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ARCHIBALD READ RICHARDSON

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Archibald Read Richardson was born on 21 August, 1881, of English parents: he was the eldest of five children. The greater part of his life before he went to Swansea in 1920 was spent in London, where he began his academic career in 1903 as an engineering student at the Royal College of Science. As a volunteer, and probably under age, he had already served in the South African War. During his college course, 1903-1907, he developed an interest in pure mathematics, and after graduating he was, from 1912 to 1914, assistant professor in mathematics at the Imperial College, working with Prof. L. N. G. Filon who had early observed his ability. In August 1914 he left for war service and returned in April 1919. In August 1920 he became Professor of Aeronautical Science at the Cadet College, Cranwell, and shortly afterwards, in October 1920, was appointed to the chair of mathematics in the University College of Swansea, the newly-created constituent college of the University of Wales. This post he held for twenty years, until he was compelled to resign in 1940 on account of increasing ill health, when he retired to Cape Town, but continued to prosecute his research until shortly before his death in 1954. He was elected a Fellow of the Royal Society in 1946.

Richardson to an eminent degree combined the qualities of a man of action with those of a scholar. His character was strong, vigorous and clear cut; it matched his constitutional robustness and vitality. He was athletic, and an active member of the Territorial Army, in the University of London Officers' Training Corps. At the outbreak of war in 1914 he went on active service with the British Expeditionary Force, rising to the rank of Lieutenant-Colonel. For his action in capturing and holding part of the Hindenburg line in the Battle of Bullecourt Richardson received the D.S.O., June 1917.

He was posted as missing after the bitter fighting of 21 March, 1918, near Farguier: during the action he received a very severe bullet wound which left him an invalid for life. A friend writes: " He was an outstandingly brave and masterly soldier, displaying the greatest forethought and perspicacity, with all his great powers concentrated on the task in hand without a thought of self. A total abstainer and non-smoker, his only indulgence was a little jam with his tea; and his relaxation out of the line was a game of chess. A great man": and again "By common consent he was the most distinguished combatant officer of the London O.T.C. before World War I: in that war he became a legendary figure. His bravery was outstanding".

Few men could have survived the severe wounds and consequent operations which he suffered. During most of the session 1924-5 Richardson was absent from Swansea as a result of an incredibly severe operation carried out by Sauerbruch in Munich for the removal of the bullet from his lung. Although the operation itself was successful the strain of it left a legacy of leukaemia, a progressive disease, which virtually turned him into an invalid for the rest of his days. It was then that his true quality was displayed, in the singleness of purpose with which he turned his immense vitality to the channels of teaching and research. Long hours spent resting on his bed by the open window were employed first in reading every single paper and work on modern algebra and its sources on which he could lay his hand, and then in applying this extensive knowledge to opening new fields of research. His letters, in later years, from Cape Town were full of mathematics and of comments on current affairs, with little mention of what he was suffering, but with constant allusions to the abilities of others, and once the typical remark " Even here I have spent nearly a year in doing statistical work on Nutrition for my medical adviser and friend, Professor Brock", who had made a study of malnutrition among the natives of Central Africa for U.N.O.

Shortly after his appointment to the chair at Swansea, Richardson was offered the post of Commandant at Cranwell—a post after his own heart—which he wisely declined. From the outset he proceeded steadily with the conduct of what was the largest department of the college; and the fact that he was able to survive and prosecute research for thirty years is proof of the soundness of his judgment in remaining at Swansea. The main cause, however, of his success in keeping the leukaemia at bay for so long lies in the devoted nursing by his wife. During the summer of 1922 Richardson married Dr. Margaret Harris, a member of the Modern Languages Department at Swansea, and an expert on German linguistics. " For me", writes his wife, " his essential quality was his goodness which was innate: . . . it was his goodness that attracted people to him and inspired such confidence. People liked to be with him because of his geniality, cheeriness and kindly sense of humour. . . . He had the gift of comforting and encouraging those in trouble. I think his greatness of spirit is shown by the rare happiness which was his in spite of 36 years of ill health and by the brimming measure of happiness he gave others. His life was a triumph".

Richardson's four earliest publications were on analysis: they were followed by two on hydromechanics, a subject in which he was a specialist at the time of his going to Swansea. But his bent lay in algebra; and he soon turned his attention to regions which at that time (1920) were largely unexplored. He created a flourishing honours and research school of algebra; and two of his students, R. Wilson and D. E. Littlewood, now occupy chairs in the University, the former as his successor and the latter at Bangor. Richardson built a tradition of good teaching, was held in high esteem by his colleagues on the Senate, and was ready to stand firm to his convictions however unpopular or novel they were.

