SRINIVASA RAMANUJAN.

I.

SRINIVASA RAMANUJAN, who died at Kumbakonam on April 26th, 1920, had been a member of the Society since 1917. He was not a man who talked much about himself, and until recently I knew very little of his early life. Two notices, by P. V. Seshu Aiyar and R. Ramachandra Rao, two of the most devoted of Ramanujan's Indian friends, have been published recently in the *Journal of the Indian Mathematical Society*; and Sir Francis Spring has very kindly placed at my disposal an article which appeared in the *Madras Times* of April 5th, 1919. From these sources of information I can now supply a good many details with which I was previously unacquainted. Ramanujan (Srinivasa Iyengar Ramanuja Iyengar, to give him for once his proper name) was born on December 22nd, 1887, at Erode in southern India. His father was an accountant (*gumasta*) to a cloth merchant at Kumbakonam, while his maternal grandfather had served as *amin* in the Munsiff's (or local judge's) Court at Erode. He first went to school at five, and was transferred before he was seven to the Town High School at Kumbakonam, where he held a "free scholarship", and where his extraordinary powers appear to have been recognised immediately. "He used", so writes an old schoolfellow to Mr. Seshu Aiyar, "to borrow Carr's *Synopsis of Pure Mathematics* from the College library, and delight in verifying some of the formulæ given there. . . . He used to entertain his friends with his theorems and formulæ, even in those early days. . . . He had an extraordinary memory and could easily repeat the complete lists of Sanscrit roots (*atmanepada* and *parasmepada*); he could give the values of $\sqrt{2}$, π, e, ... to any number of decimal places. . . . In manners, he was simplicity itself. . . ."

He passed his matriculation examination to the Government College at Kumbakonam in 1904; and secured the "Junior Subraniam Scholarship". Owing to weakness in English, he failed in his next examination and lost his scholarship; and left Kumbakonam, first for Vizagapatam and then for Madras. Here he presented himself for the "First Examination in Arts" in December 1906, but failed and never tried again. For the next few years he continued his independent work in mathematics, "jotting down his results in two good-sized notebooks": I have one of these note books in my possession still. In 1909 he married, and it became necessary for him to find some permanent employment. I quote Mr. Seshu Aiyar:

To this end, he went to Tirukoilur, a small sub-division town in South Arcot District, to see Mr. V. Ramaswami Aiyar, the founder of the Indian Mathematical Society, but Mr. Aiyar, seeing his wonderful gifts, persuaded him to go to Madras. It was then after some four years' interval that Mr. Ramanujan met me at Madras, with his two well-sized note-books referred to above. I sent Ramanujan with a note of recommendation to that true lover of Mathematics, Dewan Bahadur R. Ramachandra Rao, who was then District Collector at Nellore, a small town some eighty miles north of Madras. Mr. Rao sent him back to me saying it was cruel to make an intellectual giant like Ramanujan rot at a mofussil station like Nellore, and recommended his stay at Madras, generously undertaking to pay Mr. Ramanujan's expenses for a time. This was in December 1910. After a while, other attempts to obtain for him a scholarship having failed, and Ramanujan himself being unwilling to be a burden on anybody for any length of time, he decided to take up a small appointment under the Madras Port Trust in 1911.

But he never slackened his work at Mathematics. His earliest contribution to the *Journal of the Indian Mathematical Society* was in the form of questions communicated by me in Vol. III (1911). His first long article on 'Some Properties of Bernoulli's Numbers' was published in the December number of the same volume. Mr. Ramanujan's methods were so terse and novel and his presentation was so lacking in clearness and precision, that the ordinary reader, unaccustomed to such intellectual gymnastics, could hardly follow him. This particular article was returned more than once by the Editor before it took a form suitable for publication. It was during this period that he came to me one day with some theorems on Prime Numbers, and when I referred him to Hardy's Tract on *Orders of Infinity*, he observed that Hardy had said on p. 36 of his Tract 'the exact order of $\rho(x)$ [defined by the equation

$$\rho(x) = \pi(x) - \int_2^x \frac{dt}{\log t},$$

where $\pi(x)$ denotes the number of primes less than x], has not yet been determined', and that he himself had discovered a result which gave the order of $\rho(x)$. On this I suggested that he might communicate his result to Mr. Hardy, together with some more of his results.

