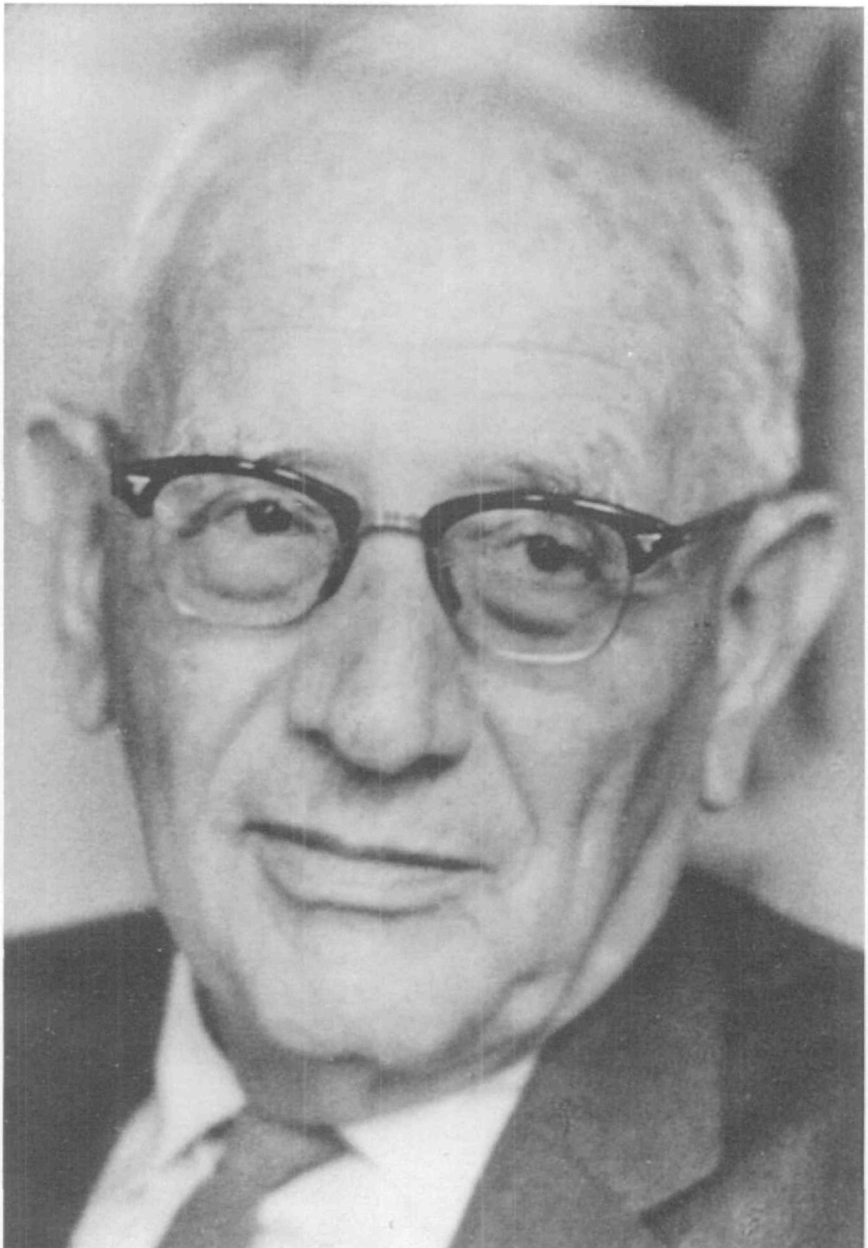

G. PÓLYA, 1887–1985

George Pólya, on an outing of the ETH, Zürich, in the mid 1930s

OBITUARY

GEORGE PÓLYA

His life

George (György) Pólya was born in Budapest on 13 December 1887, the son of Jakab (1844–1897) and Anna (née Deutsch; 1853–1939) Pólya. Jakab was a lawyer in the Budapest office of the Assicurazioni Generali of Trieste, a large international insurance firm. Before joining the company he had had a private law practice. His real interests, however, lay in economics and statistics and he continued to study these on the side, in the hope of obtaining an academic position in economics, so that he could devote full time to research. To help in obtaining such a position, in 1882 he changed the family name from Pollák to the rustic Hungarian Pólya (pronounced pō'yaw, or, more commonly in English, pōl'yaw). While earning a living in the law, he wrote a number of books and pamphlets, and he learned languages as well, including enough English to translate Adam Smith's *Wealth of Nations* into Hungarian, a translation used as a text in Hungarian schools for decades. Shortly before his untimely death from a heart attack, he succeeded in becoming Privatdozent at the University of Budapest.

Pólya's mother came from a family claiming residence in Buda since the 16th century. The family included several professional people; one, a professor of botany, was a contemporary of Pólya's mother.

George was 10 when his father died and left his wife, George, and four other children, two boys and two girls. Another girl had died in infancy. Pólya's older brother by 11 years, Eugene (Jenő), studied medicine and became a famous surgeon and professor of surgery at the University. He was eminent in his field, having developed a technique of stomach surgery that bears his name, but he loved mathematics and regretted having chosen medicine over mathematics as a career. After their father's death in 1897, the two sisters, Ilona and Flóra, had to go to work for the insurance firm in order to help support the family. Much later, the younger brother, László, was killed in the First World War. A few of the descendants of these have gone on to study mathematics seriously.

Pólya's mother urged him to take up his father's profession so he tried studying law at the University of Budapest, where he enrolled in 1905, but he found it boring, so he continued in the subject for only one semester. From law he moved to languages and literature for two years and passed an examination for a teacher's certificate that allowed him to teach Latin and Hungarian in the lower grades of the gymnasium, a certificate he never used. Fond of literature—he translated the poetry of Heine into Hungarian while still in the gymnasium—he was also interested in philosophy. Fortunately for mathematics one of his philosophy professors convinced him that the study of mathematics and physics would help in his understanding of philosophy. He therefore began a serious study of mathematics, a subject that had not interested him in his earlier schooling. His interest in physics was shared by his good friend at the University, the eminent physical chemist and, later, social scientist, Michael Polányi.

In light of Pólya's later work in problem solving it is interesting that he was not particularly attracted to the famous Eötvös Competition that had inspired so many of his contemporaries. When asked to take the contest Pólya neglected even to turn in his paper.

At the University of Budapest his physics professor was the well-known physicist, Loránd Eötvös, but the professor who was to influence him most was the mathematician Leopold (Lipót) Fejér. Fejér had an appealing style and personality; his wit and charming conversations with his students—often in coffee houses—attracted many to his circle.

So Pólya was drawn into mathematics and finally abandoned physics and philosophy. In later years he explained this by saying: 'I thought I am not good enough for physics and I am too good for philosophy. Mathematics is in between.' [1]*

At the University Pólya was one of the founders, along with Karl Polányi (Michael Polányi's older brother), of a student society called the Galileo Circle (Galilei Kör). Their slogan was 'Do not put religious, social, or political considerations above scientific research and speculation.' The Circle sponsored lectures, opened reading rooms and published a journal for young scholars. Pólya lectured to the Circle on Mach. They saw themselves as socialists, but they were not very revolutionary. And though they heard lectures on Karl Marx and other thinkers not approved of completely by the government, Pólya remembered the meetings as being dull and innocuous.

Pólya spent the 1910–11 academic year at the University of Vienna, earning money by tutoring the son of a local nobleman, and in 1912 he returned to Budapest where he received his doctorate in mathematics. The years 1912–13 were spent in Göttingen, where he met Klein (then just retired), Hilbert, Runge, Landau, Weyl (later to be a colleague in Zürich), Hecke, Courant, and Toeplitz. Originally scheduled to take an appointment at Frankfurt, he went to Paris at the beginning of 1914, and after a short stay there, he accepted, instead, a position at the Eidgenössische Technische Hochschule (Swiss Federal Institute of Technology) in Zürich, an appointment arranged by Adolf Hurwitz. His relationship with Hurwitz was short—Hurwitz died in 1919—but very close. Pólya felt that his mathematical taste was much influenced by Hurwitz, and he was fond of describing their talks about mathematics, often on walks near the ETH. He wrote only one joint paper with Hurwitz [1916, 1], really two separate papers joined together, but after Hurwitz's death Pólya edited his collected papers. He also credited Hurwitz's extensive mathematical diaries for a number of problems that later appeared in the *Aufgaben und Lehrsätze*. These diaries may have influenced Pólya in other ways: he kept a mathematical log himself throughout much of his career, recording mathematical conversations and keeping track of his own mathematical ideas.

The department head at the ETH when Pólya arrived was Arthur Hirsch. His other colleagues there included Franel, Geiser and, later, Plancherel, Kollros, Gonseth and Hermann Weyl. Even later, Bernays came to the ETH from Göttingen.

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* This tribute to George Pólya consists of a series of essays, both biographical and scientific. Each part has a list of references at the end and single digits in brackets refer to entries in the list of references for each short essay. References of the form [date, number] are to works of Pólya and appear in the Pólya bibliography given at the end of the whole series.

With Plancherel he had a close working relationship, and one joint paper of theirs on Fourier integrals [1937, 2] is considered quite important, being often cited in subsequent work on the subject. He had less in common with Weyl—Weyl was more concerned with vast generalizations than Pólya—but they were quite close personally. He wrote two papers to honor Weyl [1961, 2] and [1972, 1], though Pólya claimed that mathematically they never quite understood each other. Pólya shared with the logician, Bernays, an interest in problem solving and maintained close contact with him until Bernays' death in 1977.

The after-effects of a leg infection, following a soccer injury during his student days, kept Pólya out of the Hungarian army during the early years of the First World War. Later, when the need for soldiers became more urgent, Pólya, having fallen under the influence of the outspoken pacificist, Bertrand Russell, refused to return to Hungary and serve. Because of this he was unable to return to his homeland for many years following the First World War.

Pólya became a Swiss citizen and in 1918 he married Stella Vera Weber, the only surviving daughter of Robert Weber, professor of physics at the University of Neuchâtel. Her sisters had died of tuberculosis and Mrs. Pólya herself spent a year in Davos, a well-known center for the treatment of respiratory diseases. At the time of their meeting, Weber, who had suffered a serious stroke, was staying with his daughter at the Hotel Zürichberg in Zürich. It was Pólya's willingness to play dominos with her father that convinced the daughter that the young Pólya must be a very kind man. They had a long and happy marriage—married for 67 years at the time of Pólya's death.

Mrs. Pólya grew up in Neuchâtel, in French-speaking Switzerland, so French was always spoken in their home. Pólya's native language was, of course, Hungarian, but he spoke German in Zürich. Later he learned English, published mathematical papers in German, English, French, Hungarian, Danish and Italian, and read other languages with varying degrees of ease. He had learned Latin and Greek in school, so he could recite long passages of Virgil and Homer in the original. His favorite author for quoting, however, was Dante, in Italian. His early interest in languages never left him.

At the ETH Pólya rose through the ranks from Privatdozent in 1914 to Titularprofessor in 1920 and finally to Professor Ordinarius in 1928. In 1924 as the first international Rockefeller Fellow, he spent a year in England in order to work with G. H. Hardy, first at New College, Oxford, then at Trinity College, Cambridge. It was Hardy who had recommended Pólya when asked by the Rockefeller administrators to suggest a recipient for the grant. During that time work was begun on the classic book written with Hardy and J. E. Littlewood, *Inequalities*, published by Cambridge University Press in 1934. Intended originally for the Cambridge Tracts series, the manuscript quickly outgrew that format, running to more than 300 pages. After more than fifty years it is still a standard reference, and specific inequalities proved there have inspired numerous subsequent papers. On being asked many years later about his discovery of his very ingenious proof of the arithmetic–geometric means inequality that uses the first two terms of the Maclaurin series for e^x (Theorem 9 in the book, proof in Section 4.2), Pólya replied, 'It's the best mathematics I ever dreamt.' He apparently did dream about mathematics and even wrote of this in [1970, 2].

While in England he also met A. E. Ingham with whom he was to collaborate later. Hardy was at that time trying to reform the Tripos, viewing the problems on the examination as out-of-date and largely irrelevant to modern mathematics. In order to demonstrate his thesis, he arranged for Pólya to take the examination, expecting him to do badly, thereby showing that this kind of mathematics was not what an active research mathematician from the Continent would do well. He miscalculated. If Pólya had been officially entered, it is reported, he would have been named Senior Wrangler.

In 1923 Pólya and his slightly younger fellow countryman, Gábor Szegő, whom he had known in Budapest, started their long collaboration by signing a contract with Springer for the book that was to be the most acclaimed contribution of each of them, the masterful collection of problems called the *Aufgaben und Lehrsätze aus der Analysis*. Szegő had received his Ph.D. from the University of Vienna in 1918, six years following Pólya's, and had then been appointed Privatdozent in Berlin in 1921. Pólya had an idea for a problem book in analysis but realized he would need help in the sort of undertaking he had in mind, so he proposed to Szegő a collaboration. Pólya once remarked that the two volumes of the *Aufgaben und Lehrsätze* were the result of a true collaboration—each knew certain areas the other did not know. This was a revealing comment because Szegő's early career was closely tied to Pólya's work. Szegő's first publication in an international journal was the solution of a problem posed by Pólya in 1913 [**8**], and his first research paper was on a question posed by Pólya involving the determinant of a Toeplitz matrix [**10**]. This work led to investigations in orthogonal polynomials, the subject of Szegő's best known research.

The early years in Zürich were almost unbelievably productive for Pólya. During this time he published a large number of papers—up to a dozen each year. But even more impressive than the quantity is the amazing range. In 1918 he published papers on series, number theory, combinatorics, and on voting systems. The next year there were again papers in analysis, number theory and voting systems, but also in astronomy, probability and pedagogy. All of this was achieved at a time when he was doing some of his most profound work in the study of integral functions. And in part of this period, he and Szegő were assembling and writing the *Aufgaben und Lehrsätze*.

The importance of this two-volume set was not only in the range and depth of the problems contained in it—there must be hundreds if not thousands of citations of the work in the literature—it was also in the organization. The problems were grouped not by subject but by method of solution. This was a novelty. The concentration required to put this together resulted in a period of intense strain for Pólya. His wife even today recalls her concern for his health over those years. But major mathematical achievements were made, and in the *Aufgaben und Lehrsätze* Pólya and Szegő produced a mathematical masterpiece that assured their reputations.

