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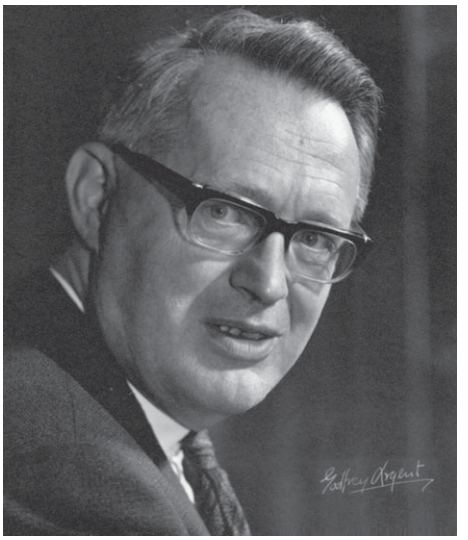
OBITUARY

David Rees, FRS 1918–2013



David Rees completed his Cambridge undergraduate studies in mathematics in summer 1939; in his first three months of postgraduate work in autumn 1939, he produced a characterization of completely 0-simple semigroups. War then intervened: he worked until the end of the war at Bletchley Park, the British codebreaking centre in Buckinghamshire, where he was part of a team that broke the Enigma code regularly for some critical months during 1940. After the war, he first worked at Manchester University, but moved to Cambridge University in 1948. In the immediate post-war period, he continued with research into semigroups and non-commutative algebra. His first paper was very influential, and he is considered by semigroup theorists to be one of the founding fathers of their subject.

At Cambridge, after attending a seminar by Douglas Northcott, Rees changed the focus of his research to commutative Noetherian rings. During an extraordinarily productive period between 1954 and 1961, he produced a string of far-reaching, foundational and deep ideas and results of lasting significance. Highlights include reductions of ideals, his Valuation Theorem, the theory of grade, the graded rings that are nowadays known as 'Rees rings', the Artin–Rees Lemma and his characterization of local rings whose completions have zero nilradical.

Rees was appointed to the Chair of Pure Mathematics at the University of Exeter in 1958 and elected FRS in 1968. He was awarded the Polya Prize of the London Mathematical Society, and an Honorary DSc by the University of Exeter, in 1993.

1. *Family background and education*

David Rees was born and brought up in Abergavenny; he was the fourth of five children of Gertrude (née Powell) and (another) David Rees, a corn merchant. The family lived above David's father's corn shop. There is history of both longevity and mathematical ability in David

Rees's father's line: his father died at the age of 88, three of his siblings had 90th birthdays and one of his great-great-grandfathers was the Reverend Thomas Rees (1774–1858), a well-known non-conformist minister, who, according to one obituarist, was considered to be the best mathematician in Wales in 1802.

David Rees was educated at King Henry VIII Grammar School in Abergavenny. At the time, the school had an excellent headmaster, Wyndham Newcombe, who was also a very good teacher of mathematics. Rees's early teenage years were affected by ill health, and he was absent from school for several terms. During those periods of illness, he studied at home independently, and his mother, armed with lists from the young David, became one of the best customers of the Abergavenny public library. This diligence stood him in good stead when he was able to return to normal schooling: under the guidance of mathematics master L. F. Porter, he was able to catch up quickly with his mathematics. He did rather well in School Certificate examinations in 1934 and 1936, and was awarded a State Scholarship and admission to Sidney Sussex College, Cambridge, where his studies were supervised by Gordon Welchman. Rees started as a Commoner, but was made an Exhibitioner after one year and, after he had come top in the Preliminary Examination for Part II at the end of his second year, he was made a Scholar. Rees was persuaded to take Parts IIB and III together in 1939, and another candidate, Hermann Bondi, with whom he had a friendly rivalry and who only had to take Part IIB at that time, managed to just beat him into second place.

2. *The beginnings of postgraduate work*

Rees began postgraduate work in September 1939, without a proper supervisor, but inspired by 'wonderful lectures' by Philip Hall. In the autumn of 1939, he had a rather successful three months, during which he produced a characterization of completely 0-simple semigroups.

Here are the relevant definitions. Let S be a semigroup, with operation written multiplicatively. A (two-sided) ideal of S is a non-empty subset A of S such that $as \in A$ and $sa \in A$ for all $a \in A$ and $s \in S$. A *zero element* of S is a (necessarily uniquely determined) element $0 \in S$ such that $0s = 0 = s0$ for all $s \in S$. The semigroup S with zero is called 0-*simple* if $\{0\}$ and S are its only ideals and there exist $s, t \in S$ such that $st \neq 0$. The semigroup S is said to be *completely* 0-*simple* if it is 0-simple and has a non-zero idempotent element e such that the only idempotent $f \in S$ for which $ef = fe = f \neq 0$ is e itself.

In his first paper [**1**], Rees gave a recipe for constructing completely 0-simple semigroups. Take a group G, and form the semigroup $G^0 = G \cup \{0\}$ with zero 0 (such a semigroup is referred to as a 0-*group*). Let Σ and Λ be non-empty indexing sets and let $M = (m_{\lambda\sigma})$ be a $\Lambda \times \Sigma$ matrix with entries in G^0. Assume that M is *regular*, that is, that no 'row' or 'column' of M consists entirely of zeros. Set $S = (\Sigma \times G \times \Lambda) \cup \{0\}$ and define a binary operation on S as follows: for all $\sigma, \tau \in \Sigma$, $\lambda, \mu \in \Lambda$ and $g, h \in G$, set

$$(\sigma, g, \lambda)(\tau, h, \mu) = \begin{cases} (\sigma, gm_{\lambda\tau}h, \mu) & \text{if } m_{\lambda\tau} \neq 0, \\ 0 & \text{if } m_{\lambda\tau} = 0; \end{cases}$$
$$(\sigma, g, \lambda)0 = 0(\sigma, g, \lambda) = 00 = 0.$$

Then, with this operation, S turns out to be a completely 0-simple semigroup. This semigroup is referred to as the $\Sigma \times \Lambda$ *Rees matrix semigroup over the* 0-*group* G^0 *with regular sandwich matrix* M. However, the main thrust of Rees's paper [**1**] is that *every* completely 0-simple semigroup is isomorphic to one constructed in this way. In his monograph ⟨**8**⟩ on semigroup theory, John Howie refers to these results as 'The Rees Theorem' and reports that that theorem 'has played a dominant rôle in the development of the subject'. An analogue (that Rees described as 'the first big theorem in semigroup theory') for completely simple semigroups (without zero) had been proved by Suschkewitsch ⟨**24**⟩.

Paper [1] was submitted in early May 1940, and represents a very successful start by Rees to postgraduate research. Given Rees's intensive work at Bletchley Park from December 1939 (see § 3), most of the work for [1] must have been completed in Rees's first three months of research. In that paper, Rees does thank 'Mr. P. Hall, both for his encouragement, while this paper was being written, and his very considerable assistance in preparing the paper for publication'. It should be noted that paper [1] is explicitly mentioned in the citation that accompanied David Rees's election as FRS. The phrases 'Rees matrix semigroup' and 'Rees Theorem' ensure that his name will live on among the semigroup community.

3. *Work at Bletchley Park during the Second World War*

By summer 1939, Gordon Welchman had been appointed to work at Bletchley Park, the British codebreaking centre in Buckinghamshire. In December 1939, Welchman knocked on the door of Rees's college rooms to tell him that he had a job for him to do. Rees naturally wanted to know details, but Welchman refused to elaborate, and only after prompting did he tell Rees to meet him a few days later at Bletchley railway station. Rees did so, and in this way was recruited to a team of codebreakers in Hut 6 at Bletchley Park.

Welchman recruited several other young mathematicians he knew from Cambridge, including some he had taught at Sidney Sussex College. Even in later life after the veil of secrecy that covered the war-time exploits of Bletchley Park had been lifted, David Rees did not like to talk about his time there. However, it is now clear that he was part of a team that broke the Enigma code regularly for some critical months during the summer and autumn of 1940.

The German operators of the Enigma machines were told which three of the five available rotors and which settings to use each day, but they had to choose the initial positions of the rotors and indicate their choices by means of the first three letters of their initial messages. John Herivel, who had also been recruited to Bletchley Park from Sidney Sussex College by Welchman, predicted in February 1940 that some German operators, when tired or stressed, might use short cuts that could be exploited by the Bletchley Park codebreakers. For three months, this lateral thinking by Herivel produced no result; but in May 1940 some of the German operators began to make the predicted mistakes, and David Rees and his fellow codebreakers were able to use the technique known as the 'Herivel tip' to break Enigma ciphers for some critical months from May 1940.

Herivel has written an account ⟨7⟩ of the Herivel tip and related matters, in which he attributes the first successful use of the tip to David Rees: see ⟨7, pp. 118–119⟩. Interestingly, the same book contains a reproduction of a statement by David Rees about the Herivel tip in which he declared that he did not recollect being the person responsible for the first successful use of it, although he conceded that 'it is possible that my memory is at fault'; see ⟨7, p. 122⟩. What is not in doubt is that the first successful use of the Herivel tip resulted in much rejoicing, shouting and standing on chairs. Rees thought very highly of Herivel's idea: he described it as 'brilliant' in the above-mentioned statement ⟨7, p. 122⟩; and he is quoted in ⟨7, p. 11⟩ as having said, in 2000, that 'of course, the Herivel tip was one of the seminal discoveries of the Second World War'. Rees told the present author in 2007 that, in his opinion, Herivel did not receive the recognition that he deserved.

In late 1941, David Rees was seconded to the Enigma Research Section at Bletchley Park, run by Dillwyn ('Dilly') Knox, and where the Abwehr Enigma used by the German Secret Service was broken. The so-called 'Double Cross Committee' used captured German agents to persuade Hitler that the D-Day landings would be south of Calais rather than in Normandy. It is said that, without the break into the Abwehr Enigma, British intelligence officers could not have known that the deception was working.