This passage brings me to the beginning of my own acquaintance with Ramanujan. But before I say anything about the letters which I received from him, and which resulted ultimately in his journey to England, I must add a little more about his Indian career. Dr. G. T. Walker, F.R.S., Head of the Meteorological Department, and formerly Fellow and Mathematical Lecturer of Trinity College, Cambridge, visited Madras for some official purpose some time in 1912; and Sir Francis Spring, K.C.I.E., the Chairman of the Madras Port Authority, called his attention to Ramanujan's work. Dr. Walker was far too good a mathematician not to recognise its quality, little as it had in common with his own. He brought Ramanujan's case to the notice of the Government and the University of Madras. A research studentship, "Rs. 75 *per mensem* for a period of two years", was awarded him; and he became, and remained for the rest of his life, a professional mathematician.

II.

Ramanujan wrote to me first on January 16th, 1913, and at fairly regular intervals until he sailed for England in 1914. I do not believe that his letters were entirely his own. His knowledge of English, at that stage of his life, could scarcely have been sufficient, and there is an occasional phrase which is hardly characteristic. Indeed I seem to remember his telling me that his friends had given him some assistance. However, it was the mathematics that mattered, and that was very emphatically his.

Madras, 16th January 1913

"Dear Sir

I beg to introduce myself to you as a clerk in the Accounts Department of the Port Trust Office at Madras on a salary of only £20 per annum. I am now about 23 years of age. I have had no university education but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at Mathematics. I have not trodden through the conventional regular course which is followed in a university course, but I am striking out a new path for myself. I have made a special investigation of divergent series in general and the results I get are termed by the local mathematicians as 'startling'

Just as in elementary mathematics you give a meaning to a^n when n is negative and fractional to conform to the law which holds when n is a positive integer, similarly the whole of my investigations proceed on giving a meaning to Eulerian Second Integral for all values of n. My friends who have gone through the regular course of university education tell me that $\int_0^\infty x^{n-1}e^{-x}dx = \Gamma(n)$ is true only when n is positive. They say that this integral relation is not true when n is negative. Supposing this is true only for positive values of n and also supposing the definition $n\Gamma(n) = \Gamma(n+1)$ to be universally true, I have given meanings to these integrals and under the conditions I state the integral is true for all values of n negative and fractional. My whole investigations are based upon this and I have been developing this to a remarkable extent so much so that the local mathematicians are not able to understand me in my higher flights.

Very recently I came across a tract published by you styled *Orders of Infinity* in page 36 of which I find a statement that no definite expression has been as yet found for the no of prime nos less than any given number. I have found an expression which very nearly approximates to the real result, the error being negligible. I would request you to go through the enclosed papers. Being poor, if you are convinced that there is anything of value I would like to have my theorems published. I have not given the actual investigations nor the expressions that I get but I have indicated to the lines on which I proceed. Being inexperienced I would very highly value any advice you give me. Requesting to be excused for the trouble I give you.

I remain Dear sir Yours truly

S. Ramanujan

P.S. My address is S. Ramanujan, Clerk Accounts Department, Port Trust, Madras, India."

I quote now from the "papers enclosed," and from later letters :—

"In page 36 it is stated that 'the no of prime nos less than $x = \int_2^x \frac{dt}{\log t} + \rho(x)$ where the precise order of $\rho(x)$ has not been determined. . . .'

I have observed that $\rho(e^{2\pi x})$ is of such a nature that its value is very small when x lies between 0 and 3 (its value is less than a few hundreds when $x = 3$) and rapidly increases when x is greater than 3. . . .