During the 1930s Pólya worked closely with Gaston Julia on a series of problems, and this meant regular trips to Paris. Early on, in his 1914 Paris visit, he had encountered Picard and Hadamard, with whom he shared common interests. On one of his visits to Paris in the 1930s Pólya spoke at the jubilee of Hadamard about Hadamard's studies of the relationship between coefficients in an expansion of a function and the properties of the function. Jerzy Neyman recalled much later that when he heard this lecture he 'formed the impression that Pólya's work was somehow inspired by that of Hadamard. Now after having inspected Pólya's collected papers, [he] was a little dubious. Could it have been that the situation was the reverse, with Pólya's results inspiring an effort by Hadamard?'

In 1933 Pólya was again selected for a Rockefeller grant, this time to visit

Princeton. Though there were no mathematicians at Princeton with whom he worked closely, he did discuss problems with Oswald Veblen and met with other mathematicians on the East Coast, including G. D. Birkhoff. Veblen had been highly recommended by Hardy, who told Pólya that Veblen had 'all the American, all the British, all the Scandinavian virtues.' That summer he visited Stanford, at the invitation of H. F. Blichfeldt, the department head at that time. A few years later, in 1936, Szegő, who had gone from Berlin to Königsberg, was forced by the Nazi regime to leave Germany. He went first to Washington University, St. Louis, and then to Stanford, where he became department head. This turned out to be relevant to Pólya's own career.

On their first trip to Stanford in 1933 the Pólyas decided they liked California, Stanford in particular. The situation in Europe became ever more threatening, so in 1940 the Pólyas left Switzerland by way of Lisbon. For the first two years in the United States Pólya had a position at Brown University, and then, after a short stay at Smith College, he left in 1942 for Stanford. That fall Pólya and Szegő were reunited and soon resumed their collaboration, this time on a series of papers on mathematical physics and on a second book, the *Isoperimetric inequalities in mathematical physics*, published as part of the Annals of Mathematics Studies by the Princeton University Press [1951, 2].

At Stanford his colleagues in the Mathematics Department included, in addition to Szegő, the mathematicians Blichfeldt, J. V. Uspensky, and W. A. Manning. These were soon to be joined by Menahem Schiffer, Stefan Bergman, and Charles Loewner, with whom Pólya had many shared mathematical interests.

With his arrival in America Pólya started what was, in a sense, a new career outside the bounds of purely mathematical research. Shortly before coming to America Pólya had written in German a draft of a book on problem solving. When he later rewrote it in English and told Hardy that he planned to call it *How to solve it*, Hardy remarked, 'It is appropriate that you go to America. It is the country of "How to" books.' Pólya went to four publishers before finding one willing to publish it, Princeton University Press, which brought it out in 1945. Later Doubleday published it in paperback and it enjoyed enormous success. Again available from Princeton, both in hardcover and paperback, *How to solve it* has never been out of print since 1945 and has sold well over 1,000,000 copies. It has been translated into 17 languages, quite possibly a record for a modern mathematics book. It was followed in 1954 and 1955 by the two-volume *Mathematics and plausible reasoning* and in 1962 and 1965 by *Mathematical discovery*, again in two volumes.

These later works were not evidence of a newly found interest in teaching and in heuristics. The organization of the *Aufgaben und Lehrsätze* showed this interest, and he had written earlier articles on problem solving. But in the 1950s he devoted considerable time to writing and lecturing on the subject, becoming a prominent voice in mathematics education in the United States and, indeed, in the world. He became emeritus professor at Stanford in 1953 but continued to teach and write. From 1955 to 1974 he taught well over one thousand high school and college teachers in a series of institutes at Stanford supported by grants from General Electric, Shell and the National Science Foundation. In the summer of 1964 he taught at an NSF-supported program in Versoix, Switzerland.

In 1946 Szegő and Pólya started a high school competition at Stanford that was modelled on the Eötvös Contest in Hungary. By the time the contest was discontinued in 1965—the Stanford department had decided to shift its emphasis to graduate teaching [1974, 2; p. 2]—the contest was being taken annually by 1200 participants in 151 centers in the western states. Pólya and J. Kilpatrick later gathered together the problems and solutions from the Stanford (later Stanford–Sylvania) Contest and these were published as a collection in [1974, 2].

Shortly after the Second World War Pólya joined Tibor Radó and Irving Kaplansky on a committee to make up the first post-war Putnam Examination. Pólya was a member of the Putnam Prize Committee from 1948 to 1950. He also served the Mathematical Association of America as Northern California Chairman in 1947 and as a member of the Board of Governors from 1958 to 1960.

At the ETH Pólya's Ph.D. students were Alfred Aeppli (1924), Eduard Benz (1935), Ernst Boller (1932), Albert Edrei (1939), Florian Eggenberger (1924), Fritz Gassmann (1926), Gottfried Grimm (1932), Reinwald Jungen (1931), Hermann Muggli (1938), Albert Pfluger (1935), Alice Roth (1938), Walter Saxer (1923), Emil Schwengeler (1925) and August Stoll (1930). Of these, Saxer and Pfluger later joined the ETH faculty. He also advised in the role of 'Korreferent' the dissertations of Hans Albert Einstein (1936), Victor Junod (1933), Wilhelm Machler (1932), Hans Arthur Meyer (1934), Egon Moecklin (1934), Hans Odermatt (1926) and James J. Stoker (1936). In addition, he assisted in the advising of the Ph.D. dissertations of Egon Lindwart (with Landau; 1914) at Göttingen and of Nikolaus Kritikos (with Fueter; 1920) at the University of Zürich. At Stanford he was an advisor for the dissertations of Michael Israel Aissen (1951), Madeleine Rose Ashton (1962), Donald W. Grace (1964), Madeline Johnsen (1946), Charles McLoud Larsen (1960), Burnett C. Meyer (1949), Grove Crawford Nooney (1953) and Andrew Van Tuyl (1947).

The years following his 'retirement' were busy ones. He was a travelling lecturer for the Mathematical Association of America from 1953 to 1956, visiting campuses in many parts of the country. In 1963 he became a consultant for the Educational Research Council of Greater Cleveland, participating in their development of curriculum materials. Part of each year was spent in Zürich, and there was other travelling as well. In 1968 the Pólyas went to Trinidad to attend a conference at the University of the West Indies on mathematics in the Commonwealth schools. They attended International Congresses of Mathematicians (Harvard in 1950, Amsterdam in 1954 and Edinburgh in 1958) and, of special importance to Pólya, the Second International Congress on Mathematical Education at Exeter, England, in 1972, where he was specially honored along with Jean Piaget. A happy surprise at the Congress was the discovery that Mrs. Pólya and Piaget had been classmates as children in dance classes given by an English dancing teacher in Neuchâtel.

While in England for the Congress in Exeter, Pólya made a second film; the first, 'Let Us Teach Guessing,' had been made in 1968. The 1972 film, 'Guessing and Proving,' was made for the Open University by the BBC. He had first thought of proving the Pythagorean theorem in a 'Lecture without Words'—he cited Mendelssohn—but instead he presented a problem on volumes of polyhedra inscribed in a sphere. It was followed by a conversation with Maxim Bruckheimer.

Pólya taught his last course at Stanford in 1978, in the Computer Science Department. It was a course in combinatorics wherein Pólya taught the first part, Robert Tarjan the second; these lectures were later written up by Donald Woods and published by Birkhäuser [1983, 1]. The previous year Stanford had given a dinner to celebrate Pólya's 90th birthday. Several friends and colleagues spoke following the dinner: Jerzy Neyman, Peter Lax (then President of the American Mathematical Society), Donald Knuth, Halsey Royden, and the Nobel laureate and Stanford physics professor, Felix Bloch, a former student of Pólya's at the ETH.

In 1980 he was named Honorary President of the Fourth International Congress on Mathematical Education in Berkeley, but he suffered a severe attack of shingles a few months prior to the Congress and was unable to attend. Following this illness Pólya was less physically active. It was about this time Pólya told his wife that he was satisfied with what he had accomplished in life. He felt that he had done with his talents everything he could have done. Following the 1980 illness, in spite of failing eyesight, he continued to edit and to supervise various projects, until a couple of months before his death. He died from the effects of a stroke, on 7 September 1985, in Palo Alto, California. His wife survives him.

At the time of his death he had been a member of the London Mathematical Society for 60 years and an honorary member for 29 years. In 1947 he had been elected corresponding member in geometry of the Académie des Sciences, Paris, occupying a seat previously held by de la Vallée-Poussin and earlier by Klein, Sylvester and Steiner. Pólya was proud to say the seat could be traced all the way back to Newton. He was also a member of the USA National Academy of Sciences (1976), the Hungarian Academy of Sciences (1976), the American Academy of Arts and Sciences (1974) and the Académie Internationale de Philosophie des Sciences, Bruxelles (1965), honorary member of the Swiss Mathematical Society (1957), the New York Academy of Sciences (1976), The Mathematical Association of Great Britain (1972), Society for Industrial and Applied Mathematics (SIAM) (1972) and of the Council of the Société Mathématique de France (1952). In 1963 he was given the Award for Distinguished Service to Mathematics by the Mathematical Association of America and in 1968 was presented with the Blue Ribbon by the Educational Film Library Association for his film 'Let Us Teach Guessing.' The California Mathematics Council awarded him life membership in 1976. He was awarded honorary doctorates by the Swiss Federal Institute of Technology (Dr.Sc., 1947), University of Alberta (LL.D., 1961), the University of Wisconsin at Milwaukee (D.Sc., 1969) and the University of Waterloo (Dr.Math., 1971). In 1962 a group of his colleagues at Stanford edited a *Festschrift* to honor Pólya on his 75th birthday [**11**]. With 60 articles by eminent colleagues and friends from around the world, its breadth matched that of Pólya's own work.

In recognition of his pioneering work in combinatorics, the Society for Industrial and Applied Mathematics established a Pólya Prize in Combinatorial Theory and Its Applications, to which is attached a handsome medal with Pólya's portrait. The Mathematical Association of America also awards each year a Pólya Prize for Expository Writing for outstanding articles in the *College Mathematics Journal*. In 1987 the London Mathematical Society announced that it is establishing a Pólya Prize to be awarded for outstanding work by a mathematician in the United Kingdom.

As important, perhaps, as his many honors is the fact that he was admired, respected and loved by his colleagues, former students, and by many friends from around the world. Evidence of this was the attendance at the memorial symposium in his honor at Stanford University in May, 1986: research mathematicians, college teachers and high school teachers all came together to hear talks on his work by Albert Baernstein, Kai Lai Chung, Albert Edrei, Paul Erdős, Ronald L. Graham, Dennis A. Hejhal and Alan H. Schoenfeld. In the Mathematics Library at Stanford there is only one portrait on the wall, that of Pólya. And the first academic building on campus named for a mathematician is Pólya Hall.

Jorge Luis Borges once said that 'when writers die, they become books, which is, after all, not too bad an incarnation.' At the memorial symposium to honor Pólya, Erdős put it more mathematically: 'In the Arabian Nights, they say, "May the king live forever." In Pólya's case, we can say, "May his theorems live forever."'

Pólya the mathematician and teacher

When asked which mathematician of the past he admired most, Pólya replied without hesitation, 'Euler'. In [1] he said: 'Among old mathematicians, I was most influenced by Euler and mostly because Euler did something that no other great mathematician of his stature did. He explained how he found his results and I was deeply interested in that.' Pólya went on to admit that Euler wrote so much, he was not familiar with all of Euler's work. He nevertheless knew much of it very well and had a number of volumes of Euler's *Opera Omnia* in his personal collection.

His interest in Euler was consistent with the kind of mathematician and teacher Pólya was. He liked mathematics that was fairly close to its concrete roots. Of the two kinds of problems described by Poincaré, those 'qui se posent' and those 'qu'on se pose,' Pólya would surely have chosen the former. Albert Pfluger, his Ph.D. student and successor at the ETH described Pólya's taste thus in [7]: 'Pólya was attracted by problems originating in physical sciences and engineering, and many of his mathematical developments are motivated by such problems. Characteristic is his special liking for the concrete, but typical, particular case by which the general idea can be seized and comprehended or a general method can be verified.' Pólya was fond of saying that mathematics is the most abstract of the sciences, so in teaching it (or writing it) one must be as concrete as possible.

Pólya saw good mathematical problems in many fields not traditionally close to mathematics. At the request of the ETH he recalled in 1978 some of the people he worked with there. In addition to names one would expect—Weyl, Plancherel, Bernays, Gonseth, Pauli and Hopf—he pointed out that he knew a number of faculty in other departments and had taught their students, in architecture, chemistry, forestry, as well as those in engineering, physics and mathematics. So he became interested in what people were doing in these departments. He said, 'The Architecture Department had an interesting library. I studied there architectural ornaments and that led to one of my papers: about the analogy of the symmetries of crystals in the plane. I think one point in it is new. I illustrated the 17 plane symmetries with ornaments. I must confess the architects were not so much interested in it. But the professor of mineralogy, Niggli [Paul Niggli], was very much interested in it and he wrote a parallel paper.' (See [6].) Pólya's paper was his famous work on symmetries in the plane [1924, 1].

Further, he pointed out that he taught chemists and mentioned that one of his successful papers is on counting certain chemical combinations. 'Again the chemists were not too much interested in it but Niggli was very much interested. He even used the ideas of my paper in his [class], which was obligatory for chemists and other people and he included some of it in his textbook.' This work in chemistry culminated in Pólya's monumental paper on groups, graphs, and chemical compounds [1937, 3]. To describe it as 'successful' is to be too modest. It is one of the most celebrated papers in the history of combinatorics. In [2] Pólya's theory is called a 'milestone, not only in graph theory and chemical enumeration, but in mathematics as a whole.'

Another colleague, Paul Jaccard, the biologist, wrote on the distribution of plants and this led to further work by Pólya [1930, 5]. His work with engineers led to [1930, 6]. Einstein's son, Hans Albert Einstein, wrote his dissertation in 1936 at the ETH on the movement of silt in rivers; Pólya was one of the advisors. (Pólya was to write on this subject himself in [1937, 1] and [1938, 2].) The Pólyas maintained contact with the younger Einstein, who became a professor of hydraulic engineering at the University of California, Berkeley. The Pólyas had lived a few doors down the street from the Einsteins in Zürich, on the Büchnerstrasse.

