$ if and only if each element of $\mathfrak{a}$ is integrally dependent on $\mathfrak{b}$. Furthermore, the set $\mathcal{I}$ of all ideals of R which have $\mathfrak{b}$ as a reduction has a unique maximal member, $\overline{\mathfrak{b}}$ say: $\overline{\mathfrak{b}}$ is the union of the members of $\mathcal{I}$, and this ideal $\overline{\mathfrak{b}}$ is precisely the set of all elements of R which are integrally dependent on $\mathfrak{b}$. The ideal $\overline{\mathfrak{b}}$ is called the *integral closure of* $\mathfrak{b}$; it has the property that the ideals of R which have $\mathfrak{b}$ as a reduction are precisely those between $\mathfrak{b}$ and $\overline{\mathfrak{b}}$.

As mentioned above, the appearances in the literature of the concepts of reduction and integral closure in the half-century since Northcott and Rees published [**18**] are manifold. One enduring reason for this is provided by the links between integral closures and the theory of tight closure introduced by M. Hochster and C. Huneke in ⟨**11**⟩. Hochster and Huneke were able to use their new theory to give a short proof (in the case where the local ring concerned has prime characteristic) of a generalization (see ⟨**11**, Theorem 5.4⟩) of the following result of J. Lipman and A. Sathaye, which is itself a generalization of a result of J. Briançon and H. Skoda ⟨**3**⟩ about the ring of convergent power series $\mathbb{C}\{Z_1, \ldots, Z_n\}$ in n indeterminates $Z_1, \ldots, Z_n$.

THEOREM 2.2.1 (J. Lipman and A. Sathaye ⟨**13**⟩). *Let Q be a regular local ring, and let $\mathfrak{a}$ be a proper ideal of Q that can be generated by t elements. Then $\overline{\mathfrak{a}^{n+t}} \subseteq \mathfrak{a}^{n+1}$ for all $n \in \mathbb{N}_0$.*

However, some of the ongoing uses of reductions of ideals in commutative algebra concern avenues of research that were initiated by Northcott himself. One such concerns the coefficients of the Hilbert function of an $\mathfrak{m}$-primary ideal $\mathfrak{q}$ in a d-dimensional local ring Q, where $\mathfrak{m}$ denotes the maximal ideal of Q and $d \geqslant 1$. For large values of the integer n, the length $L_Q(Q/\mathfrak{q}^n)$ behaves like a polynomial in n; indeed, there exist integers $e_0(\mathfrak{q}), \ldots, e_d(\mathfrak{q})$, called the *normalized Hilbert coefficients of* $\mathfrak{q}$, such that

$$L_Q(Q/\mathfrak{q}^n) = \sum_{i=0}^{d} (-1)^i e_i(\mathfrak{q}) \binom{n+d-i-1}{d-i} \quad \text{for all large } n.$$

The positive integer $e_0(\mathfrak{q})$ is the *multiplicity of* $\mathfrak{q}$. In [**43**], Northcott used reductions to prove that, when Q is Cohen–Macaulay, the $\mathfrak{m}$-primary ideal $\mathfrak{q}$ can be generated by a system of parameters if and only if $e_1(\mathfrak{q}) = 0$, and, when that is the case, $e_2(\mathfrak{q}) = \ldots = e_d(\mathfrak{q}) = 0$ also. Over the years since the appearance of [**43**], numerous mathematicians have published papers that build on this work of Northcott, and the concept of reduction number appears in quite a few of them.

2.3. *Developments facilitated by homological algebra*

The late 1950s saw dramatic developments in commutative algebra, many of which arose from the use of homological algebra as an effective tool for the study of commutative rings. Northcott's papers from the 1950s and early 1960s show that he was also involved, to a greater or lesser extent, in many fundamental developments where homological algebra can play a valuable rôle. For example, two further joint papers by Northcott and Rees, [**34**, **36**], presented arguments very relevant to the development of the theories of Cohen–Macaulay rings and Gorenstein rings.