The difference between the no of prime nos of the form $4n-1$ and which are less than x and those of the form $4n+1$ less than x is infinite when x becomes infinite. . . .

The following are a few examples from my theorems :—

(1) The nos of the form 2^p3^q less than $n = \frac{1}{2}\frac{\log(2n)\log(3n)}{\log 2 \log 3}$ where p and q may have any positive integral value including 0.

(2) Let us take all nos containing an odd no of dissimilar prime divisors viz.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 30, 31, 37, 41, 42, 43, 47 &c

(a) The no of such nos less than $n = \frac{3n}{\pi^2}$.

(b) $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots + \frac{1}{30^2} + \frac{1}{31^2} + \cdots = \frac{9}{2\pi^2}$.

(c) $\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \&c. = \frac{15}{2\pi^4}$.

(3) Let us take the no of divisors of natural nos viz.

1, 2, 2, 3, 2, 4, 2, 4, 3, 4, 2 &c (1 having 1 divisor, 2 having 2, 3 having 2, 4 having 3, 5 having 2, &c).

The sum of such nos to n terms

$$= n\,(2\gamma - 1 + \log n) + \tfrac{1}{2} \text{ of the no of divisors of } n$$

where $\gamma = \cdot 5772156649\ldots$, the Eulerian Constant.

(4) 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18 &c are nos which are either themselves sqq. or which can be expressed as the sum of two sqq.

The no of such nos greater than A and less than B

$$= K\int_A^B \frac{dx}{\sqrt{\log x}} + \theta(x)^* \quad \text{where} \quad K = \cdot 764\ldots$$

and $\theta(x)$ is very small when compared with the previous integral. K and $\theta(x)$ have been exactly found though complicated. . . ."

Ramanujan's theory of primes was vitiated by his ignorance of the theory of functions of a complex variable. It was (so to say) what the theory might be if the Zeta-function had no complex zeros. His methods of proof depended upon a wholesale use of divergent series. He disregarded entirely all the difficulties which are involved in the interchange of double limit operations: he did not distinguish, for example, between the sum of a series Σa_n and the value of the Abelian limit

$$\lim_{x \to 1} \Sigma a_n x^n,$$

or that of any other limit which might be used for similar purposes by a modern analyst. There are regions of mathematics in which the precepts of modern rigour may be disregarded with comparative safety, but the Analytic Theory of Numbers is not one of them, and Ramanujan's Indian work on primes, and on all the allied problems of the theory, was definitely wrong. That his proofs should have been invalid was only to be expected. But the mistakes went deeper than that, and many of the actual results were false. He had obtained the dominant terms of the classical formulæ, although by invalid methods; but none of them are such close approximations as he supposed.

This may be said to have been Ramanujan's one great failure. And yet I am not sure that, in some ways, his failure was not more wonderful than any of his triumphs. Consider, for example, problem (4). The dominant term, which Ramanujan gives correctly, was first obtained by

* This should presumably be $\theta(B)$.

Landau in 1908. The correct order of the error term is still unknown. Ramanujan had none of Landau's weapons at his command; he had never seen a French or German book; his knowledge even of English was insufficient to enable him to qualify for a degree. It is sufficiently marvellous that he should have even dreamt of problems such as these, problems which it has taken the finest mathematicians in Europe a hundred years to solve, and of which the solution is incomplete to the present day.

"... IV. Theorems on integrals. The following are a few examples

$$(1)\quad \int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2}\cdot\frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2}\ldots \&c\; dx$$

$$= \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma(a+\frac{1}{2})}{\Gamma(a)}\cdot\frac{\Gamma(b+1)}{\Gamma(b+\frac{1}{2})}\cdot\frac{\Gamma(b-a+\frac{1}{2})}{\Gamma(b-a+1)}.$$

...