The quality of finding good problems in many areas and then turning them into good mathematics was one of Pólya's strengths. But he went beyond finding good questions. His answers were models of clarity. At Pólya's 90th birthday dinner at Stanford H. L. Royden pointed out that Pólya was not only a mathematician, he was also a teacher. It was not sufficient for him to solve a problem; he had to study it until he saw the nature of the solution clearly and simply, in order to put it in a form easily understood by his readers or listeners.

Pólya not only delved into the problem till he achieved full understanding and then rewrote and reworked his papers for complete clarity, he also found phrases that stick in the mind, making mathematical ideas simple and memorable. For example, when describing his theorem on random walks, that a wandering point in a lattice *must* return to its starting point in one and two dimensions, given sufficient time, but that it need not do so in higher dimensions, Pólya remarked that in two dimensions, it is really true that all roads lead to Rome!

In [1947, 1], a paper concerned with estimating electrostatic capacity, Pólya discusses the problem of minimizing capacity, in the special case of thermal conductance, and gives the example of a cat preparing itself for sleeping through a cold winter night: 'he pulls in his legs, curls up, and, in short, makes his body as spherical as possible.' The cat thereby demonstrates knowledge of the theorem: 'Of all solids with a given volume, the sphere has the minimum capacity.'

Pólya used Steiner symmetrization to solve Rayleigh's problem on the shape of a drum, and this simple device was then the basis for the solution of many problems in mathematics gathered together in his book with Szegő, *Isoperimetric inequalities in mathematical physics*. Royden has remarked that he thought these techniques 'too simple and easy to be very deep' until, several years after seeing them at Stanford, at an Eastern graduate school he was taught far more complicated methods for some of these problems yet they gave far more limited results than the 'elementary' methods of Pólya and Szegő. Mark Kac was, of course, also interested in problems concerning the shape of a drum and he once inscribed one of his reprints on the subject to Pólya: 'To another drummer. Mark Kac.'

It was Pólya's concern for making the mathematics absolutely clear, even intuitive, that developed early on into an interest in education. As early as 1917, he was addressing himself to questions of teaching. On 22 November of that year he spoke at the town hall in Zürich on mathematical discovery. This was written up and published in 1919, his first paper on problem solving [1919, 6]. A diagram in this paper outlining the solution of a problem is strikingly similar to the diagram on the endpaper of the second volume of his book *Mathematical discovery* of 1962. Another early appearance of his ideas on problem solving was in the preface to the *Aufgaben und Lehrsätze*.

Pólya was, of course, not the first mathematician to consider the question of how people solve mathematical problems. Descartes, Euler, Poincaré, Mach, Bolzano and Hadamard, among others, had certainly thought about mathematical discovery and creativity and had written on them. But Pólya made guessing, looking at data, analogy, generalization and specialization part of the language of every serious teacher and student of mathematics. He often signed letters with 'G. P.' and would playfully note that it stood for 'guessing and proving.'

In 1963, when Pólya was given the Distinguished Service Award by the Mathematical Association of America, Szegő wrote in the citation: 'George Pólya is unique among mathematicians for combining, during his distinguished scientific career, deep research on a very broad front with an ever present interest in mathematical education.' [9]

This interest in problem solving not only influenced the course of his later career, the writing of *How to solve it*, *Mathematics and plausible reasoning*, and *Mathematical discovery*, among many other works; it also influenced his career as a research mathematician. His mathematical work spanned real and complex analysis, calculus of variations, probability, geometry, number theory, combinatorics and graph theory. When asked how he happened to work in so many distinct branches of mathematics—something very unusual even among first-rate mathematicians—Pólya responded [1]: 'I was partly influenced by my teachers and by the mathematical fashion of that time. Later I was influenced by my interest in discovery. I looked at a few questions just to find out how you handle this kind of question.'

Hardy apparently did not think this moving from problem to problem was such a good idea. Pólya wrote [1969, 4]: 'In working with Hardy, I once had an idea of which he approved. But afterwards I did not work sufficiently hard to carry out that idea, and Hardy disapproved. He did not tell me so, of course, yet it came out when he visited a zoological garden in Sweden with Marcel Riesz. In a cage there was a bear. The cage had a gate, and on the gate there was a lock. The bear sniffed at the lock, hit it with his paw, then he growled a little, turned around and walked away. "He is like Pólya," said Hardy. "He has excellent ideas, but does not carry them out."'

Pólya's taste was always for the intuitive and creative side of mathematics, always based on specific problems. He became impatient with what he saw in modern mathematics as excessive generalization, often in his view rather sterile. Though he liked and admired Emmy Noether as a mathematician, he disagreed openly with her on one occasion and wrote about it [1973, 3]. It was after a lecture Pólya gave in Göttingen about 40 years earlier. He wrote, 'It finally turned into a debate on generalisation and I defended the relatively concrete particular cases. Then once I interrupted Emmy: "Now, look here, a mathematician who can only generalise is like a monkey who can only climb UP a tree." And then Emmy broke off the discussion—she was visibly hurt. And then I felt sorry. I don't want to hurt anybody and especially I don't want to hurt poor Emmy Noether. I thought about it repeatedly and finally I decided that, after all, it was not one hundred per cent my fault. She should have answered: "And a mathematician who can only specialise is like a monkey who can only climb DOWN a tree." In fact, neither the up, nor the down, monkey is a viable creature. A real monkey must find food and escape his enemies and so he must incessantly climb up and down, up and down. A real mathematician must be able to generalise and to specialise. A particular mathematical fact behind which there is no perspective of generalisation is uninteresting. On the other hand, the world is anxious to admire that apex and culmination of modern mathematics: a theorem so perfectly general that no particular application of it is possible.'

Then too there is the famous Pólya–Weyl wager of 9 February 1918, that arose in discussions of the work of L. E. J. Brouwer. Weyl and Pólya considered the two theorems: (1) any bounded set of (real) numbers has a precise upper bound, and (2) every infinite set of numbers contains a countable subset. Weyl predicted that 'by the end of 1937 Pólya himself or a majority of leading mathematicians will admit that the concepts of number, set and countable, to which these theorems refer and which we generally consider basic today are quite vague; and that inquiries into the validity or falsehood of these theorems are futile...' Further, Weyl predicted Pólya or the majority of leading mathematicians would 'agree that the theorems [above], when interpreted literally in as reasonable terms as possible, are absolutely false.' There were further technical details, but the terms specified that Weyl would win if his forecast came true; otherwise Pólya would win. Witnesses included T. Carleman, A. Speiser and A. Weinstein, among others. From our present-day perspective, it becomes quite clear that Pólya won. It was admitted privately by Weyl in 1938 [1972, 1].

As a person, Pólya was gregarious, outgoing and friendly. He was a natural teacher, obviously enjoying the interaction with students. And he was always teaching. Even in conversation he would lead one into thinking about questions that one might not otherwise see. So if one was not learning mathematics from him one was learning about literature or something else. He loved especially the writings of Voltaire, Anatole France and, as mentioned earlier, Heinrich Heine. His favorite authors he could quote at length. He was also interested in pictures and in music, where his tastes were, by today's standards, rather conservative. In music, he loved the 19th century romantics, especially Chopin, Schubert, Beethoven, and their contemporaries. He also liked the operas of Puccini and remembered fondly the operas of the Hungarian composer, Erkel, who wrote on nationalistic themes and is not so well known outside Hungary.

Pólya very much enjoyed jokes and stories, which he collected, quite often classifying them by their country of origin. Of course, a joke was highly prized if it had a mathematical or logical twist. His love of good stories and his joy in passing them along to his students and friends suggests, perhaps, the influence of Fejér.

Donald Knuth recalled at the dinner at Stanford to celebrate Pólya's 90th birthday that: 'A few years ago when I was visiting the home of Professor de Bruijn in Holland, he and I asked ourselves the question: If we could be any mathematician in the history of the world (besides ourselves), who would we rather be? After some discussion we narrowed the choices down to Euler and Pólya, and finally settled on George Pólya because of the sheer enjoyment of mathematics that he has conveyed by so many examples.'

This was later echoed by de Bruijn in his retirement address from the Technological University Eindhoven when he said: 'A mathematician who possibly more than anyone else has given direction to my own mathematical activity, is G. Pólya. All his work radiates the cheerfulness of his personality. Wonderful taste, crystal clear methodology, simple means, powerful results. If I would be asked whether I could name just one mathematician who I would have liked to be myself, I have my answer ready at once: Pólya.' [3]

Frank Harary expressed his admiration for Pólya in his 'Homage to George Pólya' in a special issue of the *Journal of Graph Theory* in 1977: 'With no hesitation, George Pólya is my personal hero as a mathematician...[he] is not only a distinguished gentleman but a most kind and gentle man: his ebullient enthusiasm, the twinkle in his eye, his tremendous curiosity, his generosity with his time, his spry energetic walk, his warm genuine friendliness, his welcoming visitors into his home and showing them his pictures of great mathematicians he has known—these are all components of his happy personality.

'As a mathematician, his depth, speed, brilliance, versatility, power and universality are all inspiring. Would that there were a way of teaching and learning these traits.' [4]

Jeremy Kilpatrick, a former student and, later, coauthor of Pólya's, wrote shortly after Pólya's death about his spending long summer afternoons in the shaded study of the modest house on Dartmouth...drinking juice and eating cookies and learning lessons about being a scholar—the need to avoid self-pity and self-importance, to take pains with your work; to find the right word, the right idea; to see the fun, the humor in what you are doing. And when you returned in later years with your family and saw Pólya amusing your five-year-old with a folding toy or picking lemons for you in the yard, you might have caught a glimpse of the truth that great teachers do not simply teach us to do; they teach us to be.' [5]

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The essays that follow appear roughly in the order in which Pólya became interested in or spent significant time in the field.

PÓLYA'S WORK IN PROBABILITY

K. L. Chung

Pólya's first publication, his 1913 Dissertation [1912, 1] in Hungarian, treats a problem in probability. His lifelong interest in this field is evident from his Bibliography (see [1984, 2]). Besides the twenty papers reprinted in that volume, some ten more titles indicate probability and statistics. Some of his contributions to probability are classified as analysis, others perhaps as combinatorics. The major items which will be discussed here have long since become textbook material, [1]. Readers wishing to learn more details of the results summarized below should consult these texts.

1. *Fourier transform and convergence of distributions*

The Fourier transform f of a probability measure μ in R^1 given by

$$(1)\qquad f(t) = \int_{-\infty}^{\infty} e^{itx}\mu(dx), \quad t \in R^1,$$

is known as a characteristic function, a tool used since Laplace, Poisson and Cauchy. In [1923, 2], Pólya showed that it is indeed characteristic in the sense that it uniquely determines μ. Actually he considered an absolutely Riemann-integrable function (signed density), and said of this result: 'despite its simplicity it was nowhere explicitly stressed'. So he gave a neat proof using Fejér's theorem in Fourier series. This result has been superseded by Paul Lévy's inversion formula but the method of proof remains viable. A better known result of Pólya's, slightly hidden in [1918, 4] ('On zeroes of certain integral functions') but cited and used in [1923, 2], is his sufficient condition for a given function to be a characteristic function, as follows:

$$(2)\qquad \begin{gathered} f \text{ is real-valued and continuous in } R^1; \\ f(0) = 1, f(t) = f(-t) \text{ for all } t, f \text{ is convex for } t > 0, \lim_{t\to\infty} f(t) = 0. \end{gathered}$$

He returned to it in [1949, 3] and gave several interesting examples. This criterion is still the only useful one for constructing and recognizing specific characteristic functions, the 'positive-definite' characterization by Bochner and Khintchine being too perfect to be of practical utility. It should be pointed out that Pólya considered only probability density so that the last condition in (2) is necessary by the Riemann–Lebesgue lemma. For a general measure μ as written in (1) that last condition should be omitted.

Pólya wrote a number of papers on the 'Gaussian error-law', now commonly known as the 'normal distribution'. In [1920, 3] he coined the term 'central limit theorem' (in German) and proved a general convergence theorem for a sequence of distributions, based on the convergence of transforms. Let F be a distribution function and

$$(3)\qquad \Phi(u) = \int_{-\infty}^{\infty} e^{ux}\, dF(x)$$

be the transform. He said: Tschebyscheff and Markoff considered the derivatives of Φ at $u = 0$, whereas Liapounoff considered Φ for purely imaginary values of u. But he would consider Φ as an analytic function in the strip $-a \leqslant \operatorname{Re} u \leqslant +a$, $a > 0$. This requires the convergence of the integral in (3) in a neighborhood of the complex parameter u at the origin, which restricts the general applicability of the resulting convergence theorem. However, Pólya also proved an earlier version of Lévy's theorem (1922) at the end of [1926, 9]. Later in 1937 Lévy and Cramér published their convergence theorem in terms of characteristic functions (namely Liapounoff's usage) which is now the principal tool for limit theorems. Pólya also proved in [1920, 3] the so-called 'continuity theorem for the moment problem': namely if all moments of a sequence of distributions are finite and converge to those of a given distribution which is uniquely determined by its moments, then (weak) convergence of the sequence to the latter follows. This remains a useful and convenient method in situations where the calculation of moments is more expedient than that of the transforms. For instance I was able to obtain a central limit theorem for a stochastic iterative scheme concocted by Robbins and Monro in a medical problem.

In [1923, 2] Pólya made another approach to the Gaussian error-law by considering a functional equation suggested by the combination of errors in measurement. In terms of the characteristic function Φ of the error-law, it reduces to the following: for each $a > 0$ and $b > 0$, there exists $c > 0$ such that

$$\Phi(ct) = \Phi(at)\Phi(bt).$$

Such a law is called 'stable' by Lévy. Pólya proved that the only stable law having a finite second moment is Gaussian corresponding to $\Phi(t) = e^{-t^2}$. He deduced from this that the function $e^{-|t|^\alpha}$ cannot be a characteristic function if $\alpha > 2$, while it is a characteristic function if $0 < \alpha \leqslant 1$, an immediate consequence of his condition (2). Lévy proved that the same is true for $0 < \alpha \leqslant 2$ by using his convergence theorem, and discovered a much larger class of characteristic functions called 'infinitely divisible', and even more profoundly, founded his theory of 'additive processes'. See [**2**].