David Rees subsequently moved to the 'Newmanry', the department at Bletchley Park led, for the second half of the war, by M. H. (Max) Newman, for which the first Colossus computer was constructed to assist with codebreaking.

The list of subsequently famous mathematicians whom David Rees encountered during his service at Bletchley Park includes A. O. L. Atkin, I. J (Jack) Good, J. A. (Sandy) Green (who worked at Bletchley Park as a teenager), Peter Hilton, Max Newman, G. B. Preston and Shaun Wylie. Sandy Green and Peter Hilton were later to become coauthors of mathematical papers with David Rees, and Rees's third paper [**3**] (written after the war) was about a paper by Jack Good.

There are now available in print numerous articles detailing aspects of the war-time exploits at Bletchley Park; two recent ones are The Guardian's obituary of Peter Hilton ⟨**23**⟩ and the Royal Society's Biographical Memoir about William Tutte ⟨**31**⟩.

4. *Return to academic life*

Following the end of the war, David Rees resumed his academic studies, and soon found himself working under Max Newman in a different context: he was appointed in 1945 to an Assistant Lectureship in the Department of Mathematics at Manchester University, and Newman was the head of that department.

Rees remained at Manchester until 1948, when he was appointed to a University Lectureship at Cambridge; in 1949 he was appointed to a Fellowship at Downing College. He worked in semigroup theory and non-commutative algebra while at Manchester, and continued with these themes for his first years as a Cambridge don. He was very pleased with his joint paper [**8**] (with Sandy Green) from this time; in it they considered, for positive integers n and r with $r \geqslant 2$, the semigroup $S_{n,r}$ (again written multiplicatively) generated by n elements in which each element x satisfies $x^r = x$, but which is otherwise free, and they showed that the question as to whether $S_{n,r}$ is finite is intimately related to Burnside's Conjecture in group theory. Recall that the latter conjecture for r is the statement that, for all $n > 0$, the group B_{nr} generated by n elements in which each element x satisfies $x^r = e$, but which is otherwise free, is finite. A striking result from the Green–Rees paper [**8**] is that the Burnside conjecture for r is true if and only if $S_{n,r+1}$ is finite for all $n > 0$.

David Rees wrote just five papers on semigroup theory, but their influence on the development of that subject has been very substantial. Interested readers might like to consult the tribute ⟨**11**⟩ to Rees in *Semigroup Forum*, where he is described as 'one of the pioneers of semigroup theory', as 'one of the subject's founding fathers', and as having 'laid the foundations for a number of important avenues of future research'. However, as David Rees published about forty papers in commutative algebra, it is appropriate that the majority of this obituary be devoted to his contributions to that field.

5. *The switch to commutative algebra*

Another addition to the Mathematics faculty at Cambridge in 1948 was Douglas G. Northcott, who had spent 21 post-war months in Princeton, where he had been greatly stimulated by a seminar with the title 'Valuation theory' run by Emil Artin and Claude Chevalley, and by much informal guidance from Artin. Northcott returned to Cambridge having become a dedicated algebraist (his PhD work concerned a theory of integration for functions with values in a Banach space). In Princeton, Northcott had, at Artin's suggestion, studied the famous paper ⟨**28**⟩ by Weil, and, as a consequence, began to work in the algebra underlying what some refer to as the 'pre-Grothendieck' era of algebraic geometry. Thus Northcott became a commutative algebraist.

Back in Cambridge, Northcott organized a very successful working seminar on Weil's book ⟨**29**⟩. David Rees was a member of the audience, and he was so inspired by the seminar that he too became a commutative algebraist. (Another aspect of Northcott's seminar that was

life-changing for Rees was the presence in the audience of Joan Cushen: David and Joan were married in 1952.)

David Rees's transition from semigroup theory was gradual and his first paper in commutative algebra ([9], written jointly with Northcott) only appeared in 1954. That paper is central to the next section.

6. *Reductions and integral closures*

Paper [9], written jointly with Douglas Northcott, is, by a long way, David Rees's most-cited research paper: *Mathematical Reviews* records more than 200 citations of it. It introduced the notion of *reductions* of ideals. This concept and the related concept of *integral closure* have had a major influence on research in commutative algebra in the more than 60 years since they were introduced; indeed, even in the present century, hardly a top-level international conference in commutative algebra passes without there being several mentions of reductions.

Throughout the subsequent discussion of David Rees's work in commutative algebra, the symbol R will always denote a commutative ring that is Noetherian, that is, in which every ideal is finitely generated. We shall need to use the concept of multiplicity: the *multiplicity* $e(\mathfrak{q})$ of a proper ideal $\mathfrak{q}$ of finite colength in a d-dimensional local ring Q can be defined by the equation

$$e(\mathfrak{q}) := \lim_{n\to\infty} \frac{d!\ell_Q(Q/\mathfrak{q}^n)}{n^d},$$

where ℓ denotes length.

Let $\mathfrak{b}$ and $\mathfrak{a}$ be proper ideals of R. The ideal $\mathfrak{b}$ is said to be a *reduction of* $\mathfrak{a}$ if $\mathfrak{b} \subseteq \mathfrak{a}$ and there exists $s \in \mathbb{N}_0$ (the set of non-negative integers) such that $\mathfrak{b}\mathfrak{a}^s = \mathfrak{a}^{s+1}$. One can view such a $\mathfrak{b}$ as an approximation to $\mathfrak{a}$ that nevertheless retains some of the properties of $\mathfrak{a}$: for example, a prime ideal $\mathfrak{p}$ of R is a minimal prime ideal of $\mathfrak{b}$ if and only if it is a minimal prime ideal of $\mathfrak{a}$, and when that is the case, the multiplicity of $\mathfrak{b}$ corresponding to $\mathfrak{p}$ is equal to the multiplicity of $\mathfrak{a}$ corresponding to $\mathfrak{p}$. (The *multiplicity of* $\mathfrak{a}$ *corresponding to its minimal prime ideal* $\mathfrak{p}$ is the multiplicity $e(\mathfrak{a}R_\mathfrak{p})$ of the ideal $\mathfrak{a}R_\mathfrak{p}$ of the localization $R_\mathfrak{p}$.)

The inspiration for the definition of reduction came to David Rees while he was thinking about so-called irrelevant ideals in a (commutative Noetherian) positively graded ring $S = \bigoplus_{n\in\mathbb{N}_0} S_n$ that is generated, as an algebra over S_0, by homogeneous elements of degree 1. Set $S_+ := \bigoplus_{n\in\mathbb{N}} S_n$ (where $\mathbb{N}$ denotes the set of positive integers); let $\mathfrak{A} = \bigoplus_{n\in\mathbb{N}} \mathfrak{A}_n$ be a graded ideal of R generated by homogeneous elements of degree 1; Rees noted that $\mathfrak{A}_n = S_n$ for all sufficiently large n (that is, $\mathfrak{A}$ is *irrelevant*) if and only if there exists $v \in \mathbb{N}_0$ such that $\mathfrak{A}(S_+)^v = (S_+)^{v+1}$. This observation led to the birth of the concept of reduction.

We say that $r \in R$ is *integrally dependent on* the ideal $\mathfrak{b}$ of R if there exist $n \in \mathbb{N}$ and $c_1, \dots, c_n \in R$ with $c_i \in \mathfrak{b}^i$ for $i = 1, \dots, n$ such that

$$r^n + c_1 r^{n-1} + \cdots + c_{n-1} r + c_n = 0.$$

(Actually, Rees and Northcott used 'analytically dependent' instead of the now-standard 'integrally dependent'.) Note that every nilpotent element of R is integrally dependent on $\mathfrak{b}$.

The fundamental connections between reductions and integral closures can be summarized as follows. Let $\mathfrak{b} \subseteq \mathfrak{a}$ be ideals of R. Then $\mathfrak{b}$ is a reduction of $\mathfrak{a}$ if and only if each element of $\mathfrak{a}$ is integrally dependent on $\mathfrak{b}$. Furthermore, the set $\mathcal{J}$ of all ideals of R that have $\mathfrak{b}$ as a reduction has a unique maximal member, $\overline{\mathfrak{b}}$ say: $\overline{\mathfrak{b}}$ is the union of the members of $\mathcal{J}$, and this ideal $\overline{\mathfrak{b}}$ is precisely the set of all elements of R that are integrally dependent on $\mathfrak{b}$. The ideal $\overline{\mathfrak{b}}$ is called the *integral closure of* $\mathfrak{b}$; it has the property that the ideals of R that have $\mathfrak{b}$ as a reduction are precisely those between $\mathfrak{b}$ and $\overline{\mathfrak{b}}$. We say that $\mathfrak{b}$ is *integrally closed* if $\mathfrak{b} = \overline{\mathfrak{b}}$.

The ideal $\mathfrak{b}$ is said to be a *minimal reduction of* $\mathfrak{a}$ if $\mathfrak{b}$ is a reduction of $\mathfrak{a}$ and there is no reduction $\mathfrak{c}$ of $\mathfrak{a}$ with $\mathfrak{c} \subset \mathfrak{b}$ (the symbol '$\subset$' denotes strict inclusion).