In algebra Richardson was interested both in technique and in abstract theory. His view was that problems which have long ago been worked out in ordinary numbers presumably have their counterpart if ordinary are replaced by hypercomplex numbers. In paper [**10**] of 1926 he showed how to obtain a solution, analogous to Cramer's rule involving the quotient of two determinants, for a system of general linear equations, such as

$$\Sigma a_s x b_s = \alpha,$$

in a division algebra to which the non-commutative left- and right-hand coefficients a_s, b_s and the constant term α belong.

Starting with a binary matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

Richardson defined a left-hand determinant $da - dbd^{-1}c$, and a right-hand determinant $ad - cd^{-1}bd$, in which the element d acts as a pivot that could not in general be cancelled. He extended the definition inductively to such a pair of determinants for a matrix $[a_{ij}]$ of order $n \times n$, by choosing as pivot a non-zero element a_{sn} from among those of the n-th column, in order to define determinants of order n in terms of those of order $n-1$. It led to a solution of linear equations, and to a surprising number of properties of ordinary determinants. By admitting terms such as $dbd^{-1}c$, where the factor d^{-1} could be removed, had the algebra been commutative, he extended the idea of rational integral expressions, and was enabled to retain the theorem that a determinant in general is irresoluble into factors which are polynomials in the elements.

Such treatment characterizes many of Richardson's early papers and the problems which he suggested in seminars to his research students. One interesting result is the extension of Fermat's theorem $x^N \equiv x$, mod N: it became

$$x^2(x^{N(N^2-1)} - 1) \equiv 0, \quad \text{mod } N,$$

when x was a real quaternion integer and N a prime positive integer [**13**].

In the paper [15] he laid the foundation of an invariant theory of forms in non-commutative algebra, wherein variables undergo linear substitution. Something akin to the classical theory of binary and higher forms was developed, involving two preliminary and novel features: first the effects of a change of basis from e_i to ϵ_i by linear equations $\epsilon_p = \Sigma\, \alpha_{pq} e_q$, and secondly the study of the case prior to binary forms—that of the homogeneous linear transformation in one variable only. In it was preserved that pleasant feature of the classical binary theory, the build up of the general linear transformation from a succession of simpler transformations, or conversely the analysis of the general group by means of its subgroups. Orthogonal and diagonal substitutions were studied; and invariants, particularly those of linear forms, were found. As in several other instances the paper on concomitants [15] was the fruit of a happy collaboration with his pupil D. E. Littlewood.

There followed at this stage of collaboration an investigation that led to the important paper [16] on group characters and algebra, which opened the door for the considerable advances accomplished in the last twenty years. The work centred on the symmetric group of order n, for which, as Frobenius had shown, there is a character table of integers set in p rows and columns, p being the number of partitions of n. It may be illustrated by the case when $n = 3$ and $p = 3$, for which the table may be written

1	3	2
1	1	1
2	0	−1
1	−1	1

where the numbers 1, 3, 2 above the columns denote the number of operations (6 in all) of the group, which belong to the separate classes of conjugate operations. Now associated with any 3×3 matrix $[a_{ij}]$ $(i, j = 1, 2, 3)$ is a *permanent* $\Sigma + a_{1i} a_{2j} a_{3k}$ and a *determinant* $\Sigma \pm a_{1i} a_{2j} a_{3k}$ where the coefficients are epitomized by the first and third rows respectively of the table and where the six operations of the group are illustrated by the six terms of either form. This suggested to Richardson an intermediate function.

$$2a_{11} a_{22} a_{33} - a_{12} a_{23} a_{31} - a_{13} a_{21} a_{32},$$

on the basis of the second row, and in general a set of p functions of the elements of an $n \times n$ matrix, starting with the permanent and ending with the determinant. At the suggestion of his friend Prof. A. R. Forsyth he called these functions *immanants*. With this idea in mind Richardson laboriously calculated immanants of various matrices, particularly those whose determinants expressed one symmetric function in terms of another type. All these calculations he handed over to D. E. Littlewood for perusal, and after further consideration they found that when all these expressions were put in terms of the x_i, of which they were symmetric functions, then the immanants were recognized to be the bialternants

$$|x_s^{\lambda_t+n-t}| \div |x_s^{n-t}|, \tag{1}$$

where $s,\ t = 1,\ 2,\ \ldots,\ n;\ \lambda_1+\lambda_2+\ldots = n,\ \lambda_1 \geqslant \lambda_2 \geqslant \ldots$. This brought into prominence again the partition $\{\lambda\} = \{\lambda_1 \lambda_2 \ldots\}$ of n, which had already occurred in the table of characters. At this stage Richardson read conscientiously through the literature in a search for every reference to symmetric functions: and he obtained relations such as the following:

$$3!\{3\} \quad = s_1^3+3s_1 s_2+2s_3 = 3!\{\bar{1}\bar{1}\bar{1}\}$$

$$3!\{21\} \quad = 2s_1^3 \qquad\quad -2s_3 = 3!\{\bar{2}\bar{1}\}$$

$$3!\{111\} = s_1^3-3s_1 s_2+2s_3 = 3!\{\bar{3}\}$$

which express the immanants (now denoted by $\{\lambda_1 \lambda_2 \ldots\}$ or by $\{\overline{\mu_1 \mu_2 \ldots}\}$ where $\{\lambda\}$ and $\{\bar{\mu}\}$ are conjugate partitions) in terms of the power sums $s_r = \Sigma x_i^r$. Here the first immanant $\{3\}$ is h_3 or equally well a_{111} (in Macmahon's notation): the last $\{111\}$ is h_{111} or equally well a_3, the general immanant $\{\lambda\}$ being equal to the determinant $|h_{\lambda_s-s+t}|$, which is another form of (1) above.

Richardson found that the foundations of the whole matter were laid years ago by Jacobi: he saw the relation of the immanants with the recent work of Aitken and of A. Young and above all with the classical dissertation of Schur on invariant matrices ["Ueber einer Klasse von Matrizen", Berlin (1901)].

There was at first perhaps a feeling of disappointment for the collaborators that Schur had anticipated them to some extent, but the new insight, that these immanants and S-functions (as they came to be called in honour of Schur, their originator) were related to invariant matrices was indeed worth the loss of precedence: and, as Littlewood has said, " It was quite marvellous to see the intricate theory taking shape under our hands ".

Richardson and Littlewood continued their joint work for a few years on the problems which arose from these functions. Applications developed, which have led to a vigorous school of algebraists working out the implications of group characters for invariant theory, who have gone a long way towards fully answering the question regarding symbolic and ordinary methods, which had originated the investigation. Richardson was a pioneer rather than a consolidator; and now that the scheme was fully launched and in very competent hands, his interests turned towards more abstract theory. Despite his illness he was constantly looking forward to the problems that awaited solution and to further generalizations.

" The next century " he wrote, " must deal with the most general kinds of algebras in which the connecting relations are less restricted: for example, non-associative algebras and the theory of partially ordered algebras."

During his later years at Swansea he was at work on a book, the contents of which may be gathered from a short note that appeared in the *Mathematical Gazette* [**22**]. He was coming more and more to the conclusion that many of our ordinary theorems are really of much wider extent. For example, the group characters are not really intrinsic to groups—they are invariants associated with equivalence relations when these combine under a law of multiplication suitably postulated. Many different special branches of mathematics, such as theories of congruence, measure, of classes, ideals and rings, groups and their characters, of valuations and modular functions in general algebra, have some or all of the following ideas in common:

(a) A formal mathematical system A is defined; *e.g.* a group.

(b) General relations determine subsystems of A; *e.g.* equivalence relations.

(c) These subsystems of A are regarded as instances of a formal system B; *e.g.* the class-ring of which the elements are the classes of conjugate elements of a group.

(d) Correspondences are established between B and other systems such as the integers, algebraic numbers, A itself.

(e) In many instances such correspondences are homomorphic.

Richardson aimed at discovering an abstract theory governing these considerations [**20**]. As, however, little was known about general systems he had to discover the necessary technique and construct examples to serve as pointers to his goal. By suitable modifications of the binary nature of the relations between elements and of the laws of association, distribution and cancellation, he succeeded in establishing homographic correspondences between systems (mostly non-associative) and certain subsets of the set of all equivalence relations over the systems, while at the same time retaining many of the more important theorems of the special branches ([**21**], [**24**], [**25**]). In his abstract work he never lost touch with reality: in a characteristic paper [**20**] he even enlisted the theory of probability. " Have you seen anywhere ", he wrote in 1938, " an estimate of the *strength* of a postulate or the *extent* of a theorem ? I use the theory of probability to define these. One interesting result is that the postulate that a finite number system contains an idempotent element is about e^{-1}. For number systems containing only 4 elements I have calculated the extent of the theorems of Lagrange and Sylow." The paper was a preliminary survey, as he put it, of the extent to which certain theorems are likely to be true in an arbitrary multiplicative system.

In the course of his investigations a number of special results came to light. For instance he showed how to calculate by purely rational methods p commutative rational matrices (each linear in the variables and of order equal to that of the Galois group) which are factors of the binary p-ic [**19**], and in [**25**] he extended his methods to the general m-ary p-ic. In 1937 Temple had interpreted the Clebsch Aronhold symbols as those of a nil-potent algebra, a step which led to a renewed interest in them. Richardson appears to have made an independent discovery; for, commenting on his own factorization, he remarked (1942), " It follows then that the symbolic methods are really calculations with suitable matrices. I have suspected this for a long while and especially its connexion with tensor algebra. . . . My new method is direct—I actually factorize the form. Incidentally this ought to open up a new idea in the theory of algebraic functions. It also enlarges the notion of group characters and pseudo-representations ". By such a representation he meant a double set of commuting matrices C_i and C_i' satisfying the relation

$$(C_i C_j - \Sigma c_{ijk} C_k) \ (C_i' C_j' - \Sigma c_{ijk} C_k') = 0$$

summed for the self-conjugate classes of the groups [**25**].