2. *Random walk*

Pólya's celebrated theorem on random walks appeared in [1921, 7], but important complements are given in [1938, 2], based on a lecture he gave at a conference in Geneva in 1937, which was accompanied by a film showing 'the cartage of stones by the current'. A point (not a 'particle'!) executes a walk on the integer lattice of R^d, $d \geqslant 1$, so each step takes it to one of the $2d$ neighboring positions with probability $1/2d$ each. The steps are independent. Given an arbitrary lattice point A, will the point starting from the origin ever reach A with probability one? The answer is 'Yes' if $d = 1$ or 2, but 'No' if $d \geqslant 3$. He stressed the dimensional breakdown as 'newsworthy', not intuitively obvious. For $d = 1$ the problem is similar to that of gambler's ruin treated by Laplace. But the question raised by Pólya, that of *recurrence* as it is called today, lay hidden in the ruin problem. One might say that the demand of a quantitative answer, the probability of ruin of Peter before Paul, obscured the even more fundamental question of the certainty of ruin (of one of them). This question becomes more prominent in higher dimensions and opens up a new vista. It can also be formulated as the problem of *rencontre* (Pólya reminded the reader that he was not speaking of the classical problem of the Marquis de Montmort!) of two points executing random walks independently. He treated this second problem by analogy with the first, and added a third one in [1938, 2], that of 'novelty': will the point never pass through the same position twice? He reduced this to the first problem by an elegant 'reversal' argument. Actually the second problem can also be reduced to the first by reversing the steps of one of the strollers from the site of rencontre. This kind of reasoning has received some attention lately under the name of 'coupling'. Pólya proved his results by using a 'first passage decomposition'. When $A = 0$, if we denote the position of the point at time n by S_n, and put $p_0 = 1$,

$$p_n = P(S_n = 0), \quad w_n = P(S_k \neq 0 \text{ for } 1 \leqslant k < n;\ S_n = 0), \quad n \geqslant 1;$$

$$P(z) = \sum_{n=0}^{\infty} p_n z^n; \quad W(z) = \sum_{n=1}^{\infty} w_n z^n.$$

Then he obtains the relation (known today as 'renewal equation'):

$$P(z) = \frac{1}{1 - W(z)}. \tag{4}$$

From this he easily deduces that

(5) $$\sum_{n=1}^{\infty} w_n = 1 \Leftrightarrow \sum_{n=0}^{\infty} p_n = \infty.$$

Now he calculates p_n by Fourier inversion and Laplace's method (apparently used in his dissertation). The result is

(6) $$p_n \sim 2\left(\frac{d}{2\pi}\right)^{d/2} n^{-d/2},$$

from which it follows that the conclusion in (5) obtains if and only if $d = 1$ or 2. The general case $A \neq 0$ follows from this special case and another relation of the form $P_A(z) = W_A(z)\,P(z)$, with the same P as before.

It is clear in Pólya's walk that if return to 0 once is (almost) certain then so also is return infinitely many times. We now call such a walk 'recurrent'. Equally clear is the other side of the dichotomy: if return once is uncertain then return infinitely many times is impossible. We call such a walk 'transient'. Thus, 'not-recurrent' has a stronger implication than it logically connotes. The change of viewpoint from 'at least once' to 'infinitely often' might seem specious. It is decidedly not; indeed the switch is tantamount to converting a supermartingale to a martingale, or a superharmonic function to a harmonic one. In the present case it yields the following: if N denotes the total number of returns to 0, and $E(N)$ its mathematical expectation, then

(7) $$P(N = \infty) = 0 \text{ or } 1 \text{ according as } E(N) < \infty \text{ or } = \infty.$$

In this form the theorem constitutes an instance of the Borel–Cantelli lemma, but since the events (returns) are not independent the result is not covered by the original assertion. In this light Pólya's theorem appeared to be the first significant case of a 'zero-or-one' law for dependent events, an outstanding phenomenon in probability.

An extension of Pólya's theorem to 'generalized random walks' was obtained by Chung and Fuchs [3]. Let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of independent and identically distributed random variables in R^d, and $S_n = \sum_{k=1}^{n} X_k$. Thus in Pólya's scheme the X_k denote the successive steps, and S_n the position at time n, the walk starting from 0. A punctilious return to 0 is now ruled out in general, and what can be more natural than to substitute for this a return to an arbitrary neighborhood B of 0? Note that when the latter is a ball of radius smaller than 1, then in the lattice case return to the ball means return to 0 precisely. It is rather unexpected that the dichotomy presented in (7) remains true if N now denotes the total number of visits to B by the sequence $S_n, n = 1, 2, \ldots$. To the unspoiled it may also be surprising that the dichotomy does not depend on the size of B. To evaluate the sum represented by $E(N)$ there, Fourier inversion with Abel summability was used. The conclusions are: if $d = 1$, the random walk is recurrent if $E(X_1) = 0$; if $d = 2$, it is recurrent under this condition and the supplementary condition $E(\|X_1\|^2) < \infty$; if $d \geqslant 3$, it is always transient (unless degenerate into a lower dimension). Further results, along this direction were given by Kesten, Spitzer, Port, Stone, Dudley, ..., some of which deal with walks on groups. Extension of Pólya's theorem in the form (7) to homogeneous Markov chains with countably many states is quite easy. In fact, the method indicated above generalizes easily with suitable notation, which would have rid Pólya of the nuisance of 'periodicity' pestering him: return is possible only in an even number of steps. The general notion of recurrence vs. transience in Markov processes in discrete or continuous time plays a prime role in the theory. It has been investigated by many probabilists: Doblin, Harris, Orey, Azéma, Kaplan-Duflo, Revuz.... The case of Brownian motion in R^d deserves special mention. In [1938, 2] Pólya discussed the diffusion of molecules as well as Einstein's study of cartage. He wrote down the heat equation, a hyperbolic equation (for H. A. Einstein's cartage problem) as well as the Laplace equation, by passage to the limit of second order difference equations. Thereby he indicated a heuristic solution to his old problem by considering positive harmonic functions, pointing out in particular the solution $1/r$ (where r denotes the distance from the origin) conveys the transience in R^3. Indeed in the last footnote to the paper he discussed the probability interpretation of a Dirichlet boundary value problem with values zero or one on different parts of the boundary. This is essentially the modern method used to ascertain the recurrence or transience of Brownian motion in different dimensions. Let us note that the results on generalized random walks stated above imply the recurrence, but not the transience cases. Pólya considered the application of probability to the 'transport of solid material by the rivers' as 'serious', whereas his own scheme of 'promenade at hazard' as 'curious'. In [1970, 2] he told the story of how he came to conceive of the latter from constantly running into a couple while strolling in the park. Actually, shortly before he wrote [1919, 9] he had treated Lord Rayleigh's problem of random flight in [1921, 7], and used the same word 'Irrfahrt' in both papers. That earlier, no doubt serious, application apparently did not meet Descartes' precept, which Pólya quoted, of 'commencing with the objects that are the simplest and easiest to know', and it yielded no memorable consequences. On the other hand, half a century since Pólya wrote those charming passages, the curious has certainly become curiouser and curiouser.

3. *Urn scheme*

This is also known as the Pólya–Eggenberger scheme (see [1923, 7] and [1928, 6]). Eggenberger wrote his thesis in Zürich under Pólya in 1924, and did the practical work. The best exposition of the ideas, however, was given in [1931, 1] which reproduced the contents of a course Pólya gave in March 1929 at the Institut Henri Poincaré, for which he thanked Émile Borel for the invitation. Pólya was seeking non-Gaussian laws of errors for dependent events, and cited Markov's chained-events. The spread of epidemic was mentioned as an example, and led to the following model of 'contagion'. An urn contains initially ρ red and σ black balls, and one ball is drawn each time. After each drawing, $1+\delta$ balls of the color drawn are added to the urn, where δ is an integer and adding becomes subtracting when the number is negative. Call the drawing of a red ball 'success'. If $\delta > 0$ then success is contagious as it reinforces other successes. If $\delta = 0$ or -1, the scheme reduces to the classical ones of drawing with or without replacement, which led respectively to the central limit theorem (Bernoulli, de Moivre, Laplace, Gauss) and 'the law of small numbers' (Poisson). So Pólya's extension is certainly very 'simple'; but is it also very 'easy to know' according to Descartes' precept? Let us ask the most exigent question about the scheme: what is the probability of drawing n balls with their colors completely specified in sequential order? An easy argument gives the answer

$$(8) \qquad \frac{\rho(\rho+\delta)(\rho+2\delta)\dots(\rho+(r-1)\delta)\,\sigma(\sigma+\delta)(\sigma+2\delta)\dots(\sigma+(s-1)\delta)}{1(1+\delta)(1+2\delta)\dots(1+(r-1)\delta)(1+r\delta)(1+(r+1)\delta)\dots(1+(n-1)\delta)},$$

where r denotes the total number of red balls in the specified sequence, s that of the black ones, so that $r+s = n$. Given the initial data σ and ρ, this probability depends only on n and r, but not on the specified order in which the two colors appear, so long as none of the factors in (8) becomes negative. Thus the formula holds for all $n \geqslant 1$ when $\delta \geqslant 0$. A sequence of random variables $x_1, x_2, \ldots, x_n, \ldots$ is now called 'exchangeable' when the joint distribution of any n (arbitrary) of them in any permutation is the same. If we put x_n equal to one or zero according as the nth ball drawn in Pólya's scheme is red or black, we obtain such a sequence. This is a remarkable generalization of a sequence of independent and identically distributed random variables that constituted the primary structure of classical investigation. The theory of exchangeability has been developed by Bruno DeFinetti with implications on the foundation of statistics [**4**]. The concept implies that of (strict) stationarity, but even for $n = 1$ it is not trivial to show that the probability of drawing a red ball at the nth drawing is the same for all n. In case $\delta = -1$, Poisson had to give a combinatorial argument not so easy to follow. Pólya gave an elegant proof of stationarity in the general case by use of a multiple generating function. Clearly, here probability merges into combinatorics.

Pólya's distribution given by (8) can be reorganized by using generalized binomial coefficients into a hypergeometric form, now known by his name. If we let the parameters ρ, σ and δ grow (or not grow) with n and let n become infinite, we obtain various limiting distributions including the normal, Poisson (compounded), gamma, beta, etc. (but not all Pearson types). Applications to special processes have been made in various areas of practice, too numerous to recount here. For a curious and serious application, read the title of [**5**]. This brief discussion must not stop before a note of dissonance, sounded by Pólya himself. He noted that if $\delta > 0$ in his scheme, failure as well as success tends to reinforce itself. Now in an epidemic each victim produces many more germs to cause further infection, but why should each uninfected person also enhance the chance of others being so? Here we must hear his own voice (translated, italics mine): 'In reducing this fact [of contagion] to its simplest terms and adding to it a certain symmetry, *propitious for mathematical treatment*, we are led to the urn scheme.' Judging from the data shown in [1923, 7], theory and practice fit well enough in the case of Swiss smallpox.

4. *Miscellanies*

A number of papers reprinted in other volumes of the Collected Papers bear fruits in probability. [1932, 2] (with A. Bloch) deals with roots of equations with random $(0, \pm 1)$ coefficients. Since Pólya spent a preponderant time studying the roots of functions (originally as an approach to the Riemann Hypothesis), it is fitting that he should randomize the problem at least once. This topic was taken up by Littlewood and Offord later. [1931, 4] (with Szegő) dealing with the transfinite diameter was a sequel to his earlier work [1928, 5]. It played an important role in the development of the notion of 'capacity', through M. Riesz, Frostman and Choquet. The paper was roundly commented on by Hille. Since G. A. Hunt, P. A. Meyer and Dellacherie, capacity has become a germane part of probability. It is possible that we can still learn something from the old source. [1933, 3] dealing with heat exchange inspired Schoenberg and Karlin to the study of so-called Pólya frequency functions with statistical applications [**6**].

Among the remaining reprinted papers two early ones belong to geometrical probability and have been adequately commented on by Santalo, Coxeter and Kingman. A few others solve special problems such as balloting, coupon-collecting and clustering of randomly picked points. One is on heuristic reasoning but another

[1921, 1], which gives an intuitive derivation of the Gaussian law, was not reprinted. Pólya presented the latter in my class some years ago, together with a new statistical-methodological characterization of that law, which he announced at the Bologna Congress [1928, 8] and later discussed in detail in [1931, 1], cited in Section 3 above. I recall that he asked in class whether his assumptions for solving the problem could not be ameliorated. The last item in probability which is reprinted [1976, 1] contains a simple problem with an empirical twist, highly recommended to teachers of elementary courses.

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1. See, for example, both volumes of W. Feller's *An introduction to probability theory and its applications* (John Wiley & Sons, 1950, 1966), or my *A course in probability theory* (2nd ed., Academic Press, 1974) and *Elementary probability theory with stochastic processes* (3rd ed., Springer, Berlin, 1979).
2. P. Lévy, *Théorie de l'addition des variables aléatoires* (Gauthier-Villars [first edition published in 1937]).
3. K. L. Chung and W. H. J. Fuchs, 'On the distribution of values of sums of random variables', *Mem. Amer. Math. Soc.* 6 (1951).
4. An early exposition was given at the same conference in Geneva as Pólya did in [1938, 2]; see *Actualités scientifiques et industrielles*, 739 (Hermann & Cie.).
5. D. Blackwell and D. G. Kendall, 'The Martin boundary for Pólya's urn scheme, and an application to stochastic population growth', *J. Appl. Probability* 1 (1964) 284–296.
6. See for example, S. Karlin, *Total positivity* (Stanford University Press, 1968).