(3) If
$$\int_0^\infty \frac{\cos nx}{e^{2\pi\sqrt{x}}-1}\,dx = \phi(n),$$

then
$$\int_0^\infty \frac{\sin nx}{e^{2\pi\sqrt{x}}-1}\,dx = \phi(n)-\frac{1}{2n}+\phi\left(\frac{\pi^2}{n}\right)\sqrt{\frac{2\pi^3}{n^5}}.$$

$\phi(n)$ is a complicated function. The following are certain special values

$$\phi(0)=\frac{1}{12};\quad \phi\left(\frac{\pi}{2}\right)=\frac{1}{4\pi};\quad \phi(\pi)=\frac{2-\sqrt{2}}{8};\quad \phi(2\pi)=\frac{1}{16};$$

$$\phi\left(\frac{2\pi}{5}\right)=\frac{8-8\sqrt{5}}{16};\quad \phi\left(\frac{\pi}{5}\right)=\frac{6+\sqrt{5}}{4}-\frac{5\sqrt{10}}{8};\quad \phi(\infty)=0;$$

$$\phi\left(\frac{2\pi}{3}\right)=\frac{1}{3}-\sqrt{3}\left(\frac{3}{16}-\frac{1}{8\pi}\right).$$

$$(4)\quad \int_0^\infty \frac{dx}{(1+x^2)(1+r^2x^2)(1+r^4x^2)\ldots \&c} = \frac{\pi}{2(1+r+r^3+r^6+r^{10}+\&c)}$$

where 1, 3, 6, 10 &c are sums of natural nos.

$$(5)\quad \int_0^\infty \frac{\sin 2nx}{x(\cosh \pi x+\cos \pi x)}\,dx = \frac{\pi}{4}-2\left(\frac{e^{-n}\cos n}{\cosh\frac{\pi}{2}}-\frac{e^{-3n}\cos 3n}{3\cosh\frac{3\pi}{2}}\ldots \&c\right).$$

...

V Theorems on summation of series;* *e.g.*

(1) $\frac{1}{1^3}\cdot\frac{1}{2}+\frac{1}{2^3}\cdot\frac{1}{2^2}+\frac{1}{3^3}\cdot\frac{1}{2^3}+\frac{1}{4^3}\cdot\frac{1}{2^4}+\&c$

$$=\tfrac{1}{6}(\log 2)^3-\frac{\pi^2}{12}\log 2+\left(\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\&c\right).$$

(2) $1+9.(\tfrac{1}{4})^4+17.\left(\frac{1.5}{4.8}\right)^4+25.\left(\frac{1.5.9}{4.8.12}\right)^4+\&c=\frac{2\sqrt{2}}{\sqrt{\pi}.\{\Gamma(\frac{3}{4})\}^2}.$

(3) $1-5.(\tfrac{1}{2})^3+9.\left(\frac{1.3}{2.4}\right)^3-\&c=\frac{2}{\pi}.$

(4) $\frac{1^{13}}{e^{2\pi}-1}+\frac{2^{13}}{e^{4\pi}-1}+\frac{3^{13}}{e^{6\pi}-1}+\&c=\frac{1}{24}.$

(5) $\frac{\coth\pi}{1^7}+\frac{\coth 2\pi}{2^7}+\frac{\coth 3\pi}{3^7}+\&c=\frac{19\pi^7}{56700}$

(6) $\frac{1}{1^5\cosh\frac{\pi}{2}}-\frac{1}{3^5\cosh\frac{3\pi}{2}}+\frac{1}{5^5\cosh\frac{5\pi}{2}}-\&c=\frac{\pi^5}{768}.$

. . .

VI. Theorems on transformation of series and Integrals, *e.g.*

(1) $\pi\left(\frac{1}{2}-\frac{1}{\sqrt{1}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{5}}-\frac{1}{\sqrt{5}+\sqrt{7}}+\&c\right)$

$$=\frac{1}{1\sqrt{1}}-\frac{1}{3\sqrt{3}}+\frac{1}{5\sqrt{5}}-\&c.$$

. . .