By expressing the conditions that the matrix factors of a form be singular he showed that different factorizations over a domain of integrity correspond to different classes and forms of that domain. This led him to new theorems relating to the composition of forms, and to the solutions of the corresponding Diophantine equations; and it gave new proofs of some of L. E. Dickson's theorems on ternary forms, and related them to generalized quaternions and double theta functions.

He now turned his attention to the arithmetic of forms, and particularly to the composition of quadratic and of cubic forms. In his paper [**26**] on quadratics he attained his end by an ingenious use of matrices, starting with the unexpected addition of a skew symmetric matrix to that of a quadratic form $x'Ax$, then proceeding with commuting and scalar matrices, and ending with a battery of skew determinants and of Pfaffians, skilfully aimed at reducing to quadratics any factors of higher degree in the sets of parameters which they involved. Thus he factorized $f(x)$, a quadratic form in n indeterminates over a commutative and associative ring $\mathscr{R}$, obtaining a resolution

$$f(x) = u(\alpha') \, . \, v(\alpha),$$

where each of u and v is a quadratic of order 2^{n-1} in its parameters α_i' or α_i, and where x is bilinear in α and α' over $\mathscr{R}$. Such a resolution was not identically true in α and α'; yet it was true for an infinite module of elements in $\mathscr{R}$ which satisfy certain quadratic and bilinear relations depending on n, but independent of f. As a sequel he wrote paper [**27**] upon the composition

$$f(x) = f(\alpha') \, . \, f(\alpha),$$

which was known to be an identity when x was a suitable bilinear combination of α' and α, over $\mathscr{R}$, if and only if $n = 1, 2, 4$ or 8. He worked out the case $n = 16$ and indicated an extension to the case $n = 32$, replacing the identity by the same provisions as in his paper [**26**], with various interesting corollaries.

He also wrote two short papers on cubic forms, [**28**] in 1947 and [**29**] in 1951. " What does astonish me " he remarked in a letter " is the vast gulf separating the quadratic case from the cubic ": for there was no starting point comparable to the convenient matrix product $x'Ax$ of the quadratic. None the less he made a surprisingly effective advance into a region which is largely unexplored. With four ternary row-vectors x, α, β, γ, where, for example, $x = [x_1, x_2, x_3]$, he resolved a general ternary cubic $f(x)$, obtaining

$$f(x) = a(\alpha) \, . \, \phi(\beta, \alpha),$$

where $a(\alpha)$ is a ternary cubic in the α_i, and $\phi(\beta, \alpha)$ a bi-cubic in the β_i and α_i, over $\mathscr{R}$. It was done by constructing nine linear forms in the x_i, placing them in a 3×3 matrix $F(x)$, such that its determinant was $f(x)$: he then defined another such matrix and determinant, $A(\alpha)$ and $a(\alpha)$, by means of the condition $x \, . \, A(\alpha) = \alpha \, . \, F(x)$ which had to be true identically for all α_i and x_i. Next, having formed the adjoint matrix $A^*(\alpha)$, he proved the lemma:

If $x = \beta \, . \, A^*(\alpha)$, *then* $f(x) = a(\alpha) \, . \, \phi(\beta, \alpha)$.

It led to some elegant relations of reciprocity among the x, α, β and further parameters, and among the factors of composition. For certain special cases of the cubic $f(x)$ the bicubic $\phi(\beta, \alpha)$ factorizes into $a(\alpha)$ and a cubic in β; but in general it does not. He conjectured a similar lemma

$$f(x) = \big(a(\alpha)\big)^s \, . \, \phi(\beta, \alpha)$$

for an n-ary, n-ic $f(x)$ where s is a positive integer less than n. The second of these papers on cubics had been written early in 1951 in an interval of renewed strength after a three-year gap of increasing physical weakness. In a letter written later he remarked: " my own interests lie rather in a set of interchangeable relations between the various cubics, with a view to extending the theory of composition to cubic forms. However, it is certain that I shall never be able to do this. I have had the above results for some years and my letter to you arose from a brief spell when I was able to work to write up a short account ". He continued to ponder over and to note down his mathematical thoughts until the end.

Richardson died on 4 November, 1954: a leader by inspiration and action, who gave of his best at the call of his country and thereafter in his home and college life. He has an honoured place among algebraists in the succession of Macmahon, Young and Wedderburn: a great man has passed on.

I am very grateful for the help that has been readily offered by his wife, his colleagues, Prof. R. Wilson and Prof. D. E. Littlewood, and others who have provided material for this notice.

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