Most of [**9**] is written under the hypothesis that R is a local ring Q with infinite residue field, and so that hypothesis will be in force until further notice; also $\mathfrak{m}$ will denote the maximal ideal of Q. Rees and Northcott defined the *analytic spread* $\ell(\mathfrak{a})$ of $\mathfrak{a}$; this turns out to be equal to the dimension of $\mathcal{G}(\mathfrak{a})/\mathfrak{m}\mathcal{G}(\mathfrak{a})$, where $\mathcal{G}(\mathfrak{a})$ denotes the associated graded ring $\bigoplus_{i\in\mathbb{N}_0} \mathfrak{a}^i/\mathfrak{a}^{i+1}$ of $\mathfrak{a}$. They proved that every reduction of $\mathfrak{a}$ requires at least $\ell(\mathfrak{a})$ generators, that a reduction of $\mathfrak{a}$ is a minimal reduction of $\mathfrak{a}$ if and only if it can be generated by $\ell(\mathfrak{a})$ elements, and that each reduction of $\mathfrak{a}$ contains a minimal reduction of $\mathfrak{a}$. Thus all minimal generating sets of all minimal reductions of $\mathfrak{a}$ have exactly $\ell(\mathfrak{a})$ elements.

They went on to show that $\ell(\mathfrak{a})$ can be interpreted as follows. Elements $u_1, \dots, u_t \in \mathfrak{a}$ are said to be *analytically independent in* $\mathfrak{a}$ if, whenever $h \in \mathbb{N}$ and $f \in R[X_1, \dots, X_t]$ (the ring of polynomials over R in t indeterminates) is a homogeneous polynomial of degree h such that $f(u_1, \dots, u_t) \in \mathfrak{a}^h\mathfrak{m}$, then all the coefficients of f lie in $\mathfrak{m}$. Then, if $\mathfrak{b}$ is a reduction of $\mathfrak{a}$, $\dim_{Q/\mathfrak{m}}(\mathfrak{b}/\mathfrak{m}\mathfrak{b}) =: t$ and $\{u_1, \dots, u_t\}$ is a minimal generating set for $\mathfrak{b}$, it turns out that $\mathfrak{b}$ is a minimal reduction of $\mathfrak{a}$ if and only if $u_1, \dots, u_t$ are analytically independent in $\mathfrak{a}$. Consequently, $\ell(\mathfrak{a})$ is equal to the largest number of elements of $\mathfrak{a}$ that are analytically independent in $\mathfrak{a}$, and $\mathrm{ht}\mathfrak{a} \leqslant \ell(\mathfrak{a}) \leqslant \dim_{Q/\mathfrak{m}}(\mathfrak{a}/\mathfrak{m}\mathfrak{a})$.

As mentioned above, the appearances in the literature of the concepts of reduction and integral closure in the 60 years since Rees and Northcott published [**9**] are very numerous; far-reaching extensions, generalizations and related concepts have been studied in depth. The reader can glean some idea of the enormous influence that these ideas of Rees and Northcott have had, and continue to have, in commutative algebra by studying the book ⟨**25**⟩ by Swanson and Huneke on integral closures. That book (which, incidentally, is dedicated to Joseph Lipman and David Rees) contains, *inter alia*, a wealth of information and detail about many of Rees's contributions to commutative algebra.

7. *Rees rings*

In this section, in which we revert to consideration of the general commutative Noetherian ring R, we recall some graded rings used by Rees to good effect. Nowadays, these rings are referred to as 'Rees rings' and 'extended Rees rings'.

Let $\mathfrak{a}$ be an ideal of R. Let $\{a_1, \dots, a_h\}$ be a generating set for $\mathfrak{a}$. Let T be an indeterminate, and consider the polynomial ring $R[T]$ as a graded ring in the usual way. Then the subring $R[a_1T, \dots, a_hT]$ of $R[T]$ is equal to

$$\left\{ \sum_{i=0}^{t} r_i T^i \in R[T] : t \in \mathbb{N}_0,\ r_i \in \mathfrak{a}^i \text{ for all } i = 0, \dots, t \right\}$$

and so is independent of the choice of generators of $\mathfrak{a}$; we denote it by $R[\mathfrak{a}T]$. It inherits an $\mathbb{N}_0$-grading from $R[T]$, and it is again Noetherian. By the (*ordinary*) *Rees ring of* $\mathfrak{a}$, we shall mean the $\mathbb{N}_0$-graded ring $\mathcal{R}(\mathfrak{a}) := \bigoplus_{i\in\mathbb{N}_0} \mathfrak{a}^i$ in which the product of an element r of the ith component $\mathfrak{a}^i$ and an element s of the jth component $\mathfrak{a}^j$ (where $i, j \in \mathbb{N}_0$) is the element rs of the $(i + j)$th component $\mathfrak{a}^{i+j}$.

A *homogeneous* isomorphism between graded rings is an isomorphism that preserves degrees. There is an obvious homogeneous isomorphism between $R[\mathfrak{a}T]$ and $\mathcal{R}(\mathfrak{a})$. Notice that the graded ring $\mathcal{R}(\mathfrak{a})/\mathfrak{a}\mathcal{R}(\mathfrak{a})$ is homogeneously isomorphic to the *associated graded ring* $\mathcal{G}(\mathfrak{a}) := \bigoplus_{i\in\mathbb{N}_0} \mathfrak{a}^i/\mathfrak{a}^{i+1}$ *of* $\mathfrak{a}$. The ring $\mathcal{R}(\mathfrak{a})$ is also called the *blowing up ring of* $\mathfrak{a}$; this terminology has its roots in the fact that the projective spectrum of $\mathcal{R}(\mathfrak{a})$ is the topological space underlying the scheme obtained by blowing up $\mathrm{Spec}(R)$ with respect to $\mathfrak{a}$.

We use $R[\mathfrak{a}T, T^{-1}]$ to denote the subring

$$R[a_1T, \ldots, a_hT, T^{-1}]$$

of $R[T, T^{-1}] = R[T]_T$, and refer to this as the *extended Rees ring of* $\mathfrak{a}$. (Note that $R[a_1T, \ldots, a_hT, T^{-1}]$ is independent of the choice of finite generating set $\{a_1, \ldots, a_h\}$ for $\mathfrak{a}$.) Also $R[\mathfrak{a}T, T^{-1}]$ inherits a $\mathbb{Z}$-grading from $R[T, T^{-1}] = R[T]_T$. (We use $\mathbb{Z}$ to denote the set of all integers.) We can write $R[\mathfrak{a}T, T^{-1}] = \bigoplus_{i\in\mathbb{Z}} \mathfrak{a}^i T^i$, where we interpret $\mathfrak{a}^i$, for a negative integer i, as R. Write U for T^{-1}, and note that U is a non-zerodivisor in $R[\mathfrak{a}T, T^{-1}]$. It is also worth noticing that there is a homogeneous isomorphism

$$R[\mathfrak{a}T, T^{-1}]/UR[\mathfrak{a}T, T^{-1}] \xrightarrow{\cong} \mathcal{G}(\mathfrak{a}).$$

The 0th component of $R[\mathfrak{a}T, T^{-1}]$ is R, and Rees used to very good effect the observation that, for an $i \in \mathbb{N}_0$, the 0th component of the graded ideal $R[\mathfrak{a}T, T^{-1}]U^i$ of $R[\mathfrak{a}T, T^{-1}]$ is just $\mathfrak{a}^i$. In other words, $R[\mathfrak{a}T, T^{-1}]U^i \cap R = \mathfrak{a}^i$. By means of this observation, Rees was able to reduce some questions about powers of an ideal in a Noetherian ring to the case where the ideal is principal and generated by a non-zerodivisor. In that special case, simplifications are often available. The following proof of Krull's Intersection Theorem, based on the proof in Rees [14], illustrates his use of the above device.

THEOREM 7.1 (W. Krull's Intersection Theorem ⟨10⟩). (*Recall that R is Noetherian.*) *If $r \in \bigcap_{i=1}^{\infty} \mathfrak{a}^i$, then there exists $a \in \mathfrak{a}$ such that $r = ar$.*

Proof. We deal first with the case where $\mathfrak{a}$ is the principal ideal Ru generated by a non-zerodivisor u. Since $r \in \bigcap_{i=1}^{\infty} \mathfrak{a}^i$, we can, for each $i \in \mathbb{N}$, write $r = u^i s_i$ for some $s_i \in R$. Then $s_i = us_{i+1}$ for all $i \in \mathbb{N}$, since u is a non-zerodivisor in R. Therefore, $Rs_1 \subseteq Rs_2 \subseteq \cdots \subseteq Rs_i \subseteq \cdots$, and so there exists $j \in \mathbb{N}$ such that $Rs_j = Rs_{j+1}$. Thus $s_{j+1} = s_j b$ for some $b \in R$, from which we see that $s_j = us_{j+1} = s_j(bu)$, with $bu \in Ru$. Therefore, $r = u^j s_j = (bu)u^j s_j = (bu)r$.

In the general case, consider the (Noetherian) extended Rees ring $\mathcal{S} := R[\mathfrak{a}T, T^{-1}]$, and set $U := T^{-1}$, a non-zerodivisor of that ring. Let $r \in \bigcap_{i=1}^{\infty} \mathfrak{a}^i$. Then $r \in \bigcap_{i=1}^{\infty} \mathcal{S}U^i$, and, by the first paragraph of this proof, we can write $r = fUr$ for some $f \in \mathcal{S}$. Write $f = \sum_{i=-v}^{w} b_i T^i$, where $b_i \in \mathfrak{a}^i$ for all $i = -v, \ldots, w$. Compare components of degree 0 to see that $r = b_1 r$, and note that $b_1 \in \mathfrak{a}$. □

In the same paper [14], Rees also gave a proof of what is now known as 'the Artin–Rees Lemma'. That proof also uses the extended Rees ring.