(3) $1-\frac{x^2\,\underline{|3}}{(\underline{|1}\;\underline{|2})^3}+\frac{x^4\,\underline{|6}}{(\underline{|2}\;\underline{|4})^3}-\frac{x^6\,\underline{|9}}{(\underline{|3}\;\underline{|6})^3}+\&c$

$$=\left\{1+\frac{x}{(\underline{|1})^3}+\frac{x^2}{(\underline{|2})^3}+\&c.\right\}\left\{1-\frac{x}{(\underline{|1})^3}+\frac{x^2}{(\underline{|2})^3}-\&c\right\}.$$

. . .

(6) If $\alpha\beta=\pi^2$, then $\frac{1}{\sqrt[4]{\alpha}}\left\{1+4\alpha\int_0^\infty\frac{xe^{-\alpha x^2}}{e^{2\pi x}-1}dx\right\}$

$$=\frac{1}{\sqrt[4]{\beta}}\left\{1+4\beta\int_0^\infty\frac{xe^{-\beta x^2}}{e^{2\pi x}-1}dx\right\}.$$

---

* There is always more in one of Ramanujan's formulæ than meets the eye, as anyone who sets to work to verify those which look the easiest will soon discover. In some the interest lies very deep, in others comparatively near the surface; but there is not one which is not curious and entertaining.

(7) $n\left(e^{-n^2} - \frac{e^{-\frac{n^2}{3}}}{3\sqrt{3}} + \frac{e^{-\frac{n^2}{5}}}{5\sqrt{5}} - \&c\right)$

$$= \sqrt{\pi}\,(e^{-n\sqrt{\pi}} \sin n\sqrt{\pi} - e^{-n\sqrt{3\pi}} \sin n\sqrt{3\pi} + \&c).$$

(8) If n is any positive integer excluding 0

$$\frac{1^{4n}}{(e^{\pi}-e^{-\pi})^2} + \frac{2^{4n}}{(e^{2\pi}-e^{-2\pi})^2} \dots \&c = \frac{n}{\pi}\left\{\frac{B_{4n}}{8n} + \frac{1^{4n-1}}{e^{2\pi}-1} + \frac{2^{4n-1}}{e^{4\pi}-1} \dots \&c\right\}$$

where $B_2 = \frac{1}{6}$, $B_4 = \frac{1}{30}$, &c.

VII. Theorems on approximate integration and summation of series.

. . .

(2) $1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \dots + \frac{x^x}{\lfloor x}\,\theta = \frac{e^x}{2}$

where $\theta = \frac{1}{3} + \frac{4}{135(x+k)}$ where k lies between $\frac{8}{45}$ and $\frac{2}{21}$.

(3) $1 + \left(\frac{x}{\lfloor 1}\right)^5 + \left(\frac{x^2}{\lfloor 2}\right)^5 + \left(\frac{x^3}{\lfloor 3}\right)^5 + \&c = \frac{\sqrt{5}}{4\pi^2} \cdot \frac{e^{5x}}{5x^2 - x + \theta}$

where θ vanishes when $x = \infty$.

(4) $\frac{1^2}{e^x - 1} + \frac{2^2}{e^{2x}-1} + \frac{3^2}{e^{3x}-1} + \frac{4^2}{e^{4x}-1} + \&c$

$$= \frac{2}{x^3}\left(\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \&c\right) - \frac{1}{12x} - \frac{x}{1440} + \frac{x^3}{181440}$$

$$+ \frac{x^5}{7257600} + \frac{x^7}{159667200} + \&c \text{ when } x \text{ is small.}$$

(*Note.*—x may be given values from 0 to 2).