Rees had introduced the concept of grade in ⟨**16**⟩: in that paper he showed that all maximal regular sequences in a proper ideal $\mathfrak{a}$ of a commutative Noetherian ring R have the same length, and called their common length the *grade of* $\mathfrak{a}$. He achieved this result, which has turned out to be absolutely fundamental in commutative algebra, by a pioneering use of his application of homological algebra in ⟨**15**⟩, for he showed that grade $\mathfrak{a}$ is equal to the least integer i such that $\operatorname{Ext}^i_R(R/\mathfrak{a}, R) \neq 0$. In [**34**], Northcott and Rees provided an elementary approach to the theory of grade that avoids the use of homological algebra. That paper also contains a systematic approach to the theory of semi-regular rings (nowadays called Cohen–Macaulay rings in recognition of seminal works by F. S. Macaulay ⟨**14**⟩ and I. S. Cohen ⟨**8**⟩), a theme that had interested Northcott earlier (see [**16**, **30**]), and was to interest him later (see [**44**]).

Northcott and Rees also contributed to the basic theory of Gorenstein rings, because the last of their joint papers, [**36**], contains the theorem that a local ring in which every ideal generated by a system of parameters is irreducible must be Cohen–Macaulay, and this theorem was an important ingredient in H. Bass's characterization of Gorenstein local rings in his seminal 'ubiquity' paper ⟨**2**⟩.

In [**35**], Northcott provided a short and elementary proof of the (already known) fact that the polynomial ring $K[X_1, \ldots, X_n]$ in n indeterminates with coefficients in a field K has global dimension n. In [**38**], he again studied a polynomial ring $S := R[X_1, \ldots, X_n]$, but in this paper R is only assumed to be a commutative ring; his main result in [**38**] is that, if $\mathfrak{P} \in \operatorname{Spec}(S)$ and $\mathfrak{p} := \mathfrak{P} \cap R$, then $S_{\mathfrak{P}}$ is a regular local ring if and only if $R_{\mathfrak{p}}$ is a regular local ring. Nowadays, one has the standard theory of fibre rings of flat ring homomorphisms with which to approach such questions.

Another example of Northcott's work in this area is provided by his paper [**33**], where he showed that, for a d-dimensional Cohen–Macaulay local ring Q, there exists a positive integer r such that every decomposition, as an irredundant intersection of irreducible ideals, of each ideal of Q that can be generated by a system of parameters has precisely r terms. Although Northcott's proof did not use homological algebra, nowadays the integer r is called the *type of* Q, and is known to be equal to the dth Bass number of Q with respect to its maximal ideal $\mathfrak{m}$, that is, the dimension of $\operatorname{Ext}^d_Q(Q/\mathfrak{m}, Q)$ as a vector space over the residue field of Q.

2.4. *The Eagon–Northcott complex and generic perfection*

A glance at Northcott's full list of publications shows that he was involved in comparatively few collaborations. However, some of them deserve mention in this obituary. The collaboration between Northcott and David Rees has already been described in some detail in § 2.2 and § 2.3 above. Also noteworthy is the collaboration between Northcott and J. A. ('Jack') Eagon that resulted from substantial visits made by Eagon to Sheffield, the last of which was in 1972–1973. One result of this collaboration was the famous Eagon–Northcott complex, which was presented in [**53**] and which is described as follows.

Let R be a commutative ring (with identity), and let

$$A := \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1r} \\ a_{21} & a_{22} & \ldots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1} & a_{s2} & \ldots & a_{sr} \end{bmatrix}$$

be an $s \times r$ matrix with entries in R, where $s \leqslant r$. Let I be the ideal of R generated by the $s \times s$ minors of A.

Let K be the exterior algebra over R generated by $X_1, X_2, \ldots, X_r$. Then, for each integer k with $1 \leqslant k \leqslant s$, the kth row of A determines a differentiation Δ_k on K for which

$$\Delta_k(X_{i_1} X_{i_2} \ldots X_{i_n}) = \sum_{\mu=1}^{n} (-1)^{\mu+1} a_{k i_\mu} X_{i_1} \ldots \widehat{X_{i_\mu}} \ldots X_{i_n}$$

for all choices of n distinct integers $i_1, \ldots, i_n$ from $\{1, 2, \ldots, r\}$, where $\widehat{X_{i_\mu}}$ indicates omission of X_{i_μ}.