PÓLYA'S WORK IN ANALYSIS

R. P. Boas

When one looks at Pólya's work as a whole one is struck by his great power and versatility. He proved many difficult, fascinating and important theorems, and devised methods that still retain their effectiveness today. It is interesting to see how, in many cases, he found his problems. Some of them seem to spring up from nowhere, but he often makes use of the principle (enunciated in the preface to *Aufgaben und Lehrsätze*) that to determine a line one can start at a point and follow a direction, or interpolate between two points, or draw a parallel. The difference between Pólya and other people was that his generalizations were usually both deep and difficult; or at least, unexpected. He was also quick to see where something interesting was going on, and step in with an improvement, or even a complete theory. Hardy is supposed to have said once that Pólya had brilliant ideas but didn't follow them up. There was some truth in this unkind remark. The *Collected Papers* include many brief contributions that contain the germs of substantial theories that were developed later by others. Nevertheless, it would be unreasonable to complain, considering that at the height of his career Pólya was publishing two or three major papers in analysis every year, and doing the same thing in probability. He could hardly have had time to follow up everything that he initiated. In parts of analysis his best ideas were tossed out as problems (many of which led to substantial theories), or absorbed into *Aufgaben und Lehrsätze* or into *Inequalities*. He was also very helpful to others. I can illustrate this trait from my own experience. In the early 1940s I was interested in what is now known as the Whittaker constant W. It was known that if f is an integral function of exponential type τ there is a number W such that if f and all its derivatives have zeros in the unit disk, then $f \equiv 0$ if $\tau < W$ but not necessarily if $\tau > W$; and that $\log 2 \leqslant W \leqslant \pi/4$. I wrote to Pólya about some ideas I had about this problem, and he replied with a letter that improved both upper and lower bounds, with suggestions for how to make further improvements; but he never published anything on the subject himself.

In discussing Pólya's work in Analysis, it seems best to begin with one of the fields to which he made the most numerous and most significant contributions, namely complex analysis.

1. *Connections between the sequence of coefficients in a power series and properties of the analytic function determined by the power series.* In principle, the sequence of coefficients contains all the properties of the function; the problem is to make the sequence surrender information about some particular property. The most interesting theorems connect an easily-stated property of the coefficients with an easily-stated property of the function. Pólya's own survey of this field is in [1928, 7].

Fatou conjectured, and Pólya proved [1916, 1; 1929, 1] that the circle of convergence is 'usually' a natural boundary: by changing the signs of the coefficients one can always make the sum of the series non-continuable. In fact [1917, 7], this happens for almost all sequences of signs in the sense that the set of sequences for which it happens is the complement of a nowhere dense perfect set. This must have been one of the first applications of point-set topology to complex analysis. Pólya also showed [1950, 4] that it is possible to change the signs so that the sum of the series satisfies no algebraic differential equation.

Another easily-stated property of the sequence of coefficients is the presence of many gaps. According to Fabry's famous gap theorem, a series is non-continuable if the density of zero coefficients is 1. Pólya established the definitive character of Fabry's theorem by showing [1939, 1; 1942, 1] that no weaker hypothesis than Fabry's will allow the same conclusion. If the density of zero coefficients is less than 1 but still positive, Pólya showed [1923, 4; 1929, 1] that singular points still occur, but less frequently; in particular, if the minimum density of the vanishing coefficients is δ, there is no arc of regularity whose length exceeds $1-\delta$ times the circumference. (This had been proved by Fabry, but in a less lucid formulation.) In [1923, 5] Pólya proved a much deeper result, the corresponding theorem for Dirichlet series on the line of convergence.

Pólya also extended Fabry's theorem in another way, by considering Dirichlet series whose exponents are not necessarily real [1927, 10, 11]: here, if $n/\lambda_n \to 0$, the domain of the function is necessarily convex; Fabry's theorem is an immediate consequence. This is one of the few known significant results about Dirichlet series with complex exponents. For the special case of power series with $\liminf n/\lambda_n = 0$, Pólya showed that the domain is simply-connected. In [1934, 3] Pólya extended these investigations to functions that do not have single-valued continuations. The results are rather complicated; the following special case gives an idea of their character. A function defined by a power series $\sum a_n z^n$ is of the form $F\{-\log(1-z)\}$, where F is an integral function at most of order 1 and mean type, if and only if $a_n = H(n)$, where

$$H(z) = \{1/\Gamma(z+1)\} \sum_{k=0}^{\infty} A_k \Gamma^{(k)}(z)/k!,$$

with $\limsup |A_n|^{1/n} < \infty$.

Instead of considering zeros in the sequence of coefficients, one can connect the changes of sign in the sequence with the location and nature of the singular points. The idea appears in its purest form in [1932, 1], where the sum of the power series has poles, and no other singularities, on the unit circle; the spacing of the poles is connected with the variations in sign of the sequence of coefficients. This, together with results about trigonometric polynomials and about Dirichlet series, follows from a theorem about Mellin transforms proved in [1930, 3]; it takes someone of Pólya's skill and breadth of knowledge to perceive the connections.

Many relations between the coefficients and the singular points of a power series are expressed in terms of recurrent determinants formed from the coefficients. Paper [1928, 4] provides a condition of this kind for the sum of the series to have at least one regular point on the boundary.

Still different results on singular points are connected with the Hadamard product $\sum a_n b_n z^n$ of $A(z) = \sum a_n z^n$ and $B(z) = \sum b_n z^n$. The singular points of the product can in general occur only at products of singular points of A and B; Pólya devoted three papers [1927, 2, 3; 1933, 5] to the question of when a product of singular points will necessarily be a singular point of the product.

The coefficients of many familiar power series are integers. The question of the behavior of the sum of such a series is, as Pólya remarked in his survey [1922, 3], an attractive one, because it combines the apparently unrelated concepts of rational functions, integers, and conformal mapping. Pólya conjectured [1916, 3] and Carlson subsequently proved [**1**], that a power series with integral coefficients and radius of convergence 1 represents either a rational function or a function with the unit circle as a natural boundary. In [1923, 1] and [1928, 4] Pólya returned to this subject and found still deeper results.

The methods that Pólya used in proving many of the results mentioned so far were systematized in a long paper [1929, 1], which Pólya himself partially summarized in [1929, 3]. This very influential paper studies various kinds of densities of sequences of numbers; convex sets and their supporting functions; and integral functions of exponential type and their conjugate indicator diagrams (which describe the rate of growth of functions in different directions). Finally this material is applied to theorems about the distribution of values of integral functions of infinite order, and the location of singular points of functions whose power series have a finite radius of convergence. All these topics have turned out to be capable of generalizations and extensions that go far beyond those actually presented in the paper. The central theorem is Pólya's representation of an integral function f of exponential type (one whose modulus grows no faster than $\exp(A|z|)$) as

$$\frac{1}{2\pi i}\int_C F(w)\, e^{zw}\, dw,$$

where F is the Borel–Laplace transform of f, and C surrounds the conjugate indicator diagram. This representation has turned out to be important in contexts far beyond those to which Pólya originally applied it. For example, expansion of the kernel e^{zw} in series leads to expansions of functions of exponential type in terms of various data. Pólya applied this idea on a small scale in [1926, 1]; the same idea was applied to more substantial problems by his students and by many others.

A related paper with Plancherel [1937, 2] was ostensibly concerned with generalizations to n dimensions of the Paley–Wiener theorem (a function f is of the form

$$\int_{-\tau}^{\tau} e^{izt} g(t)\, dt,\ g \in L^2,$$

if and only if f is an integral function of exponential type at most τ and belongs to $L^2(-\infty, \infty)$); however, the same paper contains many important results on one-dimensional problems.

The last chapter of [1929, 1] appears as [1933, 5], which contains a full discussion of Pólya's results on the singular points of an Hadamard product.

Fabry's gap theorem and its generalizations do not, of course, apply to integral functions, for which there is only one singular point. What, then, does a gap condition for the power series of an integral function imply? Pólya answered this question in [1928, 4], although the proofs appear only in [1929, 1]: for each theorem about singular points on the circle of convergence of a power series there is a parallel theorem on Julia lines of integral functions.

The theorems on power series with integral coefficients were extended in [1931, 2] to power series whose coefficients are integers of an algebraic field. Related problems concern power series $\sum a_n z^n$ with only finitely many different a_n [1931, 2] and those for which $n!a_n$ are integers [1921, 6; 1922, 4]. In [1922, 2] Pólya studied power series whose sums are algebraic functions (and shows how to construct such series). In [1935, 3] and [1950, 4] Pólya studied power series whose sums satisfy an algebraic differential equation. In particular, he proved that a formal power-series solution must actually converge. Paper [1920, 1] discusses rational functions whose power series have rational coefficients.

2. *The general character of an analytic function as revealed by its behavior on a set of isolated points.* The subject originates with Pólya's discovery [1915, 2] that 2^z is the 'smallest' (in a well-defined sense) transcendental integral function with integral values at the positive integers. Pólya's further contributions to this topic are in [1916, 3] and [1920, 4; 1928, 4]; these results have inspired much further work, which peaked several decades later. The function 2^z is (in the terminology introduced later by Pólya) of exponential type $\log 2$. For somewhat larger type, an integral function with integral values at the positive integers must be of the form $P(z)\,2^z + Q(z)$, where P and Q are polynomials; this was discovered by A. Selberg [**2**]. Activity in this field continued at least into the 1970s (for a survey, see Vol. 1 of the *Collected Papers*, pp. 771–772).

Paper [1941, 3] is a fundamental contribution to the interpolatory theory of integral functions: what kinds of function can take, with some of their derivatives, prescribed values at prescribed points? Papers [1933, 6] and [1937, 2] deal with the problem of how slow the growth of an integral function can be when it is bounded on a regularly spaced sequence of points. The problem that inspired [1933, 6] was a conjecture by Littlewood that an integral function of order less than 2 cannot be bounded at the lattice points unless it is a constant. This had been proved by J. M. Whittaker [**3**], but Pólya's method is at the basis of many subsequent generalizations. Paper [1933, 6] is presented as a commentary on a problem set by Pólya in 1931: an integral function of order 1, type 0, bounded at the integers, is constant (this had been proved earlier by Valiron [**4**], although Pólya was not aware of this). This again has led to an extensive array of generalizations.

3. *General theory of analytic functions.* Paper [1926, 3] contains the first proof of the theorem that if g and h are integral functions and $g(h(z))$ is of finite order, then either h is a polynomial and g is of finite order, or g is of zero order and h is of finite order. Paper [1926, 4] contains an elegant proof of the '$\cos \pi\rho$' theorem (originally

conjectured by Littlewood [5] and proved by Valiron [6] and Wiman [7]): if f is of order ρ, type 0, where $0 < \rho < 1$, or of smaller order, then

$$\limsup\,[\log m(r)]/[\log M(r)] \geqslant \cos \pi\rho,$$

where m and M are the minimum and maximum moduli of f.

Paper [1927, 4] and [1928, 2], with Hardy and Ingham, are concerned primarily with theorems of Phragmén–Lindelöf type for functions that are analytic (or subharmonic) in a strip. The general question is, what can be deduced about the growth of f from the growth of

$$\phi(x,y) = \frac{1}{2y}\int_{-y}^{y} |f(x+iw)|^p\, dw$$

in a strip $\alpha \leqslant x \leqslant \beta$ (or its closure). Some of the results are surprising, and most of the proofs are difficult.

4. *Zeros of polynomials and other analytic functions.* Pólya was particularly fond of theorems that connect properties of an integral function with properties of the sets of zeros of polynomials that approximate the integral function. He dealt with this general problem in [1913, 8, 9; 1914, 3]; a great deal of later work starts from there. In particular, in these papers he showed that all multiplier sequences that transform polynomials with real zeros into polynomials with real zeros are generated by the special entire functions of types I and II in [1914, 5] and [1915, 1]. These functions are defined as follows:

$$\text{(I)} \qquad \Phi(x) = \frac{\alpha_r}{r!} x^r e^{\gamma x} \prod_{\nu=1}^{\infty} (1+\gamma_\nu x), \quad \gamma \geqslant 0, \quad \gamma_\nu \geqslant 0;$$

$$\text{(II)} \qquad \Psi(x) = \frac{\beta_r}{r!} x^r\, e^{-\gamma x^2+\delta x} \prod_{\nu=1}^{\infty} (1+\delta_\nu x)\, e^{-\delta_\nu x}, \quad \delta_\nu \text{ real}, \quad \gamma \geqslant 0$$

(the zero function is considered to belong to both classes). These functions are now often referred to as Pólya–Schur or Laguerre–Pólya functions. Functions of class I are characterized as limits of polynomials with only real zeros, all of the same sign; functions of class II, as limits of polynomials with only real zeros. A power series $\sum_{n=0}^{\infty} (\gamma_n/n!)\, x^n$ belongs to class II or I according as the polynomial $\sum_{k=0}^{n} x^{n-k} \gamma_k \binom{n}{k}$ has, for all n, either all its zeros real or all its zeros real with the same sign.

There are now many applications of the Laguerre–Pólya functions: they underlie the general inversion theory of convolution transforms (Hirschman and Widder [8]); the theory of variation-diminishing transforms [9], interpolation by spline functions [10], and many other applications. Some more direct applications are given in [1914, 6; 1915, 4; 1927, 6]; these papers require separate notice because they have to do with two other themes that occur frequently in Pólya's work.

Paper [1927, 6] is one of a series dealing with zeros of trigonometric integrals; as Pólya explained in [1918, 4] and [1927, 6], his motivation for this work was that the Riemann ξ-function is represented by a trigonometric integral, so that a sufficiently good theorem about the zeros of trigonometric integrals would establish the Riemann hypothesis. That this hope is almost certainly illusory hardly diminishes the interest of the theorems that Pólya found. In [1918, 4] he started with the Fourier transform of a function supported on $[-1, 1]$, and showed that in certain cases it has only real zeros. Paper [1920, 5] presents almost everything that one would want to know about zeros of exponential polynomials, but without proofs; the proofs are available only in a Zürich thesis by Pólya's student E. Schwengeler [11]. This work led into the modern theory of the zeros of integral functions of exponential type, and also into [1933, 6] and [1937, 2], although Pólya did not contribute very much directly to this subject. In the main, Pólya concentrated on trigonometric integrals over $(-\infty, \infty)$, identifying progressively more general classes of integrals that have only real zeros [1918, 4; 1923, 8; 1926, 5, 8; 1927, 6, 7]. Although some functions that closely resemble the ξ-function do have only real zeros, they do not do much for the Riemann hypothesis; but these papers and their generalizations have been useful for other purposes: notably in physics, as Kac brings out in his comments on [1926, 8] in Vol. 2 of the *Collected Papers*.

In [1927, 7] Pólya raised the question of whether a certain family of inequalities (now known as the Turán inequalities) are satisfied; since they form a necessary condition for the truth of the Riemann hypothesis, that hypothesis would be disproved if any one failed. The first progress on this question was made in 1966 when the inequalities were proved for a sufficiently large index [**12**]; the question was finally settled in 1986 [**13**].