(5) $\frac{1}{1001} + \frac{1}{1002^2} + \frac{3}{1003^3} + \frac{4^2}{1004^4} + \frac{5^3}{1005^5} + \&c$

$$= \frac{1}{1000} - 10^{-440} \times 1{\cdot}0125 \text{ nearly.}$$

(6) $\int_0^a e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2} - \frac{e^{-a^2}}{2a +}\,\frac{1}{a +}\,\frac{2}{2a +}\,\frac{3}{a +}\,\frac{4}{2a +} \&c.$

(7) The coefficient of x^n in $\frac{1}{1 - 2x + 2x^4 - 2x^9 + 2x^{16} - \&c}$

$$= \text{the nearest integer to } \frac{1}{4n}\left\{\cosh(\pi\sqrt{n}) - \frac{\sinh(\pi\sqrt{n})}{\pi\sqrt{n}}\right\}.\,*$$

* This is quite untrue. But the formula is extremely interesting for a variety of reasons.

IX. Theorems on continued fractions, a few examples are :—

(1) $$\frac{4}{x+}\frac{1^2}{2x+}\frac{3^2}{2x+}\frac{5^2}{2x+}\frac{7^2}{2x+}\&c = \left\{\frac{\Gamma\left(\frac{x+1}{4}\right)}{\Gamma\left(\frac{x+3}{4}\right)}\right\}^2$$

. . .

(4) If $$u = \frac{x}{1+}\frac{x^5}{1+}\frac{x^{10}}{1+}\frac{x^{15}}{1+}\frac{x^{20}}{1+}\&c$$

and $$v = \frac{\sqrt[5]{x}}{1+}\frac{x}{1+}\frac{x^2}{1+}\frac{x^3}{1+}\&c$$

then $$v^5 = u\,.\,\frac{1-2u+4u^2-3u^3+u^4}{1+3u+4u^2+2u^3+u^4}.$$

(5) $$\frac{1}{1+}\frac{e^{-2\pi}}{1+}\frac{e^{-4\pi}}{1+}\frac{e^{-6\pi}}{1+}\&c = \left(\sqrt{\frac{5+\sqrt{5}}{2}}-\frac{\sqrt{5}+1}{2}\right)\sqrt[5]{e^{2\pi}}.$$

(6) $$\frac{1}{1-}\frac{e^{-\pi}}{1+}\frac{e^{-2\pi}}{1-}\frac{e^{-3\pi}}{1+}\&c = \left(\sqrt{\frac{5-\sqrt{5}}{2}}-\frac{\sqrt{5}-1}{2}\right)\sqrt[5]{e^{\pi}}.$$

(7) $\frac{1}{1+}\frac{e^{-\pi\sqrt{n}}}{1+}\frac{e^{-2\pi\sqrt{n}}}{1+}\frac{e^{-3\pi\sqrt{n}}}{1+}\&c$ can be exactly found if n be any positive rational quantity. . . ."

27 February 1913

". . . I have found a friend in you who views my labours sympathetically. This is already some encouragement to me to proceed. . . . I find in many a place in your letter rigorous proofs are required and you ask me to communicate the methods of proof. . . . I told him* that the sum of an infinite no of terms of the series $1+2+3+4+\ldots = -\frac{1}{12}$ under my theory. If I tell you this you will at once point out to me the lunatic asylum as my goal. . . . What I tell you is this. Verify the results I give and if they agree with your results . . . you should at least grant that there may be some truths in my fundamental basis. . . .

To preserve my brains I want food and this is now my first consideration. Any sympathetic letter from you will be helpful to me here to get a scholarship either from the University or from Government. . . .

1. The no of prime nos. less than $$e^a = \int_0^\infty \frac{a^x\,dx}{xS_{x+1}\,\Gamma(x+1)}$$

where $$S_{x+1} = \frac{1}{1^{x+1}}+\frac{1}{2^{x+1}}+\ldots.$$

* Referring to a previous correspondence.