Next, let $Y_1, Y_2, \ldots, Y_s$ be s new indeterminates, and, in the ring $R[Y_1, Y_2, \ldots, Y_s]$ of polynomials, let Φ_n be the R-module consisting of all forms of (total) degree n. The Eagon–Northcott complex R^A has the form

$$\cdots \longrightarrow R^A_{r-s+1} \xrightarrow{d_{r-s}} R^A_{r-s} \longrightarrow \cdots \longrightarrow R^A_1 \xrightarrow{d_0} R^A_0 = R \longrightarrow 0 \longrightarrow \cdots,$$

where $R^A_{q+1} = K_{s+q} \otimes_R \Phi_q$ for each $q = 0, 1, \ldots, r-s$, and, for $q > 0$, whenever $i_1, \ldots, i_{s+q}$ are integers such that $1 \leqslant i_1 < \ldots < i_{s+q} \leqslant r$ and $v_1, \ldots, v_s$ are non-negative integers such that $v_1 + \ldots + v_s = q$, then

$$d_q(X_{i_1} \ldots X_{i_{s+q}} \otimes Y_1^{v_1} \ldots Y_s^{v_s}) = \sum_{\substack{j=1 \\ v_j > 0}}^{s} \Delta_j(X_{i_1} \ldots X_{i_{s+q}}) \otimes Y_1^{v_1} \ldots Y_j^{v_j - 1} \ldots Y_s^{v_s};$$

for $q = 0$,

$$d_0(X_{i_1} \dots X_{i_s} \otimes 1) = \det \begin{bmatrix} a_{1i_1} & a_{1i_2} & \dots & a_{1i_s} \\ a_{2i_1} & a_{2i_2} & \dots & a_{2i_s} \\ \vdots & \vdots & \ddots & \vdots \\ a_{si_1} & a_{si_2} & \dots & a_{si_s} \end{bmatrix}.$$

When the original matrix A has only a single row, the Eagon–Northcott complex R^A is, essentially, the Koszul complex generated by that row. Among the applications of this complex is the elegant result that, when R is Noetherian and I is proper, then $\operatorname{grade} I \leqslant r - s + 1$, and, when $\operatorname{grade} I = r - s + 1$, then the R-module R/I has projective dimension $r - s + 1$ and the Eagon–Northcott complex R^A provides a free resolution of R/I of shortest possible length.

The work in the area of the Eagon–Northcott complex yielded an example of a 'generically perfect ideal', in the terminology of [**64**], also a joint paper by Eagon and Northcott. In fact, generically perfect ideals and modules, and 'strongly generically perfect' modules formed a recurring theme in several of Northcott's papers from the 1960s and early 1970s, including [**64**, **66**, **69**]. Let Λ be a commutative Noetherian ring, and let $S := \Lambda[X_1, \dots, X_n]$, where $X_1, \dots, X_n$ are indeterminates over Λ. Let M be a non-zero Noetherian S-module. If R is any (commutative) Noetherian Λ-algebra, then $R \otimes_\Lambda M$ becomes a module over $R \otimes_\Lambda S$ (which is naturally isomorphic to $R[X_1, \dots, X_n]$) in an obvious way. We use 'pd' to denote projective dimension. We say that M is *strongly generically perfect over* Λ if M is flat over Λ, and there is an integer g such that, for every commutative Noetherian Λ-algebra R such that $R \otimes_\Lambda M \neq 0$, we have

$$\operatorname{grade}(0 :_{R[X_1,\dots,X_n]} R \otimes_\Lambda M) = \operatorname{pd}_{R[X_1,\dots,X_n]} R \otimes_\Lambda M = g.$$

We say that M is merely *generically perfect over* Λ if M is perfect, that is $\operatorname{grade}_S M = \operatorname{pd}_S M$, and flat over Λ. Melvin Hochster pursued this line of investigation and showed in ⟨**9**⟩ that if M is generically perfect over Λ, then it is strongly generically perfect over Λ.

Subsequently, in ⟨**10**⟩, Hochster extended results from this theory to non-Noetherian situations; to do so, he developed a theory of grade for non-Noetherian rings that led to Northcott's description of 'polynomial grade' in Chapter 5 of his book *Finite free resolutions* [**79**].

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Prior to his death, Douglas Northcott had prepared some historical notes about his early life and his experiences during the Second World War, and he left instructions for these to be forwarded to the Royal Society after his death. I am very grateful to the Royal Society, and to Douglas Northcott's daughter Pamela Raynor, for allowing me to use these notes.

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Rodney Y. Sharp
Department of Pure Mathematics
University of Sheffield
Hicks Building
Sheffield
S3 7RH
United Kingdom

R.Y.Sharp@sheffield.ac.uk