LEMMA 7.2 (The Artin–Rees Lemma [14, Lemma 1]). (*Recall that R is Noetherian.*) *Let $\mathfrak{a}, \mathfrak{b}$ be two ideals of R. Then there exists $k \in \mathbb{N}$ such that $\mathfrak{a}^n \cap \mathfrak{b} = \mathfrak{a}^{n-k}(\mathfrak{a}^k \cap \mathfrak{b})$ for all $n \geqslant k$.*

Proof. Set $\mathcal{S} := R[\mathfrak{a}T, T^{-1}]$, the extended Rees ring of $\mathfrak{a}$, and let $\mathfrak{B} = \mathfrak{b}R[T, T^{-1}] \cap \mathcal{S}$, an ideal of S. Thus an element $\sum_{i=-v}^{w} r_i T^i$ of $R[T, T^{-1}]$ belongs to $\mathfrak{B}$ if and only if $r_i \in \mathfrak{a}^i \cap \mathfrak{b}$ for all $i = -v, \ldots, w$. Hence $\mathfrak{B}$ is a graded ideal of the Noetherian ring $\mathcal{S}$, and so has a finite generating set of homogeneous elements, $\{b_{i_1}T^{i_1}, \ldots, b_{i_q}T^{i_q}\}$ say, where $b_{i_j} \in \mathfrak{a}^{i_j} \cap \mathfrak{b}$ for all $j = 1, \ldots, q$. Let $k = \max\{i_1, \ldots, i_q\}$. Then $\mathfrak{a}^n \cap \mathfrak{b} = \mathfrak{a}^{n-k}(\mathfrak{a}^k \cap \mathfrak{b})$ for all $n \geqslant k$. (The reader might find it helpful to note that $(\mathfrak{a}^i \cap \mathfrak{b})\mathfrak{a}^{n-i} \subseteq (\mathfrak{a}^{i+1} \cap \mathfrak{b})\mathfrak{a}^{n-(i+1)}$ for integers n, i with $0 \leqslant i < n$.) □

David Rees explained the name of the lemma as follows: David had his proof of the lemma in 1954, but he did not submit it for publication until May 1955; paper [14] appeared in 1956,

in the very month in which Emil Artin lectured, at a conference in Japan, about his discovery of the same argument and result; M. Nagata was asked to adjudicate as to who should receive the credit, and responded that 'it is obviously the Artin–Rees Lemma'.

There is a version of the lemma for modules, which states that if N is a submodule of a finitely generated R-module M, then there exists $k \in \mathbb{N}$ such that $\mathfrak{a}^n M \cap N = \mathfrak{a}^{n-k}(\mathfrak{a}^k M \cap N)$ for all $n \geqslant k$. This result means that the topology induced on N by the $\mathfrak{a}$-adic topology on M is the $\mathfrak{a}$-adic topology on N. The interested reader might like to consult ⟨**13**, Theorem 8.5⟩.

As well as being well suited to the study of powers of a fixed ideal $\mathfrak{a}$ of R, the extended Rees ring $R[\mathfrak{a}T, T^{-1}]$ can be used to explore the integral closures of the powers of $\mathfrak{a}$, because it turns out that $\overline{R[\mathfrak{a}T, T^{-1}]U^i} \cap R = \overline{\mathfrak{a}^i}$ for each $i \in \mathbb{N}_0$ (where, once again, $U = T^{-1}$). However, Rees's Valuation Theorem, which is the subject of the next section, also provides information about the integral closures of powers of $\mathfrak{a}$.

8. *The Rees valuations*

In a series of papers [**11**, **12**, **15**, **16**, **17**] published during an exceptionally productive period from 1955 to 1957, David Rees established what he called his 'Valuation Theorem', which can be viewed as describing the integral closures of the powers of an ideal $\mathfrak{a}$ of R in terms of certain uniquely determined discrete valuation rings (DVRs). These DVRs are nowadays referred to as 'the Rees valuations rings', while the associated discrete valuations are called 'the Rees valuations'.

Intimately related to the Rees valuations is the asymptotic Samuel function, defined as follows.

Definition 8.1. Let $\mathfrak{a}$ be a proper ideal of the (Noetherian) ring R. The *order function of* $\mathfrak{a}$ is the function $w_{\mathfrak{a}} : R \longrightarrow \mathbb{N}_0 \cup \{\infty\}$ for which

$$w_{\mathfrak{a}}(r) = \begin{cases} m & \text{if } r \in \mathfrak{a}^m \setminus \mathfrak{a}^{m+1}, \\ \infty & \text{if } r \in \bigcap_{i=1}^{\infty} \mathfrak{a}^i. \end{cases}$$

Note that if R is an integral domain, then $\bigcap_{i=1}^{\infty} \mathfrak{a}^i = 0$, and 0 is the only element $r \in R$ for which $w_{\mathfrak{a}}(r) = \infty$.

Lemma and Definition 8.2 (Rees [**11**, Lemma 1.2]). *With the notation of Definition* 8.1, *for each* $r \in R$, *the limit*

$$\lim_{n\to\infty} \frac{w_{\mathfrak{a}}(r^n)}{n} =: \overline{w_{\mathfrak{a}}}(r)$$

exists, provided that ∞ *is permitted as a limit. The resulting function* $\overline{w_{\mathfrak{a}}} : R \longrightarrow \mathbb{R} \cup \{\infty\}$ *is called the* asymptotic Samuel function.

The name is in recognition of P. Samuel's initiation of the study of the asymptotic theory of ideals in ⟨**21**⟩. Samuel's work had a formative influence on Rees.

The definition of the order function and the definition of integral closure can be extended in obvious ways to the case where the underlying ring, A say, is not Noetherian, and the analogue of Lemma 8.2 still holds: see McAdam ⟨**14**, Proposition 11.1⟩. Indeed, for a $k \in \mathbb{N}_0$, an ideal I of A and $a, a' \in A$, one can show that, if $a \in \overline{I^k}$, then $\overline{w_I}(a) \geqslant k$, while if $\overline{w_I}(a') > k$, then $a' \in \overline{I^k}$. See McAdam ⟨**14**, Proposition 11.2⟩.

For the statement of Rees's Valuation Theorem, we require the concept of discrete integer-valued valuation of R, including in the case where R is not a domain. For basic properties of DVRs and the associated discrete valuations, the reader is referred to ⟨**13**, Chapter 4⟩.

DEFINITION 8.3. By a *discrete integer-valued valuation of* R, we shall mean the composition v^* of the natural ring homomorphism $R \longrightarrow R/\mathfrak{p}$ for some minimal prime ideal $\mathfrak{p}$ of R and a (conventional) discrete integer-valued valuation v of the quotient field of $R/\mathfrak{p}$ that is non-negative on $R/\mathfrak{p}$. (Strictly speaking, the values of v and v^* lie in $\mathbb{Z} \cup \{\infty\}$.) Note that, for an $r \in R$, we have $v^*(r) = \infty$ if and only if $r \in \mathfrak{p}$.

THEOREM 8.4 (Rees's Valuation Theorem [**16**]). (*Recall that R is Noetherian.*) *Let $\mathfrak{a}$ be a proper ideal of R. Then there exist discrete integer-valued valuations $v_1^*, \ldots, v_h^*$ of R (in the sense of Definition 8.3), and positive integers $e_1, \ldots, e_h$, such that*

$$\overline{w_{\mathfrak{a}}}(r) = \lim_{n\to\infty} \frac{w_{\mathfrak{a}}(r^n)}{n} = \min\left\{\frac{v_1^*(r)}{e_1}, \ldots, \frac{v_h^*(r)}{e_h}\right\} \quad \textit{for all } r \in R.$$

Also, if $\operatorname{ht}\mathfrak{a} > 0$ *and none of $v_1^*, \ldots, v_h^*$ can be omitted from all these expressions, then $v_1^*, \ldots, v_h^*$ are uniquely determined up to equivalence of valuations.*

Where do the Rees valuations come from? Key points in an argument that proves their existence are that, in a DVR, every ideal is integrally closed, and the Mori–Nagata Theorem, the statement of which uses the concept of Krull domain.

DEFINITION 8.5. A *Krull domain* is an integral domain D such that

(i) for each prime ideal $\mathfrak{p}$ of D of height 1, the localization $D_{\mathfrak{p}}$ is a DVR;
(ii) $D = \bigcap_{\mathfrak{p}\in\operatorname{Spec}(D),\operatorname{ht}\mathfrak{p}=1} D_{\mathfrak{p}}$;
(iii) each non-zero $a \in D$ belongs to only finitely many of the prime ideals of D of height 1.

THEOREM 8.6 (The Mori–Nagata Theorem ⟨**16**, **17**⟩). (*Recall that R is Noetherian.*) *Suppose that R is an integral domain. Then its integral closure $\overline{R}$ is a Krull domain.*

This result is due to Y. Mori in the case where R is local and to M. Nagata in the general case. (Note that $\overline{R}$ need not be Noetherian.)

In the following hints about how the Mori–Nagata Theorem can be used to prove Rees's Valuation Theorem, attention will be concentrated on the case where R is a domain, because in that case it is easier to see where the Rees valuations come from.

Let u be a non-zero, non-unit element of a Krull domain D and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_h$ be the prime ideals of D of height 1 that contain u; for each $i = 1, \ldots, h$, let v_i be the valuation associated with the DVR $D_{\mathfrak{p}_i}$, and set $v_i(u) =: e_i$. Then one can show that

$$\overline{w_{Du}}(r) = \lim_{n\to\infty} \frac{w_{Du}(r^n)}{n} = \min\left\{\frac{v_1(r)}{e_1}, \ldots, \frac{v_h(r)}{e_h}\right\} \quad \text{for all } r \in D;$$

see McAdam ⟨**14**, Lemma 11.3⟩. Thus one has what might be called a 'Rees Valuation Theorem' for the proper, non-zero principal ideal Du in the Krull domain D.