Pólya devoted a great deal of attention to the question of how the behavior in the large of an integral or meromorphic function influences the distribution of the zeros of successive derivatives. His principal papers on this topic are [1914, 6; 1915, 4; 1921, 4; 1937, 4]; the survey [1943, 1] covers almost everything that was known up to 1942. He introduced the term 'final set' of an integral or meromorphic function for the set of limit points of the set of zeros of successive derivatives (counting also the points that are zeros of infinitely many derivatives). In [1922, 1] he determined the final sets of meromorphic functions: the final set is the polygon whose points are equidistant from the two nearest poles. It is much more difficult to determine the final sets of integral functions. Some results up to the early 1970s are mentioned in the comments on [1943, 1] in Vol. 2 of the *Collected Papers*; since then there has been considerable progress. For example, Pólya had showed [1937, 4] that for an integral function that is real on the real axis, and has only finitely many non-real zeros, the final set is a subset of the real axis provided that the order is less than 4/3; his conjecture that 4/3 can be replaced by 2 was established very recently [**14**]. Pólya remarked [1921, 4; 1922, 1] that when f is an integral function, real on the real axis, and f, f' and f'' have no zeros, then f is an exponential function. This has been generalized in various ways; in [**13**] it was shown that if f, f', f'' and f''' have only real zeros then either f is an exponential, or of the form $A(e^{icz}-e^{id})$ (c, d real), or a Pólya–Schur function; [**15**] also contains further results for meromorphic functions. It appears plausible that a Pólya–Schur function, of order greater than 1, real on the real axis, has the whole real axis as its final set; under some additional restrictions, this was established in [**16**].

In [1913, 3] Pólya gave the first correct proof of Laguerre's famous theorem, which states (loosely) that if f is the Laplace transform of $\phi, x > x_0$, and ϕ has V changes of sign, then f has at most V zeros on $x > x_0$. More precise results were found some 20 years later [**17**]. This is only one of many 'sign rules' for zeros; in particular, Sylvester's rule [1914, 1; Theorem IV] remained an isolated curiosity and was only explained in [1958, 3], 44 years later.

Paper [1916, 6] is an 'omnibus' theorem on the reality of zeros of a polynomial. It is worth quoting in full. Let $f(x) = \sum_{k=0}^{n} a_k x^k$ and $g(x) = \sum_{k=0}^{n+m} b_k x^k$ have only real zeros, $b_0, b_1, \ldots, b_n \geqslant 0$. Then the curve $b_0 f(y) + b_1 x f'(y) + \ldots + b_n x^n f^{(n)}(y) = 0$ has n real intersections with every line $sx - ty + u = 0$, where $s \geqslant 0$, $t \geqslant 0$, $s+t > 0$, u real. The special cases $x = 1$, $y = 0$, and $x = y$ are well known.

Paper [1932, 2] is noteworthy as having been the first paper on the zeros of a 'random' polynomial.

5. *Signs of derivatives and analytic behavior.* S. Bernstein **[18]** was the first to observe that a C^∞ function on a real interval, with all its derivatives positive, is real-analytic there; he also showed that the same conclusion follows if a sufficiently dense sequence of derivatives are positive. Much later, in 1940, D. V. Widder **[19]** discovered that if the derivatives of even order alternate in sign then the function is the restriction of an integral function of exponential type. This discovery inspired a substantial amount of activity, in which Pólya participated [1941, 1, 2; 1942, 2, 3] and which he surveyed in detail in [1943, 1]. The topic is still alive; see the comments on [1942, 2] and [1943, 1] in Vol. 2 of the *Collected Papers*; also **[20]**.

6. *Conformal mapping.* Most of Pólya's work in this field is connected with the notion of transfinite diameter. In [1928, 4] this is connected with the recurrent determinants of the coefficients, and applied there and in [1928, 5] to Koebe's $\frac{1}{4}$ theorem, to properties of the coefficients adjacent to Hadamard gaps, and the continuability of power series. Pólya reformulated the usual geometric statements about univalent functions, and then generalized them to maps of multiply-connected regions. In [1931, 4], Pólya and Szegő extended the concept of transfinite diameter to three dimensions, thereby opening up a new field of research; they made a conjecture that was proved by Pólya and Schiffer: the transfinite diameter of a convex curve is no less than one-eighth of the perimeter [1959, 2]. In [1958, 3] Pólya and Schoenberg made a major contribution to mappings onto convex domains by univalent functions. This seminal paper inspired a great deal of further research.

7. *Real analysis.* It is arguable that Pólya's most important contributions to this area are in his share of the book [1934, 4] *Inequalities* by Hardy, Littlewood and Pólya. This was the first systematic study of the inequalities that are used by every working analyst. Although there are more recent books that contain more material in certain directions, this one is still a fundamental reference—even though Hardy was once heard to complain that whenever he needed an inequality, the precise one that he wanted was not there. Pólya published little about inequalities himself, but did propose many problems about inequalities and series. Paper [1923, 3] is about the structure of real sequences; it is here that Pólya introduced what are now called Pólya peaks, which form an essential tool in many problems about integral functions. Paper [1950, 1] contains an inequality that has many applications to eigenvalues of operators.

Paper [1913, 5] is of interest as the first construction of a Peano curve with at most triple points (this being the smallest possible number).

Pólya was much interested in mean value theorems, whether for isolated functions or for solutions of differential equations. Paper [1921, 5] is a little-known gem in three dimensions. The theorems in [1922, 3, 4] on mean value theorems corresponding to a linear homogeneous differential equation are still used extensively. In [1931, 5] Plancherel and Pólya studied the mean value

$$\lim_{R\to\infty}\int_{x-R}^{x+R} f(u)\,du = \phi(x).$$

They showed that if $\phi(x)$ exists for all x, it is necessarily a linear function. They also discussed a similar problem in two dimensions, where the limit is necessarily harmonic. Paper [1934, 1] deals with analogues of Rolle's theorem for partial differential operators.

Papers [1926, 7; 1927, 3; 1938, 1, 4] deal with moment sequences and the total indeterminancy of the Hamburger moment problem for functions of bounded variation, as well as with infinite systems of linear equations.

8. *Approximation theory and numerical analysis.* Papers [1914, 4; 1968, 1] deal with Graeffe's method for solving polynomial equations approximately. Until recently the method was not highly regarded because of the large amount of computation it requires, but with modern high-speed machines it is becoming useful again. Paper [1933, 2] was a pioneering investigation on numerical quadratures and is still an important result.

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COMMENTS ON NUMBER THEORY

D. H. LEHMER

Pólya knew quite a lot of number theory and remained interested in the subject. As late as 1968, at the age of 80, he contributed an article about the sign of the error term in the prime number theorem to a collection of papers in honor of Edmund Landau [1968, 2]. However, only a few of his papers were entirely number theoretic. In reading his papers about power series with integer coefficients, for example, one is struck with the occasional applications of a number theory principle or a theorem from number theory.

Pólya was especially interested in the Legendre symbol (n/p) which is 1 or -1 according as n is a quadratic residue or a non-residue of an odd prime p. In 1912 Fekete announced the conjecture that the polynomial

$$\left(\frac{1}{p}\right)+\left(\frac{2}{p}\right)x+\left(\frac{3}{p}\right)x^2+\ldots+\left(\frac{p-1}{p}\right)x^{p-2}$$

has no real root between 0 and 1. Pólya [1919, 2] showed six years later that the conjecture was false for $p = 67$ and for infinitely many other primes. In the same paper he announced the conjecture bearing his name:

The excess of the number of integers $\leqslant x$ which have an odd number of prime factors over the number of integers with an even number of prime factors is non-negative.

In symbols: $L(x) \leqslant 0$ for $x > 1$, where $L(x) = \sum_{n\leqslant x}\lambda(n)$ and where $\lambda(n)$ is the Liouville function

$$\lambda(p_1^{\alpha_1}\ldots p_t^{\alpha_t}) = (-1)^{\alpha_1+\ldots+\alpha_t}.$$

Pólya himself verified his conjecture for $x \leqslant 1500$. This conjecture had a life of 40 years. By the 1950s it had been tested to 10^6 and for isolated values of x well beyond 10^6. Then in 1958 Haselgrove [**2**] disproved the conjecture by showing the existence of infinitely many x for which $L(x) > 0$. Not satisfied by this existence theorem, R. S. Lehman [**3**] found in 1960 that $L(906180359) = 1$. In 1980 Tanaka [**4**] examined all the numbers less than 10^9 and found the least failure of the Pólya conjecture to be 906150257.

Returning to the Legendre symbol, Pólya's name is attached to the useful inequality [1918, 3]

$$\left|\sum_{m=a}^{b}\left(\frac{m}{p}\right)\right| < \sqrt{p}\log p.$$

In fact this is easily extended to

$$\sum_{m=a}^{b}\chi(m) = O(\sqrt{k}\log k)$$

where $\chi(m)$ is any non-principal Dirichlet character modulo k.

As is well known, Pólya was very much concerned with plausible and heuristic reasoning. When visiting Berkeley in 1959 he became aware of the results of an unpublished study that I had made of the number $\Pi_d(x)$ of pairs of primes $(p, p+d)$

for $x < 37{\cdot}10^6$. As a consequence he wrote an informal account [1959, 3] based on these data of the steps in a chain of reasoning that led him to the formula

$$\Pi_d(x) = 2C_2 \prod_{p|d} \left(\frac{p-1}{p-2}\right) \frac{x}{(\log x)^2},$$

where C_2 is the 'twin prime constant'

$$C_2 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) = 0.6601618\ldots.$$

This formula was conjectured 30 years earlier by Hardy and Littlewood [**1**]. The paper ends with this moral: 'Mathematicians and physicists think alike; they are led, and sometimes misled, by the same pattern of plausible reasoning.'

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PÓLYA'S GEOMETRY

Doris Schattschneider

One of Pólya's greatest strengths was his ability to recognize the importance of geometry in solving a variety of problems. His appreciation for and use of geometry is found throughout his work.

One of his earliest papers [1913, 5] exploits the geometry of a scalene right triangle to construct a Peano curve in which the unit interval [0, 1] is continuously mapped onto the triangle and its interior. The construction is not well-known, yet is exceedingly easy to describe, and provides a lovely example of a branching algorithm. A scalene right triangle has the property that the altitude from its right angle to its hypotenuse splits the triangle into two smaller right triangles T_1, T_2 of unequal size, both of which are similar to the original triangle. The foot of that altitude is the apex of altitudes that split each of T_1 and T_2 into two similar but unequal triangles; and so on. Each $a \in [0, 1]$ can be represented in dyadic (binary) form as $0.a_1 a_2 a_3 \ldots$, where each a_i is 0 or 1; thus $a = \sum_{i=1}^{\infty} a_i 2^{-i}$. This representation of a determines a path along the infinite binary tree of altitudes constructed in the triangle. Start at the foot of the first altitude; if $a_1 = 0$, choose the altitude of the smaller of T_1 and T_2. Travel to the foot of that altitude, where a_2 determines the next choice ($a_2 = 0 \Rightarrow$ smaller; $a_2 = 1 \Rightarrow$ larger), and so on. Figure 1 (taken from Pólya's paper) shows the beginning of the path determined by $a = 0.1101\ldots$. For each a, the path determined by any dyadic representation of a converges to a unique point in or on the original triangle, and the function that maps a to that point is the desired Peano curve. Pólya shows that the curve passes at most three times through any point of the triangle.

Geometric symmetry, and particularly the question of enumeration of symmetry classes of objects, was an early and continuing interest of Pólya. Although not the first

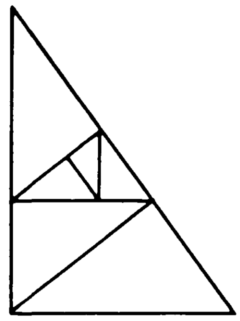

FIG. 1

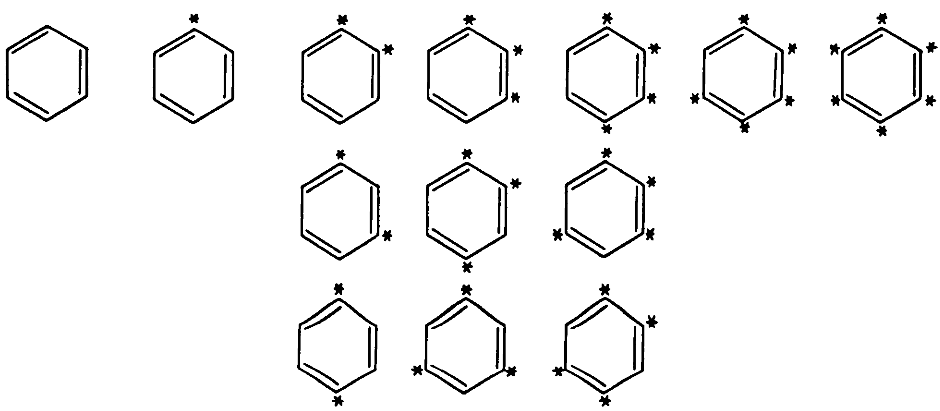

FIG. 2

to enumerate the 17 plane crystallographic groups (Fedorov had published this in 1891), Pólya was the first to illustrate each of these with a representative tiling [1924, 1]. Some tilings he took from classical sources and others he made up. In following his own familiar pedagogical maxim 'make a picture', he provided the reader with concrete figures in which to examine the differences in the symmetries of the tilings as well as to discern how, in a single tiling, the congruent copies of the tiles were related by symmetries. One reader of Pólya's paper for whom the illustrations conveyed all of the essential information was the Dutch artist M. C. Escher. The paper had been brought to Escher's attention by his brother, a geologist at the University of Leiden. In the three years 1937–39, Escher energetically produced over 25 colored periodic drawings of interlocked creatures, and corresponded with Pólya about his work [**3**].

Geometric symmetry, blended by Pólya with the theory of permutation groups and generating functions, produced one of his landmark theorems [1935, 4], and indeed, launched a whole theory which has come to be known as *Pólya enumeration*. The problem he attacked had come from the theory of *isomers* in organic chemistry, individual chemical species with identical molecular formulae but displaying differing physiochemical properties (such as different arrangements of atoms). Chemists had sought, but failed to find, an algebraic technique that would enumerate isomerism classes of an organic chemical compound. A simple example (Figure 2) is the enumeration of the potential isomers formed when benzene (which has a hexagonal molecular 'frame') is substituted by a univalent radical $*$.