2. The no of prime nos. less than $n =$

$$\frac{2}{\pi}\left\{\frac{2}{B_2}\left(\frac{\log n}{2\pi}\right)+\frac{4}{3B_4}\left(\frac{\log n}{2\pi}\right)^3+\frac{6}{5B_6}\left(\frac{\log n}{2\pi}\right)^5+\text{etc}\right.$$

where $B_2 = \frac{1}{6}$; $B_4 = \frac{1}{30}$ &c, the Bernoullian nos. . . .

For practical calculations

$$\int_\mu^n \frac{dx}{\log x} = n\left(\frac{1}{\log n}+\frac{\lfloor 1}{(\log n)^2}+\ldots+\frac{\lfloor k-1}{(\log n)^k}\theta\right)$$

where $\theta = \frac{2}{3}-\delta+\frac{1}{\log n}\left\{\frac{4}{135}-\frac{\delta^2(1-\delta)}{3}\right\}$

$$+\frac{1}{(\log n)^2}\left\{\frac{8}{2835}+\frac{2\delta(1-\delta)}{135}-\frac{\delta(1-\delta^2)(2-3\delta^2)}{45}\right\}+\text{etc}$$

where $\delta = k-\log n$. . . .

The order of $\theta(x)$ which you asked in your letter is $\sqrt{\left(\frac{x}{\log x}\right)}$.

. . .

(1) If $F(x) = \frac{1}{1+}\frac{x}{1+}\frac{x^2}{1+}\frac{x^3}{1+}\frac{x^4}{1+}\frac{x^5}{1}+\&c$

then $\left\{\frac{\sqrt{5}+1}{2}+e^{-\frac{2\alpha}{5}}F(e^{-2\alpha})\right\}\left\{\frac{\sqrt{5}+1}{2}+e^{-\frac{2\beta}{5}}F(e^{-2\beta})\right\} = \frac{5+\sqrt{5}}{2}$,

with the conditions $\alpha\beta = \pi^2$. . . .

$$e.g.\ \frac{1}{1+}\frac{e^{-2\pi\sqrt{5}}}{1+}\frac{e^{-4\pi\sqrt{5}}}{1}+\text{etc}\ldots = e^{\frac{2\pi}{\sqrt{5}}}\left\{\frac{\sqrt{5}}{1+\sqrt[5]{5^{\frac{3}{4}}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{5}{2}}-1}}-\frac{\sqrt{5}+1}{2}\right\}$$

The above theorem is a particular case of a theorem on the c.f.

$$\frac{1}{1+}\frac{ax}{1+}\frac{ax^2}{1+}\frac{ax^3}{1+}\frac{ax^4}{1+}\frac{ax^5}{1}+\&c.$$

which is a particular case of the c.f.

$$\frac{1}{1+}\frac{ax}{1+bx+}\frac{ax^2}{1+bx^2+}\frac{ax^3}{1+bx^3+}\&c$$

which is a particular case of a general theorem on c.f.

(2) i. $4\int_0^\infty \frac{xe^{-x\sqrt{5}}}{\cosh x}dx = \frac{1}{1+}\frac{1^2}{1+}\frac{1^2}{1+}\frac{2^2}{1+}\frac{2^2}{1+}\frac{3^2}{1+}\frac{3^2}{1}+\&c$

ii. $4\int_0^\infty \frac{x^2e^{-x\sqrt{3}}}{\cosh x}dx = \frac{1}{1+}\frac{1^3}{1+}\frac{1^3}{3+}\frac{2^3}{1+}\frac{2^3}{5+}\frac{3^3}{1+}\frac{3^3}{7}+\&c$

(3) $1-5.\left(\frac{1}{2}\right)^5+9.\left(\frac{1.3}{2.4}\right)^5-13.\left(\frac{1.3.5}{2.4.6}\right)^5+\&c = \frac{2}{\{\Gamma(\frac{3}{4})\}^4}$