Now return to the situation of Rees's Valuation Theorem in the special case where R is a domain, and set $\mathcal{S} := R[\mathfrak{a}T, T^{-1}]$, the extended Rees ring of $\mathfrak{a}$. By the Mori–Nagata Theorem, $\overline{\mathcal{S}}$ is a Krull domain: take this for D in the above discussion, and take $U := T^{-1}$ for u. We can conclude that there exist discrete integer-valued valuations $v_1, \ldots, v_h$ of the quotient field of

$\mathcal{S}$, non-negative on $\mathcal{S}$ and positive integers $e_1, \dots, e_h$, such that

$$\overline{w_{\overline{\mathcal{S}U}}}(r) = \min\left\{\frac{v_1(r)}{e_1}, \dots, \frac{v_h(r)}{e_h}\right\} \quad \text{for all } r \in R.$$

It can be shown that $\overline{w_{\overline{\mathcal{S}U}}}(r) = \overline{w_{\mathcal{S}U}}(r)$ for all $r \in R$; see Rees [**16**, Lemma 2.2]. Furthermore, since $\mathcal{S}U^n \cap R = \mathfrak{a}^n$ for all $n \in \mathbb{N}$, we have $\overline{w_{\mathcal{S}U}}(r) = \overline{w_{\mathfrak{a}}}(r)$ for all $r \in R$. These observations together yield a proof of the existence of Rees Valuations for $\mathfrak{a}$ in the case where R is a domain. This proof (which follows the route taken by McAdam in ⟨**14**, Chapter XI⟩) is not the original proof of Rees: in [**47**, p. 2], Rees points out that he could not use the full version of the Mori–Nagata Theorem 8.6 in 1955 because ⟨**17**⟩ was not then available to him. Also, Rees proved the general case of the Valuation Theorem by reduction to the case where $\mathfrak{a}$ is principal; in [**16**] he used (what we now call) the extended Rees ring to effect such a reduction, in the spirit of [**14**] and § 7. In [**16**, p. 222], he seems to be metaphorically 'kicking himself' for overlooking this approach in his earlier paper [**15**]! However, one could note that [**16**] carries a received date *earlier* than that of [**14**], and conclude that Rees had not realized the full potential of his 'Rees ring' arguments at the time [**15**] was submitted.

A key point in the extension of the Valuation Theorem from a Noetherian domain to a general commutative Noetherian ring R is the fact that, for an ideal $\mathfrak{a}$ of R and $r \in R$, we have $r \in \overline{\mathfrak{a}}$ if and only if $r + \mathfrak{p} \in \overline{\mathfrak{a} + \mathfrak{p}/\mathfrak{p}}$ for each minimal prime ideal $\mathfrak{p}$ of R.

Remark 8.7. The asymptotic Samuel function $\overline{w_{\mathfrak{a}}}$ of Lemma 8.2 is related to the integral closures of powers of $\mathfrak{a}$. We have already noted earlier in the section that if $r \in \overline{\mathfrak{a}^c}$ for a $c \in \mathbb{N}$, then $\overline{w_{\mathfrak{a}}}(r) \geqslant c$; the Valuation Theorem can be used to prove the converse statement in our Noetherian ring R. Thus, for $c \in \mathbb{N}$ and $r \in R$, it is the case that $r \in \overline{\mathfrak{a}^c}$ if and only if $\overline{w_{\mathfrak{a}}}(r) \geqslant c$. This consequence of the Valuation Theorem is useful in applications, such as to questions about whether two ideals $\mathfrak{a}, \mathfrak{b}$ of R are *projectively equivalent*, that is, such that $\overline{\mathfrak{a}^s} = \overline{\mathfrak{b}^t}$ for some $s, t \in \mathbb{N}$.

We now summarize Rees's approach to the proof of the uniqueness aspect of his Valuation Theorem (as stated in Theorem 8.4) on the assumptions that $\operatorname{ht}\mathfrak{a} > 0$ and none of $v_1^*, \dots, v_h^*$ is redundant. Let $w : R \longrightarrow \mathbb{Q} \cup \{\infty\}$ be defined by

$$w(r) = \min\left\{\frac{v_1^*(r)}{e_1}, \dots, \frac{v_h^*(r)}{e_h}\right\} \quad \text{for all } r \in R.$$

In [**11**, § 1], Rees let R_w denote $\{r \in R : w(r) < \infty\}$ and defined a subset S of R_w to be *w-consistent* if, for all $t \in \mathbb{N}$ and (not necessarily distinct) elements $r_1, \dots, r_t$ of S, we have

$$w(r_1 \dots r_t) = w(r_1) + \cdots + w(r_t).$$

Rees used Zorn's Lemma to see that each w-consistent subset of R_w is contained in a maximal such. Under the assumption that none of $v_1^*, \dots, v_h^*$ is redundant, he showed that there are exactly h maximal w-consistent subsets $S_1, \dots, S_h$ of R_w, and that these can be labelled so that, for each $i = 1, \dots, h$, we have $S_i = \{r \in R_w : w(r) = v_i^*(r)/e_i\}$. This imaginative approach thus shows that $S_1, \dots, S_h$ depend only on the function w. Rees was further able to recover v_i (up to equivalence of valuations) from knowledge of just S_i, and, in this way, to complete the proof of the uniqueness.

Rees thought that [**16**] was his best paper, but another of which he was particularly proud, and in which valuations also featured, was [**26**]. In that, he settled a problem that had been posed by Zariski in ⟨**32**⟩ and was related to Hilbert's 14th problem. The latter problem can be stated as follows: if S denotes the ring of polynomials in n indeterminates over a field k, and if F is a subfield of the field of fractions of S that contains k, must the ring $S \cap F$ be finitely generated over k? In ⟨**32**⟩, Zariski asked the following question: if F is a finitely generated field

extension of a field k, and S is a finitely generated integrally closed integral domain over k whose field of fractions contains F, must the ring $S \cap F$ be finitely generated over k? Zariski himself proved in ⟨**32**⟩ that if the transcendence degree of F over k is 1 or 2, then $S \cap F$ is indeed finitely generated over k.

In [**26**], Rees constructed an example that showed that the answer to Zariski's problem is negative. For this, he used delicate and very impressive geometric arguments involving an extended Rees ring, over the homogeneous coordinate ring of a projective complex elliptic curve C, of the ideal defining a point on C.

Hilbert's 14th problem was settled, again negatively, by Nagata in ⟨**18**⟩.

9. *The theory of grade*

The concept of grade, fundamental to the theory of Cohen–Macaulay rings, is also due to David Rees. Elements $u_1, \dots, u_t$ in R are said to form a *regular sequence* if they generate a proper ideal and $((u_1, \dots, u_{i-1}) : u_i) = (u_1, \dots, u_{i-1})$ for all $i = 1, \dots t$. (The particular case of this equation when $i = 1$ is interpreted as $(0 : u_1) = 0$, that is, u_1 is a non-zerodivisor on R.) In [**18**, **19**], Rees proved that, for a proper ideal $\mathfrak{a}$ of R, each maximal R-sequence contained in $\mathfrak{a}$ has length equal to the least integer i such that $\operatorname{Ext}^i_R(R/\mathfrak{a}, R) \neq 0$. Consequently, all maximal R-sequences contained in $\mathfrak{a}$ have the same length, and Rees defined this length to be the *grade of* $\mathfrak{a}$. His methods enabled him to deduce quickly that every R-sequence contained in $\mathfrak{a}$ can be extended to a maximal such, which must have grade $\mathfrak{a}$ terms. He also provided versions of the results for a non-zero finitely generated R-module M for which $M \neq \mathfrak{a}M$.

It should be noted that this work represents a pioneering use of homological algebra as a tool in commutative algebra, for Rees's paper [**18**] was submitted early in 1956, the year in which Cartan's and Eilenberg's foundational book ⟨**5**⟩ was first published; Serre's groundbreaking paper ⟨**22**⟩, which had a formative influence on Rees, had appeared only one year earlier.

It is easy to see that if $u_1, \dots, u_t \in R$ form a regular sequence in R, then

$$\operatorname{ht}(u_1, \dots, u_i) = i \quad \text{for all } i = 1, \dots, k.$$

Thus grade $\mathfrak{b} \leqslant \operatorname{ht} \mathfrak{b}$ for each proper ideal $\mathfrak{b}$ of R. Rees defined an ideal $\mathfrak{g}$ of R to be a *general ideal of height* k (although Rees actually used 'rank' rather than the now-standard 'height') if there is a regular sequence $g_1, \dots, g_k$ of length k such that $\mathfrak{g} = (g_1, \dots, g_k)$. Suppose that this is the case. Let $X_1, \dots, X_k$ be indeterminates. Rees proved, in [**19**, Theorem 2.1], that the homogeneous surjective ring homomorphism

$$(R/\mathfrak{g})[X_1, \dots, X_k] \longrightarrow \mathcal{G}(\mathfrak{g}) = \bigoplus_{n \in \mathbb{N}_0} \mathfrak{g}^n/\mathfrak{g}^{n+1},$$

which has 0th component equal to the identity map on $R/\mathfrak{g}$, and maps X_i to $g_i + \mathfrak{g}^2 \in \mathfrak{g}/\mathfrak{g}^2$ (the first component of $\mathcal{G}(\mathfrak{g})$) for all $i = 1, \dots, k$, is an isomorphism. He applied this result to prove that all the associated prime ideals of all powers of the general ideal $\mathfrak{g}$ have the same grade as $\mathfrak{g}$. This can be viewed as a generalization of F. S. Macaulay's Theorem ⟨**12**, § 50⟩ that a power of an ideal (in a polynomial ring over a field) of height k that can be generated by k elements is *unmixed*, that is, is such that all its associated prime ideals have the same height.

Rees also showed, in the same paper [**19**], in the case where R is local, that a proper ideal $\mathfrak{b}$ of R is a general ideal if and only if there is a homogeneous isomorphism

$$(R/\mathfrak{b})[X_1, \dots, X_k] \xrightarrow{\cong} \mathcal{G}(\mathfrak{b}) = \bigoplus_{n \in \mathbb{N}_0} \mathfrak{b}^n/\mathfrak{b}^{n+1},$$

which has 0th component equal to the identity map on $R/\mathfrak{b}$.