We can explain Pólya's theorem by obtaining this enumeration algebraically, using 'Pólya enumeration'. If G is a group of permutations on n symbols, the *cycle index* of G is a polynomial in n variables $x_1, \ldots, x_n$ which encodes information on how elements of G can be written as a product of disjoint cycles. Each element $\pi \in G$ determines a monomial $m(\pi)$ in the variables $x_1, \ldots, x_n$ as follows. Write π as a product of disjoint cycles, and to each cycle of length i associate the variable x_i; $m(\pi)$ is the product of these variables. The cycle index of G is the polynomial $P_G(x_1, \ldots, x_n) = |G|^{-1} \sum_{\pi \in G} m(\pi)$. For example, if G is the symmetry group of a regular hexagon (with vertices numbered cyclically $1, \ldots, 6$), then G is the dihedral group D_6 and the element $\pi = (135)(246)$, which corresponds to a rotation of 120°, has $m(\pi) = x_3^2$. The cycle index for D_6 is

$$\tfrac{1}{12}(x_1^6 + 4x_2^3 + 3x_1^2 x_2^2 + 2x_3^2 + 2x_6).$$

Labels are attached to the atoms of a molecular structure, and the figure-counting generating function records the number of ways in which this can be done. Pólya's theorem states that the isomerism enumeration generating function is obtained by substituting in the cycle index (of the permutation group that leaves invariant the molecular frame) the figure-counting generating function. We illustrate with the benzene example (Figure 2) in which there is one type of substituent [so the vertices of the hexagonal frame have no label (w), or have the * label (b)]. In this case each variable x_i in the cycle index for D_6 is replaced by $b^i + w^i$ from the figure-counting series. The function so obtained is

$$\tfrac{1}{12}[(b+w)^6 + 4(b^2+w^2)^3 + 3(b+w)^2(b^2+w^2)^2 + 2(b^3+w^3)^2 + 2(b^6+w^6)]$$
$$= b^6 + b^5 w + 3b^4 w^2 + 3b^3 w^3 + 3b^2 w^4 + b w^5 + w^6,$$

whose coefficients give the numbers of distinct isomers.

The far-reaching implications (and applications) of this theorem were apparent to Pólya, who gave a lengthy discussion and proof in the mathematical literature [1937, 3], and in addition gave explanations and illustrations of the enumeration method in several other scientific journals [1936, 1, 2, 3, 4]. Two recent papers by chemists describe the importance of Pólya's result and its many extensions (see **[2]** and **[4]**).

The role of geometric symmetry in minimizing geometric quantities and the resulting analogous minimization of physical quantities is lucidly explained by Pólya in his chapter 'Circle, sphere, symmetrization and some classical physical problems' [1961, 1] in a text for engineering students. He begins with the observation of Lord Rayleigh that of all membranes of equal area, the circle has not only the shortest perimeter, but also the lowest principal frequency. He then shows how the process of Steiner symmetrization of a geometric figure, while preserving area or volume, has a profound minimizing effect on physical quantities such as the Dirichlet integral, the principal frequency, and electrostatic capacity. The analogy between isoperimetric problems and certain physical quantities is also considered by Pólya in [1948, 4; 1951, 2; 1954, 4].

A very different sort of geometric problem, that of sightlines in an orchard, dates from an early paper [1918, 5] and appears as problem 239 in Pólya and Szegő's *Problems and theorems in analysis* [1925, 3]. A given circular orchard consists of trees of uniform thickness which are planted in a uniform array (like an integer lattice). How thick must the trees grow if, from the center of the orchard, every tree is visible, but the view beyond the orchard is completely blocked? A very nice account of the problem which gives a precise solution (Pólya gave only upper and lower bounds on the radius of the trees) can be found in [1]. Geometric symmetry plays a small role here as well—only a 45° wedge of the orchard need be considered.

Pólya used geometry in at least two distinct ways in his various pedagogical writings (notably, [1945, 2] and [1962, 1]). First, geometry was a favorite source of illustrative examples from which he could clearly demonstrate his teaching and problem-solving maxims. His generous use of figures shows the dynamic process of solution of geometric problems: look at a simple or special case, examine part of the problem or a related problem, 'turn it over and over, consider it under various aspects, study all sides' [1962, 1; Vol. 1, p. 111], look at it in a higher (or lower) dimension, embed it in a familiar figure, transform it and obtain information from the transformation process or from the transformed state. A second, quite different, use of geometry by Pólya was as metaphor and in representation of the solution process itself. Chapter 7 of *Mathematical discovery* is entitled 'Geometric Representation of the Progress of the Solution'. Here the geometric words *connection*, *bridge*, *chain* and *thread* serve as descriptions of the ways in which solutions are built. What follows then is a detailed solution of a problem (in solid geometry) from which Pólya develops a schematic diagram (a digraph!) representing the multilevel process and progress of solution. This schematic representation of the problem and its solution is emblazoned on the endpapers of the book.

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PÓLYA'S ENUMERATION THEOREM

R. C. READ

During his long life George Pólya made notable contributions to many different branches of mathematics; but to combinatorialists he is chiefly known for his enumeration theorem—the 'Hauptsatz' of his 1937 paper [1937, 3] or [1987, 1]. This paper is remarkable in many ways, not the least being that it is a long paper devoted almost exclusively to a single theorem and its applications.

Pólya's theorem, as it is generally called, solves a very general type of combinatorial problem, which can be expressed in everyday terms as follows. Suppose we have a number of 'boxes' and a store of objects—called 'figures'—exactly one of which is to be placed in each box. (It may happen that two or more boxes receive the same figure.) The resulting structure—boxes plus figures—is called a 'configuration'. We further suppose that every figure has a 'content', usually a non-negative integer, and we define the content of a configuration to be the sum of the contents of the figures in the boxes. Now, if the boxes are all distinct, it is a simple matter to determine the number of configurations having a given content.

What will happen if the boxes are not all distinct? In that case a certain rearrangement of the boxes in a configuration will give a new configuration which is equivalent, in some sense, to the original. More precisely, we have a certain group G of permutations of the boxes, and we say that two configurations are equivalent if one can be obtained from the other by permuting the boxes by some element of G. We then ask 'What is the number of inequivalent configurations having given content?' This is the problem to which Pólya's theorem provides the answer.

Pólya made elegant use of counting series and generating functions in his paper. The 'figure counting series' for a Pólya type problem is the power series in which the coefficient of x^n is the number of figures having content n; this is usually known from the statement of the problem. The 'configuration counting series' is defined similarly for configurations of content n; this series therefore summarizes all the answers to the problem. What is the connection between these two series that enables us to calculate the second from the first?

Clearly it must depend on the group G. Any permutation can be expressed as a product of disjoint cycles, and this expression is unique apart from the order of the cycles. For every permutation g of G we form a monomial $s_1^{j_1} s_2^{j_2} \ldots$ where j_i is the number of cycles of length i in g. The average of these monomials over all elements of G is what Pólya called the 'cycle-index' of G. Pólya's theorem then states that the configuration counting series is obtained by substituting the figure counting series in the cycle index, by which is meant that if we denote the figure counting series by $f(x)$, we replace every occurrence of s_i in the cycle index by $f(x^i)$.

This is the theorem that makes possible the routine solution of a wide range of problems of both theoretical interest and practical importance. Let us look at some of these applications.

A typical application is seen in the enumeration of rooted trees. In a rooted tree, one vertex—the root—is distinguished from the others; suppose there are k edges incident with this vertex. At the other end of these edges we have again a rooted tree. Thus we have k 'boxes', in each of which we can place a rooted tree. In the basic problem of this kind the edges at the root, and hence the boxes, can be permuted in any way, and hence, in applying Pólya's theorem we take the group to be the full symmetric group S_k. The theorem then gives us the counting series for these trees (with k edges at the root) in terms of the counting series, $T(x)$ say, for all rooted trees. Summing this result over all values of k, and allowing for the extra vertex, the root, we reconstruct the counting series $T(x)$, which is then defined recursively. Such recursive definitions of a counting series are common in many applications of Pólya's theorem, and though they rarely give rise to an explicit formula for the number of configurations being enumerated, they are usually quite suitable for the numerical calculation of these numbers.

Chemical compounds can be represented by their structural formulae which, to the graph theorist, are simply rather special types of graphs. In particular, acyclic chemical compounds, having no rings, correspond to trees. By methods similar to that outlined in the last paragraph, Pólya carried out the enumeration of various kinds of acyclic compounds, such as the alkanes (or paraffins), substituted alkanes, and many other families of compounds, determining the numbers of isomers (different compounds having the same numbers of atoms of various kinds). By appropriately choosing the group G he was able to effect this enumeration for the case where the shape of the molecule was taken into account (stereo-isomers) as well as when it was not. In his 1937 paper and in some other papers published in preceding years [1935, 4; 1936, 2; 1936, 3; 1936, 4] Pólya also enumerated some kinds of cyclic chemical compounds, obtained by adding alkyl radicals, or other tree-like figures, to the atoms of various simple ring structures.

These and many other applications of the theorem make up the bulk of Pólya's paper. In the last section, however, Pólya applied his considerable analytical powers to derive asymptotic results for the many enumerative problems solved in the earlier sections. In so doing he was paving the way for much of the asymptotic enumeration that was to be carried out by later research workers.

Pólya applied his theorem to many problems besides those set out in his main paper. In 1940 he used it to solve a problem in logic [1940, 1], and it is known that he had successfully enumerated unlabelled graphs with given numbers of vertices and edges. Strangely enough, he never published his work on this problem, although it is an elegant and practical application of this theorem. This enumeration can be effected as follows. For graphs with a given number, p, of vertices, we regard every pair of vertices in the graph as being a 'box' into which we can put one of two figures, namely 'edge' or 'no edge', with contents 1 or 0 respectively. Since the graphs are unlabelled there are no distinctions between vertices, which can therefore be permuted by any element of the symmetric group S_p. These permutations induce a group $S_p^{(2)}$ of the pairs of vertices, that is, of the boxes for this problem. The cyclic index of $S_p^{(2)}$ can be computed without too much trouble, and the required enumeration then follows directly by means of Pólya's theorem. Although Pólya did not publish this result himself, he communicated it to Harary, who published it in [2].

In the fifty years since the publication of Pólya's paper many advances have been made in the kind of enumerative combinatorics with which the paper was concerned. Chemists have found that Pólya's theorem can be used in their field, not just for finding the numbers of isomers of families of compounds, but for many other problems of a practical nature. Graph theorists with an interest in enumeration have made great use of Pólya's theorem, and in their endeavours to enumerate more and more complicated graphs have considerably extended Pólya's work. Generalizations of Pólya's theorem have been derived by de Bruijn [1], Harary and Palmer [3], Robinson [5] and many others. Indeed, in the last two or three decades enumerative graph theory has become a recognized branch of combinatorics (see [4] for further details).

This then was Pólya's outstanding contribution to the theory of enumeration, a remarkable theorem in a remarkable paper, and a landmark in the history of combinatorial analysis.

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PÓLYA'S CONTRIBUTIONS IN MATHEMATICAL PHYSICS

M. M. SCHIFFER

Pólya's interest in complex analysis, conformal mapping and potential theory led him to the study of boundary value problems for partial differential equations and the theory of various functionals connected with them. This interest was strengthened by his teaching at the ETH in Zürich and at Stanford University which brought him in contact with engineers and their problems in classical physics and applied mathematics. In most cases boundary value problems for partial differential equations can be solved only in very special cases and for all other cases only approximate results can be obtained.

So, Pólya developed various techniques for estimating difficult functionals in terms of easier accessible quantities, be it other functionals or geometric quantities like area or volume. He obtained a large number of important and elegant inequalities and methods of approximation. Instead of enumerating such results, we shall concentrate on a few characteristic ideas to illustrate his approach.

Symmetrization. In many cases, boundary value problems for a given domain can be solved easily when the domain has a high symmetry; for example, when it is a circle or a sphere. From such a solution, general insights into the solutions for arbitrary domains can be obtained. As a typical example, let us take the case of the eigenvalues of a vibrating membrane. That is, consider a plane domain D with boundary C and the partial differential equation $\nabla^2 u + \lambda u = 0$ with the boundary condition $u = 0$ on C. In general, the only solution is the trivial one: $u \equiv 0$. However, for a series of positive numbers $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \ldots$, there exist solutions $u_\nu(x, y)$ which are not identically zero. These λ_ν are called the eigenvalues and the functions u_ν the eigenfunctions of the problem. These solutions are known only for a small number of domains D, in particular for the circle. In 1894 Lord Rayleigh conjectured that among all domains of given area the circle has the lowest eigenvalue λ_1 [**4**]. He justified this conjecture by considering all domains for which λ_1 was known and also by a simple variational argument. However, only in 1923 did Faber give the first proof of this conjecture [**1**], while in 1924 Krahn gave an independent proof [**2**]. On the basis of this result, the difficult eigenvalue λ_1 can be estimated from below in terms of the much more accessible geometric measure of area. This fact plays a role in acoustics and potential theory.

Inequalities between different domain functionals occurred already in classical antiquity. The most famous one stated that among all plane domains D with given area A, the circle has the least perimeter L; that is, for all domains D, we have the inequality $L^2 \geqslant 4\pi A$. This is the famous isoperimetric inequality, which may be considered as the first result in the calculus of variations. Because of it, we call now, in general, inequalities between various domain functionals 'isoperimetric inequalities'. Much effort was made to give a rigorous proof for the classical isoperimetric inequality. One of the most ingenious methods of proof is due to the Swiss mathematician Jakob Steiner (see [1961, 1] and [**5**]). He introduced the concept of symmetrization of a plane domain D with respect to a given line. Suppose that D lies in the (x, y)-plane over an interval $a \leqslant x \leqslant b$ of the abscissa. Through every point x in this interval, we draw a vertical line which intersects the domain D in one or more intervals of total length $\lambda(x)$. On another copy of the interval (a, b), we draw through each point x a vertical line of length $\lambda(x)$ which is centered on that axis, that is, $-\frac{1}{2}\lambda(x) \leqslant y \leqslant \frac{1}{2}\lambda(x)$. The endpoints of these segments determine a closed curve C^* which is symmetric with respect to the x-axis and which determines a domain D^* with the same symmetry. We call D^* the symmetrization of D with respect to the x-axis. It is easy to see that D^* has the same area as the original domain D but that the length L^* of its boundary is less than or equal to the perimeter L of D. Thus, the domain of given area with the least perimeter must have the highest symmetry, that is, it must be a circle. Precisely the same method of symmetrization works in space with respect to a plane. It follows that among all bodies with given volume V, the sphere has the least surface area S, that is, we have the isoperimetric inequality: $V^2 \leqslant (1/36\pi) S^3$.