It was remarked earlier that grade $\mathfrak{b} \leqslant \operatorname{ht} \mathfrak{b}$ for each proper ideal $\mathfrak{b}$ of R. In the same paper [**19**], Rees defined R to be a *U-ring* if grade $\mathfrak{b} = \operatorname{ht} \mathfrak{b}$ for each proper ideal $\mathfrak{b}$ of R. Rees showed [**19**, Theorem 3.1] that R is a U-ring if and only if every general ideal of R is unmixed. (This result might explain the 'U' in 'U-ring'.) Macaulay ⟨**12**, § 48⟩ had proved that a polynomial ring with coefficients in a field has this property, and I. S. Cohen ⟨**6**, Theorem 21⟩ had proved that a regular local ring also has the property. Nowadays, U-rings are known as *Cohen–Macaulay rings*.

The study of Cohen–Macaulay rings was facilitated by the following result, also due to Rees.

THEOREM 9.1 (Rees [**19**, Theorem 4.3]). *Let Q be a (Noetherian) local ring with maximal ideal $\mathfrak{m}$. Then Q is Cohen–Macaulay if and only if* grade $\mathfrak{m} = \operatorname{ht} \mathfrak{m}$.

Thus the one single equality grade $\mathfrak{m} = \operatorname{ht} \mathfrak{m}$ implies that grade $\mathfrak{b} = \operatorname{ht} \mathfrak{b}$ for every proper ideal $\mathfrak{b}$ of Q. Rees went on to show in [**16**, Theorem 4.4] that, in a Cohen–Macaulay local ring $(Q, \mathfrak{m})$, an $\mathfrak{m}$-primary ideal that can be generated by $\dim Q$ elements must be a general ideal.

A *system of parameters* in a d-dimensional local ring $(Q, \mathfrak{m})$ is a set of d elements that generates an $\mathfrak{m}$-primary ideal. Rees established the following characterization of Cohen–Macaulay local rings.

THEOREM 9.2 (Rees [**19**, Theorem 4.5]). *Let Q be a (Noetherian) local ring. Then the following conditions are equivalent*:

(i) *Q is Cohen–Macaulay*;
(ii) *$e(\mathfrak{q}) = \ell_Q(Q/\mathfrak{q})$ for every ideal $\mathfrak{q}$ of Q generated by a system of parameters*;
(iii) *$e(\mathfrak{q}) = \ell_Q(Q/\mathfrak{q})$ for one ideal $\mathfrak{q}$ of Q generated by a system of parameters.*

Rees's paper [**19**] carries a received date of 13 February 1956. It is interesting to note that Northcott's and Rees's third joint paper, [**22**], carrying a received date of 9 October 1956, provided an elementary approach to the theory of grade, and Rees's Theorem 9.1, that avoids the use of homological algebra.

Northcott and Rees also contributed to the basic theory of Gorenstein rings, because the (fourth and) last of their joint papers, [**25**], contains the theorem that a local ring in which every ideal generated by a system of parameters is irreducible must be Cohen–Macaulay, and this theorem was an important ingredient in H. Bass's characterization of Gorenstein local rings in his seminal 'ubiquity' paper ⟨**1**⟩.

There is a version of the Cohen–Macaulay condition for finitely generated R-modules. Substantial books have been written about Cohen–Macaulay rings and modules; see Bruns and Herzog ⟨**4**⟩ and Yoshino ⟨**30**⟩. It is sobering to reflect on the fact that all this mathematics depends on David Rees's invention of the concept of grade.

10. *Exeter 1958–1983*

David Rees's research described in §§ 6–9 was almost all achieved during his extraordinarily prolific period from 1952 to 1957, while he was at the University of Cambridge. (He was awarded the degree of DSc by the University of Cambridge in 1959. One consequence of his war service at Bletchley Park was that he did not have the opportunity to submit for a PhD degree.) In 1958, he was appointed to the Chair of Pure Mathematics at the University of Exeter. The time-consuming practical aspects of the move of his family, with three young daughters

(a fourth was later born in Exeter), from Cambridge to Exeter had to be faced, and Rees's output of papers slowed from its astonishing rate of 1956 and 1957; there was even a missed opportunity.

Let $\mathfrak{a}$ be a proper ideal of R. It was mentioned in § 7 that the extended Rees ring $R[\mathfrak{a}T, T^{-1}]$ is a convenient tool for studying the powers of $\mathfrak{a}$. Rees became interested in studying the 'asymptotic behaviour' of the sequence $(\operatorname{Ass} R/\mathfrak{a}^n)_{n=1,2,\dots}$ as $n \to \infty$. In 1958, he had a proof that the set $\bigcup_{n\in\mathbb{N}} \operatorname{Ass} R/\mathfrak{a}^n$ is finite. (Alert readers might note that a proof of this statement can be reduced, by use of the extended Rees ring, to the case where $\mathfrak{a}$ is principal and generated by a non-zerodivisor.) Rees wrote a paper about this result; the referee asked for some changes, but Rees did not have time to attend to the rewriting on account of the move to Exeter. As a consequence, his result did not get published. Some readers will perhaps be aware that, some 20 years later, M. Brodmann's Theorem that the sequence $(\operatorname{Ass} R/\mathfrak{a}^n)_{n=1,2,\dots}$ is ultimately constant was published ⟨**3**⟩.

The year 1961 saw another substantial output of papers by David Rees, following his move to Exeter. One of his papers from that year, [**28**], contains a striking result about multiplicities of $\mathfrak{m}$-primary ideals $\mathfrak{a}$ and $\mathfrak{b}$, with $\mathfrak{b} \subset \mathfrak{a}$, in a local ring $(Q, \mathfrak{m})$. It was pointed out in § 6 that, if $\mathfrak{b}$ is a reduction of $\mathfrak{a}$, then the multiplicities of $\mathfrak{a}$ and $\mathfrak{b}$ are equal, that is, $e(\mathfrak{b}) = e(\mathfrak{a})$. In [**28**], Rees proved a partial converse. The local ring Q is said to be *formally equidimensional* (or *quasi-unmixed*) if $\dim \widehat{Q}/\mathfrak{P} = \dim Q$ for every minimal prime ideal $\mathfrak{P}$ of the completion $\widehat{Q}$ of Q.

THEOREM 10.1 (Rees [**28**, Theorem 3.2]). *If the local ring $(Q, \mathfrak{m})$ is formally equidimensional, and if $\mathfrak{b} \subset \mathfrak{a}$ are two $\mathfrak{m}$-primary ideals of Q with $e(\mathfrak{b}) = e(\mathfrak{a})$, then $\mathfrak{b}$ is a reduction of $\mathfrak{a}$.*

Rees's proof involved yet another application of the extended Rees ring.

Another of David Rees's papers published in 1961 is [**31**], in which he provided an elegant necessary and sufficient condition for a reduced local ring to be analytically unramified, that is, to have reduced completion. (Paper [**31**] carries a received date in June 1959, and so David was indeed thinking about research problems soon after arriving in Exeter.)

Recall that the nilradical $\sqrt{0}$ of R is the ideal of nilpotent elements, and that R is said to be *reduced* if $\sqrt{0} = 0$, that is, if 0 is the only nilpotent element of R. Let $\mathfrak{a}$ be a proper ideal of a local ring $(Q, \mathfrak{m})$, and let $n \in \mathbb{N}$. Recall from the Northcott–Rees theory of reductions described in § 6 that every integrally closed ideal of Q contains $\sqrt{0}$. It follows that, if there were a $k \in \mathbb{N}$ such that $\overline{\mathfrak{a}^{n+k}} \subseteq \mathfrak{a}^n$ for all $n \in \mathbb{N}$, then we would have to have

$$\sqrt{0} \subseteq \bigcap_{n=1}^{\infty} \overline{\mathfrak{a}^{n+k}} \subseteq \bigcap_{n=1}^{\infty} \mathfrak{a}^n = 0,$$

in view of Krull's Intersection Theorem 7.1. Thus the existence of such a k would force Q to be reduced. Furthermore, in the case when $\mathfrak{a}$ is $\mathfrak{m}$-primary, it can be shown without difficulty that $\overline{\mathfrak{a}\widehat{Q}} = \overline{\mathfrak{a}}\widehat{Q}$, and then the existence of such a k would lead to the conclusion that $\widehat{Q}$ is reduced, that is, that Q is analytically unramified. This discussion gives some hints about how one might prove the easier implication in the following theorem of Rees.

THEOREM 10.2 (Rees [**31**]). *Let Q be a (Noetherian) local ring. Then Q is analytically unramified if and only if, for each proper ideal $\mathfrak{a}$ of Q, there exists $k \in \mathbb{N}_0$ such that $\overline{\mathfrak{a}^{n+k}} \subseteq \mathfrak{a}^n$ for all $n \in \mathbb{N}$.*

In [**31**], Rees noted that the above theorem has, as an easy consequence, the corollary that every localization of an analytically unramified local ring Q is again analytically unramified: if

$\mathfrak{p} \in \operatorname{Spec}(Q)$ and $\mathfrak{A}$ is a proper ideal of $Q_{\mathfrak{p}}$, let $\mathfrak{a}$ be the contraction of $\mathfrak{A}$ to Q; by Theorem 10.2, there exists $k \in \mathbb{N}_0$ such that $\overline{\mathfrak{a}^{n+k}} \subseteq \mathfrak{a}^n$ for all $n \in \mathbb{N}$; therefore, for all $n \in \mathbb{N}$, we have

$$\overline{\mathfrak{A}^{n+k}} = \overline{\mathfrak{a}^{n+k}Q_{\mathfrak{p}}} = \overline{\mathfrak{a}^{n+k}}Q_{\mathfrak{p}} \subseteq \mathfrak{a}^n Q_{\mathfrak{p}} = \mathfrak{A}^n;$$

now use Theorem 10.2 again to deduce that $Q_{\mathfrak{p}}$ is analytically unramified. I am impressed by the elegance of this argument, because a direct 'bare hands' attempt to prove that analytic unramification is preserved by localization seems to me to be fraught with difficulties.

Also in [**31**], Rees provided the following necessary and sufficient condition, phrased in terms of finite generation of integral closures, for a reduced local ring to be analytically unramified.

THEOREM 10.3 (Rees [**31**]). *Let Q be a reduced local ring having full ring of quotients K, so that $K = S^{-1}R$ where S is the set of non-zerodivisors in R. Then Q is analytically unramified if and only if, for all $n \in \mathbb{N}$ and $a_1, \dots, a_n \in K$, the integral closure in K of $A := R[a_1, \dots, a_n]$ is a finitely generated A-module.*

The only entries with dates between 1962 and 1977 in Rees's list of publications are two items in conference proceedings. This marked contrast with Rees's very prolific period between 1954 and 1961 no doubt reflects the pressures associated with running a university department; also, David Rees served terms as Dean of the Faculty and as Deputy Vice-Chancellor. He was regarded as a kindly, avuncular figure by the younger mathematicians who worked under him.

Another relevant comment is that the 1960s and 1970s were important years for David's family, with all four of his daughters approaching and progressing through their teenage years during that time. Although David was not a practical man (his attempts to set up the projector for occasional slide shows were always fraught, and the control panel of the washing machine was a long-standing source of mystery to him), he was a loving, caring and supportive husband and father.

1978 saw the publication of a joint paper [**35**] by Rees and the present author. It involved the so-called 'mixed multiplicities', and as these also feature in several of David's later papers, a little background might be helpful. Let $(Q, \mathfrak{m})$ be a d-dimensional local ring and let $\mathfrak{q}_1, \dots, \mathfrak{q}_k$ be $\mathfrak{m}$-primary ideals of Q. For each k-tuple of non-negative integers $(n_1, \dots, n_k)$, the Q-module $Q/\mathfrak{q}_1^{n_1} \dots \mathfrak{q}_k^{n_k}$ has finite length, and it turns out that there exists a rational polynomial $p \in \mathbb{Q}[X_1, \dots, X_k]$ (where $X_1, \dots, X_k$ are indeterminates) of total degree d such that

$$\ell_Q(Q/\mathfrak{q}_1^{n_1} \dots \mathfrak{q}_k^{n_k}) = p(n_1, \dots, n_k) \quad \text{for all sufficiently large } n_1, \dots, n_k.$$

We call p the *multigraded Hilbert polynomial of* $\mathfrak{q}_1, \dots, \mathfrak{q}_k$. Write the homogeneous component of p of degree d as

$$\sum_{d_1+\dots+d_k=d} \frac{1}{d_1! \cdots d_k!} e(\mathfrak{q}_1^{[d_1]}, \dots, \mathfrak{q}_k^{[d_k]}) X_1^{d_1} \cdots X_k^{d_k},$$

where each $e(\mathfrak{q}_1^{[d_1]}, \dots, \mathfrak{q}_k^{[d_k]})$ is a uniquely determined rational number, called the *mixed multiplicity of* $\mathfrak{q}_1, \dots, \mathfrak{q}_k$ *of type* $(d_1, \dots, d_k)$. In fact, these mixed multiplicities turn out to be non-negative integers. Sometimes we write $e(\mathfrak{q}_1^{[d_1]}, \dots, \mathfrak{q}_k^{[d_k]})$ as

$$e(\mathfrak{q}_1, \dots, \mathfrak{q}_1, \mathfrak{q}_2, \dots, \mathfrak{q}_2, \dots, \mathfrak{q}_k, \dots, \mathfrak{q}_k),$$

where $\mathfrak{q}_1$ is listed d_1 times, $\mathfrak{q}_2$ is listed d_2 times and so on.

In the special case in which $k = 1$, the polynomial p is referred to simply as the *Hilbert polynomial of* $\mathfrak{q}_1$; it follows from the definition of the multiplicity given in § 6 that its leading coefficient is $e(\mathfrak{q}_1)/d!$, so that $e(\mathfrak{q}_1^{[d]})$ is just $e(\mathfrak{q}_1)$.

The special case in which $k = 2$ was studied by Bhattacharya in ⟨**2**⟩. In [**28**, Lemma 2.4], Rees showed that, for the Bhattacharya polynomial, $e(\mathfrak{q}_1^{[d]}, \mathfrak{q}_2^{[0]}) = e(\mathfrak{q}_1)$ (and, by symmetry,

$e(\mathfrak{q}_1^{[0]}, \mathfrak{q}_2^{[d]}) = e(\mathfrak{q}_2)$). That argument of Rees can easily be adapted to prove that, in the general case of k $\mathfrak{m}$-primary ideals of Q, we have $e(\mathfrak{q}_1^{[d]}, \mathfrak{q}_2^{[0]}, \dots, \mathfrak{q}_k^{[0]}) = e(\mathfrak{q}_1)$, and so on.

The Rees–Sharp paper [**35**] is concerned with the case where $k = 2$, that is, with the coefficients of the Bhattacharya polynomial and their relationships with ordinary multiplicities. We write $\mathfrak{q}_1 =: \mathfrak{q}$ and $\mathfrak{q}_2 =: \mathfrak{t}$. Then the comments above lead to the conclusion that, for all positive integers r and s,

$$e(\mathfrak{q}^r \mathfrak{t}^s) = e(\mathfrak{q}) r^d + \cdots + \binom{d}{i} e(\mathfrak{q}^{[i]}, \mathfrak{t}^{[d-i]}) r^i s^{d-i} + \cdots + e(\mathfrak{t}) s^d.$$

In particular, on taking $r = s = 1$, we see that

$$e(\mathfrak{q}\mathfrak{t}) = e(\mathfrak{q}) + \cdots + \binom{d}{i} e(\mathfrak{q}^{[i]}, \mathfrak{t}^{[d-i]}) + \cdots + e(\mathfrak{t}).$$

A comparison of this expression with the binomial expansion for $(e(\mathfrak{q})^{1/d} + e(\mathfrak{t})^{1/d})^d$ led B. Teissier to conjecture in ⟨**26**, Chapitre 1, § 2⟩ that $e(\mathfrak{q}^{[i]}, \mathfrak{t}^{[d-i]})^d \leqslant e(\mathfrak{q})^i e(\mathfrak{t})^{d-i}$ for all $i = (0,)1, \dots, d-1(, d)$. The validity of this conjecture would imply that

$$e(\mathfrak{q}\mathfrak{t})^{1/d} \leqslant e(\mathfrak{q})^{1/d} + e(\mathfrak{t})^{1/d},$$

which is analogous to the classical Minkowski inequality. Teissier further showed in ⟨**27**, 1.3⟩ that the above conjecture would be valid if it could be shown that, for $d \geqslant 2$,

$$e(\mathfrak{q}^{[i]}, \mathfrak{t}^{[d-i]})^2 \leqslant e(\mathfrak{q}^{[i-1]}, \mathfrak{t}^{[d-i+1]}) e(\mathfrak{q}^{[i+1]}, \mathfrak{t}^{[d-i-1]}) \quad \text{for all } i = 1, 2, \dots, d-1,$$

and, moreover, that these latter inequalities would be proved if they could be proved in the special case in which $d = 2$. So Teissier was interested in the following quite specific question: if $\mathfrak{q}$ and $\mathfrak{t}$ are $\mathfrak{m}$-primary ideals in a two-dimensional local ring $(Q, \mathfrak{m})$, do we have $e(\mathfrak{q}^{[1]}, \mathfrak{t}^{[1]})^2 \leqslant e(\mathfrak{q}^{[0]}, \mathfrak{t}^{[2]}) e(\mathfrak{q}^{[2]}, \mathfrak{t}^{[0]})$? Teissier made progress on these questions in the case where Q is a reduced Cohen–Macaulay algebra over an algebraically closed field of characteristic 0, or, more generally, when one has resolutions of singularities of surfaces available.

David Rees became fascinated by these questions and drew my attention to the question in the two-dimensional case. I realized how to answer this in the case where $\mathfrak{q}$ is generated by two elements. I told David, and he quickly brought to bear his machinery of reductions to provide a complete proof of all the above inequalities and Teissier's above-mentioned conjecture, without restriction on the local ring Q, so that the 'Minkowski inequality' for multiplicities is valid in complete generality. Thus my contribution was tiny, but nevertheless David, ever generous, wanted me to be a joint author of [**35**]; however, it was true that, without my contribution, there would have been no proof at that time of the Minkowski inequality. I quickly realized that David enjoyed thinking about research problems rather more than writing up the results, and so I did most of the writing of [**35**].

At the beginning of this section, mention was made of Rees's interest in the sequence $(\operatorname{Ass} R/\mathfrak{a}^n)_{n=1,2,\dots}$, where $\mathfrak{a}$ is a proper ideal of R. He was also interested in the sequence $(\operatorname{Ass} R/\overline{\mathfrak{a}^n})_{n=1,2,\dots}$. It follows from Ratliff ⟨**19**, Theorem 2.5⟩ and McAdam and Eakin ⟨**15**, Proposition 7⟩ that, if $\operatorname{ht} \mathfrak{a} \geqslant 1$, then the sequence $(\operatorname{Ass} R/\overline{\mathfrak{a}^n})_{n=1,2,\dots}$ is ultimately constant. In his 1981 paper [**37**, § 4], Rees noted that one consequence of his Valuation Theorem is that, in complete generality, $\bigcup_{n\in\mathbb{N}} \operatorname{Ass} R/\overline{\mathfrak{a}^n}$ is finite. The result that the sequence $(\operatorname{Ass} R/\overline{\mathfrak{a}^n})_{n=1,2,\dots}$ is increasing (in the sense that $\operatorname{Ass} R/\overline{\mathfrak{a}^n} \subseteq \operatorname{Ass} R/\overline{\mathfrak{a}^{n+1}}$ for all $n \in \mathbb{N}$) and ultimately constant, without the restriction that $\operatorname{ht} \mathfrak{a} \geqslant 1$, appeared in Ratliff ⟨**20**, (2.4) and (2.7)⟩.