Now, Pólya made the ingenious remark that Steiner's method allows a large number of applications to the theory of partial differential equations (see [1948, 4] and [1961, 1]). Consider a domain D in the (x, y)-plane and a function $f(x, y)$ defined in D which vanishes on its boundary C. This determines a body B in the (x, y, z)-space which is bounded by the surface $z = f(x, y)$ and the plane piece D. Let Π be a plane which is perpendicular to the plane $z = 0$ and symmetrize B with respect to Π. This gives a body B^* of the same type which is bounded by a surface $z = f^*(x, y)$ and by a flat base D^* in the (x, y)-plane. Then B^* has the same volume as B but a lesser surface area. Observe also that D^* has the same area as D. By the calculus formula for the surface area, we have therefore:

$$\iint_D (1 + f_x^2 + f_y^2)^{\frac{1}{2}} dx\, dy \geqslant \iint_{D^*} (1 + f_x^{*2} + f_y^{*2})^{\frac{1}{2}} dx\, dy.$$

Now consider the function $\varepsilon f(x, y)$ with an arbitrary smallness parameter $\varepsilon > 0$. The above inequality holds again and, by passage to the limit $\varepsilon \to 0$, we conclude that

$$\iint_D (f_x^2 + f_y^2)\, dx\, dy \geqslant \iint_{D^*} (f_x^{*2} + f_y^{*2})\, dx\, dy.$$

Now, the integral

$$D[f] = \iint_D (f_x^2 + f_y^2)\, dx\, dy$$

occurs in many applications in classical physics as the Dirichlet energy integral. Thus, Pólya has found a method of symmetrization for the Dirichlet integral in a domain D for arbitrary functions $f(x, y)$ in it which vanish on its boundary C.

An immediate application of this idea was a new elegant proof of Rayleigh's conjecture. Another was a proof of a conjecture of Poincaré that among all conducting surfaces that enclose a given volume, the sphere has the least electrostatic capacity [3]. This had already been proved by G. Szegő by an ingenious argument [6], but now it followed quite easily from a general method.

A large number of further inequalities relating to Dirichlet's integral could be obtained. In collaboration with Szegő, Pólya wrote a monograph *Isoperimetric inequalities in mathematical physics* [1951, 2] which is now a classic and has stimulated many further investigations in this field. Combining various methods with great analytical skill the authors obtain estimates and inequalities for numerous functionals. We mention capacity, torsional rigidity, virtual mass, polarization, transfinite diameter and interior radius. Various new kinds of symmetrizations are invented; for example, circular symmetrization with respect to a given ray. The whole work displays the taste of the authors for the concrete and explicit result, for elegance and ingenious methods.

Transplantation. Many functionals that occur in classical physics and engineering can be characterized by extremum problems in appropriate function spaces. Consider, for example, the torsional rigidity P of a plane domain D. It is defined in terms of the stress function $f(x,y)$ of D which satisfies in D the partial differential equation $\nabla^2 f+2=0$ and which vanishes on the boundary C of D. In terms of $f(x,y)$, the functional P is given as

$$P=\left(2\iint_D f(x,y)\,dx\,dy\right)^2\cdot\left(\iint_D (\nabla f)^2\,dx\,dy\right)^{-1}.$$

It is then easily seen that for every continuously differentiable function $F(x,y)$ in D which vanishes on C, we have

$$P\geqslant\left(2\iint_D F(x,y)\,dx\,dy\right)^2\cdot\left(\iint_D (\nabla F)^2\,dx\,dy\right)^{-1}.$$

Thus, we can define P directly by means of a maximum problem within a given function class and can use every element in that class to obtain a lower bound for it.

But we can use this characterization to a much greater advantage. We may imbed the given domain D in a one-parameter family of domains $D(t)$, where $D(t)$ is obtained from $D=D(1)$ by stretching of the entire (x,y)-plane $x'=tx, y'=y, t>0$. Now let $f(x,y)$ be the correct stress function of the domain $D(t_0)$. Then $f(t_0 t^{-1}x, y)$ is defined in $D(t)$, vanishes on the boundary $C(t)$ and is thus an admissible competing function in the extremum problem which defines $P(t)$. We call this function the transplant from $D(t_0)$ to $D(t)$ of the extremum function for $D(t_0)$. An easy calculation shows that the maximum property of $P(t)$ implies

$$\frac{t}{P(t)}\leqslant t_0\frac{\tau t_0^2\iint_{D(t_0)} f_x(x,y)^2\,dx\,dy+\iint_{D(t_0)} f_y(x,y)^2\,dx\,dy}{\left(2\iint_{D(t_0)} f(x,y)\,dx\,dy\right)^2}$$

with $\tau=t^{-2}$. The right-hand-side of this estimate has the form $a\tau+b$ and for $\tau_0=t^{-2}$ we have equality in this estimate. Let us plot the term $tP(t)^{-1}$ versus the abscissa τ; we can assert that through each point on that curve passes a straight line such that all curve points lie below it. Thus this curve is convex and these straight lines are its supporting lines. The slope of the supporting line at the point t_0 is

$$a=\tau_0^{-\frac{3}{2}}\iint_{D(t_0)} f_x^2\,dx\,dy\cdot P(t_0)^{-2}\geqslant 0,$$

so that the curve is non-decreasing. If in the family there is a domain $D(t_0)$ for which the stress function is known, we obtain estimates for all $P(t)$.

This example shows clearly the general idea of how we can utilize the extremum characterization of a functional to study the parameter dependence of the functionals for a parameter-dependent family of domains. One uses the extremum function of a given domain in the family as a test function for all other domains in the family by proper transplantation. This idea was carried out in many cases in an extensive paper in collaboration with M. Schiffer [1954, 1]. The most useful extremum problems considered were the Dirichlet and the Thomson principles. Since these principles give, in general, lower and upper bounds, respectively, for the functionals considered, they work very well in combination to bound the functional from below and above. The functionals discussed in the paper were, for example, virtual mass, capacity, membrane eigenvalues and outer radius. The paper stimulated much further research along these lines and was followed by a number of extensions and generalizations of its results and methods. In particular, J. Hersch and his group at the ETH in Zürich were especially active and successful in this direction.

The Method of Difference Equations. Many boundary value problems in the theory of partial differential equations can be solved numerically and approximately by studying appropriate difference equations. Pólya made important contributions in this connection. He combined the approximation by difference equations in an ingenious way with the Rayleigh–Ritz method (see [1952, 3] and [1954, 3]). He interpolated the discrete function values obtained by the difference equation method at the grid points in a linear or bilinear way. Thus he obtained functions defined at all points in the domain considered and used these functions as particularly good test functions in the Rayleigh-quotient. These ideas are closely related to the method of finite elements which has developed to a favorite tool in applied and engineering mathematics. Again he combined various general techniques to obtain upper and lower bounds for the eigenvalues in question.

Space does not permit enumerating his many further ideas, conjectures and methods in this field. But we should mention the number of his colleagues and collaborators whom he has attracted to this subject and who under his inspiration have continued in his tradition: to mention only a few, Hersch, Pfluger, Payne, Weinberger and Weinstein.

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GEORGE PÓLYA AND MATHEMATICS EDUCATION

Alan H. Schoenfeld

It has been noted that ordinary mortals can see a long way when they stand on the shoulders of giants. Pólya invoked this image in acknowledging his debt to Descartes, whose introspections about his own mathematical thinking (see, for example, Descartes' *Rules for the Direction of the Mind*) served as a major inspiration for Pólya's thoughts on the topic (see, for example, *How to solve it*, the two volumes of *Mathematics and plausible reasoning*, and the two volumes of *Mathematical discovery*). This homage was made with Pólya's typical humility. The fact is that those who explore the nature of mathematical thinking stand atop a pyramid of giants—and their feet are firmly set on Pólya's shoulders.

How strong is Pólya's influence? Education lies in the public arena, so there are two aspects of that influence to explore: (1) the impact of Pólya's work and ideas in the real world, and (2) the solidity of his work as a base for making scientific progress on issues related to understanding and teaching the nature of mathematical thinking. We consider both in order.

How to solve it, Pólya's first book-length foray on heuristics and education, appeared in 1945. (This was hardly his first educational work: Pólya and Szegő's *Aufgaben und Lehrsätze aus der Analysis I* appeared in 1925, and it was preceded by writings on heuristics.) The flyleaf of *How to solve it* contained the outline of Pólya's four-stage approach to the problem solving process: understanding the problem, devising a plan, carrying out the plan, and looking back (checking the solution). More than three decades later, the National Council of Teachers of Mathematics (NCTM) declared, in its 1980 *Agenda for Action* [3], that 'Problem solving must be the focus of school mathematics in the 1980s'. To help this process along, NCTM devoted its 1980 Yearbook [2] to *Problem Solving in School Mathematics*. If you open the Yearbook you will find Pólya's four-stage approach to problem solving reproduced in its inside covers. Continue reading and you will find that the vast majority of articles are based on Pólya's ideas about mathematical thinking. Such obvious homage, combined with Pólya's position as honorary president of the Fourth International Congress on Mathematical Education (Berkeley, 1980), testify to Pólya's pre-eminence as a mathematics educator.

Moreover, Pólya's influence extends far beyond the mathematics education community. Just as the 'back to basics' movement in mathematics during the 1970s was symptomatic of the 'basic skills' movement cutting through education at large, we find that the 'problem solving movement' cuts a wide swath in the 1980s. In addition to the predictable citations of Pólya's work in the *American Mathematical Monthly*, the *Journal for Research in Mathematics Education*, and other journals with an emphasis on mathematics education, one also finds recent citations of Pólya's writings in the *American Political Science Review*, *Annual Review of Psychology*, *Artificial Intelligence*, *Computers and Chemistry*, *Computers and Education*, *Discourse Processes*, *Educational Leadership*, *Higher Education*, *Human Learning*—to name just a few.

The scientific status of Pólya's work on problem-solving strategies has been more problematic. While in general the quality of one's contributions to mathematics is pretty clear, the quality of one's contributions to the psychology of thinking is less so. (Consider, for example, the rises and falls of Freud's reputation through the years.) It is true that Pólya's writings on 'modern heuristic' have generally struck a resonant chord with mathematicians, and have inspired numerous mathematics educators to teach problem solving *via* heuristics—but it is also true that such attempts, for the most part, have had minimal success. The math-ed literature is chock full of heuristic studies with 'promising' results. That is, students and instructors alike felt that a heuristics-based approach to course work was worthwhile, but there was rarely convincing evidence to show that the students' problem-solving performance had actually improved as a result of that approach. On the basis of instructional results, Pólya's theoretical ideas can be challenged.

Perhaps more importantly, those ideas have been challenged by a set of competing ideas from another discipline. In contrast to the fuzziness of qualitative psychology and of some educational experimentation, researchers in artificial intelligence (AI) offered what they would call real science. If one adopts the hardnosed AI point of view, no statement about cognition is proved until you have a runnable computer program that embodies that statement—so no problem solving theory is accepted until you have a program that solves problems using the theory. By that standard, Pólya's ideas fall short. As one leading AI researcher put it, 'We tried to write problem solving programs using Pólya's heuristics, and they failed; we tried other methods, and they succeeded. Thus we suspect the strategies he describes are epiphenomenal rather than real—and even if they are real, they're far less important than the ones we use in our programs.'

In both mathematics education and in AI, then, there has been empirical reason to question the solidity of the foundations established by Pólya. In recent years, however, there is increasing evidence of the solidity of those foundations. There is reason to believe that on both counts—in mathematics education and in artificial intelligence—the next decade will swing scientific opinion back in Pólya's direction. In essence, the difficulties with the implementation of Pólya's ideas were that (a) they were not specified in adequate detail for implementation, and (b) they appeared to be superseded by more 'general' methods. Recent work in cognitive science has provided the means of addressing both of these issues. First, cognitive science has provided methods for fleshing out the details of Pólya's strategies, making them more accessible for problem solving instruction. There are now studies providing clear evidence that students can learn problem solving *via* heuristics, with significant improvements in their problem-solving performance. (See, for example, [**4**].) In addition, the general methods of AI have turned out to be much weaker than had been thought; methods once thought general and powerful have turned out to have limited scope and power. Research from the past decade indicates that problem-solving strategies are much more tightly bound to domain-specific subject matter understandings than early AI researchers had claimed. In consequence, current research focuses on the elaboration of problem-solving strategies tied to bodies of subject matter. With increased sophistication in characterizing 'the knowledge structures required to operate on semantically rich domains' (for example, mathematical problem solving), the field has reached the point where it may be possible to program computer-based knowledge structures capable of supporting heuristic problem-solving strategies of the type Pólya described. Should that be the case—and this author predicts it will—research will provide the tools to implement Pólya's intuitions about problem solving, which will serve as part of the foundation for a true 'science of thought'.

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1983

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1984

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1987

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2* *The Pólya picture album: encounters of a mathematician* (with G. L. Alexanderson, ed.), Birkhäuser.

A list of problems posed and solved is unusual in a bibliography, but because of Pólya's special interest in and contributions to problem-solving, we list here problems and solutions he published in a variety of journals. Some of these have been often cited in the literature and led to a number of subsequent investigations.

P = problem S = solution

Amer. Math. Monthly: 51 (1944), 96, P 4108; 51 (1944), 167, P 4111; 51 (1944), 533, P 4138; 51 (1944), 593, P 4142; 53 (1946), 279–282, S 4138; 53 (1946), 591, P E748; 54 (1947), 107, P E756; 54 (1947), 473, S E756; 54 (1947), 340, P E780; 54 (1947), 346, P 4255; 54 (1947), 479, P 4264; 55 (1948), 162, P E780.
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