In [**37**], Rees defined $r \in R$ to be *asymptotically prime to* $\mathfrak{a}$ if $\mathfrak{a} + rR \neq R$ and $(\overline{\mathfrak{a}^n} : r) = \overline{\mathfrak{a}^n}$ for all $n \in \mathbb{N}$; he went on to define a sequence $u_1, \dots, u_r$ to be an *asymptotic prime sequence over* $\mathfrak{a}$ if u_i is asymptotically prime to $\mathfrak{a} + u_1 R + \cdots + u_{i-1} R$ for all $i = 1, \dots, r$. This concept inspired others (although the word 'prime' was dropped from the name); see McAdam ⟨**14**, Chapter VI⟩. In [**37**, Theorem 4.2], Rees proved that, when R is local, the length of each

maximal asymptotic prime sequence over $\mathfrak{a}$ is bounded above by $\dim R - \ell(\mathfrak{a})$ (where $\ell(\mathfrak{a})$ denotes the analytic spread of $\mathfrak{a}$, as in § 6), and that, if R is formally equidimensional, then every maximal asymptotic prime sequence over $\mathfrak{a}$ has length exactly $\dim R - \ell(\mathfrak{a})$.

11. *Mathematics in 'retirement'*

David Rees retired from his post as Professor of Pure Mathematics at the University of Exeter in 1983, but a glance at the end of his list of publications shows that he certainly did not retire from research in commutative algebra until much later. Many of his papers (and his book) published after 1982 were concerned, at least in part, with generalizations and extensions of topics he had studied earlier in his career.

His book [**47**] is based on eleven two-hour lectures which he presented during a three-month visit to Nagoya University in Japan during 1982–1983, at the invitation of Professor Hideyuki Matsumura. That visit inspired a large number of young Japanese commutative algebraists. In [**47**], Rees studied filtrations on R, which are defined as follows.

Definition 11.1. A *filtration* on R is a function $w : R \longrightarrow \mathbb{R} \cup \{\infty\}$ such that

(i) $w(1) \geqslant 0$ and $w(0) = \infty$;
(ii) $w(x - y) \geqslant \min\{w(x), w(y)\}$ for all $x, y \in R$; and
(iii) $w(xy) \geqslant w(x) + w(y)$ for all $x, y \in R$.

Such a filtration w is said to be *homogeneous* if $w(x^n) = nw(x)$ for all $x \in R$ and $n \in \mathbb{N}_0$.

Lemma 11.2 (Rees [**11**, Lemma 2.11]). *Let w be a filtration on R. Then, for each $r \in R$, the limit*

$$\lim_{n\to\infty} \frac{w(r^n)}{n} =: \overline{w}(r)$$

exists, provided that ∞ is permitted as a limit. The resulting function $\overline{w} : R \longrightarrow \mathbb{R} \cup \{\infty\}$ is a homogeneous filtration on R.

Rees associated with a filtration w on R, as in Definition 11.1, what we might call the *extended Rees ring* $\mathcal{E}(w)$ *of* w. Let T be an indeterminate, and let $\mathcal{E}(w)$ denote the subring of $R[T, T^{-1}] = R[T]_T$ consisting of all sums $\sum_{i=t}^{h} r_i T^i \in R[T, T^{-1}]$ with $w(r_i) \geqslant i$ for all $i = t, \ldots, h$. (Note that t and h here could be negative.) Rees defined the filtration w to be a *Noether* filtration if the extended Rees ring $\mathcal{E}(w)$ of w is a graded Noetherian ring. In [**47**, Chapter 4], Rees proved a version of his Valuation Theorem 8.4 for a Noether filtration on R. Since a basic example of a Noether filtration on R is the order function $w_{\mathfrak{a}}$ of an ideal $\mathfrak{a}$ of R (see Definition 8.1), it seems fair to assume that [**47**, Chapter 4] represents David's considered opinion about the best way to approach his Valuation Theorem 8.4.

Four of David's publications from the 1980s and 1990s, namely [**40**, **44**, **49**, **50**], are contributions to volumes of conference proceedings. In his retirement, he was a sought-after speaker at conferences. He and I were the only two British participants at the three-week microprogram on Commutative Algebra at the Mathematical Sciences Research Institute at Berkeley, CA, in 1987, and we spent quite a bit of time together. He was the 'father of the conference', not only in being the oldest mathematician among the participants, but also in being the originator, through his work of the 1950s and 1960s, of ideas relevant to quite a few of the presented lectures. (He still smoked a pipe at that time, and I have a clear memory of him flapping furiously at his jacket pocket, trying to ensure that his pipe was really out, as

we went into the lectures.) Many young participants were keen to tell him about their latest results, but often those results were not a surprise to David. On more than one occasion, I heard him say 'Ah, I have a different way of doing that ...'.

The concept of a joint reduction features in more than one of his papers written in retirement. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ be (not necessarily distinct) ideals of R, and let $r_i \in \mathfrak{a}_i$ for each $i = 1, \ldots, k$. We say that $(r_1, \ldots, r_k)$ is a *joint reduction of* $(\mathfrak{a}_1, \ldots, \mathfrak{a}_k)$ if the ideal $\sum_{i=1}^{k} r_i \mathfrak{a}_1 \cdots \mathfrak{a}_{i-1} \mathfrak{a}_{i+1} \cdots \mathfrak{a}_k$ is a reduction of the ideal $\mathfrak{a}_1 \cdots \mathfrak{a}_k$. In the special case in which $\mathfrak{a}_1 = \cdots = \mathfrak{a}_k =: \mathfrak{a}$, the k-tuple $(r_1, \ldots, r_k)$ is a joint reduction of $(\mathfrak{a}_1, \ldots, \mathfrak{a}_k)$ if and only if $(r_1, \ldots, r_k)\mathfrak{a}^{k-1}$ is a reduction of $\mathfrak{a}^k$; it is easy to see that this is the case if and only if the ideal $(r_1, \ldots, r_k)R$ is a reduction of $\mathfrak{a}$. Thus the concept of joint reduction can be viewed as a generalization of the concept of reduction.

Joint reductions were introduced by Rees in [**40**], and in [**41**] he proved that, in a local ring $(Q, \mathfrak{m})$ of dimension $d > 0$ with infinite residue field, given a d-tuple $(\mathfrak{q}_1, \ldots, \mathfrak{q}_d)$ of (not necessarily distinct) $\mathfrak{m}$-primary ideals of Q, there exists a joint reduction $(r_1, \ldots, r_d)$ of $(\mathfrak{q}_1, \ldots, \mathfrak{q}_d)$ and, moreover, we have

$$e(\mathfrak{q}_1, \ldots, \mathfrak{q}_d) = e((r_1, \ldots, r_d)Q),$$

so that the latter is independent of the choice of joint reduction.

Judith D. Sally (who, together with Melvin Hochster and Craig Huneke, organized the 1987 Berkeley Microprogram on Commutative Algebra) collaborated with Rees in [**48**] to produce, *inter alia*, a different proof of the existence of joint reductions, also in the case where Q has infinite residue field. Craig Huneke recalls another visit to the USA in the 1980s by David Rees that significantly influenced his (Huneke's) direction of research. In particular, ⟨**9**⟩, which in part came out of conversations with David, uses a method of proof that is essentially the same as one in [**38**], another highly regarded paper by David Rees.

Several authors, including Rees himself, have extended the concept of reduction to modules. One can show that, if R is a domain and $\mathfrak{a}$ is an ideal of R, then $\overline{\mathfrak{a}} = \bigcap_V \mathfrak{a}V \cap R$, where the intersection is taken over all DVRs V between R and its field of fractions. Rees used a module-theoretic analogue of this in his definition of integral dependence of modules in [**45**]. Other authors have used different definitions of integral dependence of modules; it is worth noting that in ⟨**25**, p. 303⟩, Swanson and Huneke remark that 'every choice of definition has its own problems'. The subject is rather technical. However, there is a module-theoretic version of Rees's Theorem 10.1 in which the rôle of multiplicity is played by the so-called Buchsbaum–Rim multiplicity; see Swanson and Huneke ⟨**25**, Corollary 16.5.7⟩. Buchsbaum–Rim multiplicities were studied by Rees and D. Kirby in the second [**51**] of the three joint papers that they wrote when David Rees was in his late seventies.

12. *Concluding remarks*

In later life, David Rees received many honours. As well as being elected FRS in 1968, he was made an Honorary Fellow of Downing College in 1970; in 1993 he was awarded the Polya Prize of the London Mathematical Society, and an honorary DSc by the University of Exeter. In 1988, Professor Peter Vámos, David's successor as Professor of Pure Mathematics at the University of Exeter, and I organized a conference in Exeter to mark David's 70th birthday; ten years later, we organized another meeting in Exeter to mark David's 80th year.

Considered by semigroup theorists to be one of the founding fathers of their subject, and having introduced into commutative algebra a string of far-reaching, foundational and deep ideas and results of lasting significance, there is no doubt that David Rees was a towering figure among 20th-century British algebraists.

David Rees was survived by his wife Joan by just 12 days. They are both survived by their four daughters Mary Rees (whose election as FRS in 2002 brought much joy to the Rees family),

Rebecca Rees, Sarah Rees and Deborah Grzywacz, and by three grandchildren. Mary and Sarah are Professors of Mathematics at the Universities of Liverpool and Newcastle-upon-Tyne, respectively.

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