ABRAHAM ROBINSON 1918–1974

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A. D. YOUNG, S. KOCHEN, S. KÖRNER, AND P. ROQUETTE

Prefatory Remarks

Abraham Robinson was more than just a gifted and versatile mathematician, he was a cultivated human being and all branches of scholarship and human endeavour excited his interest and challenged his passion for revealing common structural relationships between them.

He first achieved a reputation as an outstandingly brilliant applied mathematician, particularly in the fields of aerodynamics and structures and then he established himself as one of the world's great pure mathematicians and a leader in the fields of mathematical logic and philosophy. Such range and versatility are exceptionally rare these days, one would have to look to the mathematical giants of previous centuries to find comparable versatility.

The very breadth of his interests and achievements have made the task of writing this Notice particularly difficult, another Robinson is really needed to do him justice. It was accordingly decided that one of us would concentrate on his work and achievements as an applied mathematician and the others would discuss his achievements in pure mathematics with particular reference to mathematical logic, arithmetic and philosophy. This corresponds roughly to the development of his career, and we have therefore kept as far as possible to a chronological arrangement of events. This was not entirely possible, however, as there were periods when he was active both as a pure and applied mathematician.

To set the scene therefore we begin with a brief synopsis of his career.

Synopsis of Abraham Robinson's career

1918	Born at Waldenburg, Germany.
1939	Graduated from Hebrew University, Jerusalem.
1939–1940	Sorbonne.
1940	Escaped to England, member of Free French Force.
1941	Joined R.A.E., Farnborough, as Scientific Officer.
1946	M.Sc., Hebrew University.
1946	Senior Lecturer in Mathematics, College of Aeronautics, Cranfield.
1949	Ph.D., University of London.
1950	Deputy Head of Department of Aerodynamics, College of Aeronautics, Cranfield.
1951–1957	Associate Professor and later Professor of Applied Mathematics, University of Toronto.
1957	D.Sc., University of London.
1957–1962	Professor of Mathematics, Hebrew University, Jerusalem.
1962–1967	Professor of Mathematics and Philosophy at the University of California, Los Angeles.
1967–1974	Professor of Mathematics at Yale University, later in 1971 Sterling Professor of Mathematics at Yale University.

[BULL. LONDON MATH. SOC., 8 (1976), 307–323]

Honours and Important Positions

1968–1970	President of the Association for Symbolic Logic.
1972	Elected Fellow of the American Academy of Arts & Sciences.
1973	Brouwer Medal awarded by the Dutch Mathematical Society.
April 1974	Elected posthumously member of the National Academy of Sciences.

Early Years

Abraham Robinson, Abby to his friends, was born in 1918 in Waldenburg, Germany (later Walbrzych, Poland). His father who died just before he was born was also named Abraham Robinson and was a writer and philosopher of some distinction.

When Abby was fourteen, the family of mother and two sons migrated to Palestine; his father had been a keen Zionist and presumably the increasing menace of Hitlerism hastened the move. In due course Abby became a student of mathematics at the Hebrew University in Jerusalem, his teacher was Professor A. Fraenkel who first introduced him to symbolic logic and of whom he spoke warmly. It is interesting to note that Abby's first publication [**1**] was in 1939, when he was only 21, and it was in the field of symbolic logic.

He was clearly an outstanding student, for in 1939 he won a scholarship to the Sorbonne where he continued his studies. The War, however, broke out soon after his arrival in France and with the invasion of France by the Nazis in 1940 Abby had to escape and managed to get to England by the last boat to leave Bordeaux. In England he volunteered for the Free French Air Force but in company with some other scientists from France he was soon released on being sent to the Royal Aircraft Establishment, Farnborough, to make himself useful in the war effort.

The Applied Mathematician

by Alec Young

At Farnborough he had to expand his interests into the realms of applied, as distinct from pure, mathematics and in particular he had to absorb the sciences and techniques of structural mechanics and aerodynamics. He did so brilliantly and with remarkable speed.

He was initially assigned to the Structures and Mechanical Engineering Department, the head of which was Dr. Pugsley (later Professor Sir Alfred Pugsley, F.R.S.). Abby quickly established himself as an outstanding mathematician but he did not feel satisfied that his background was adequate for the work he was called upon to do. A colleague, Dr. (later Professor) Jahn, with whom he shared lodgings, recalled how Abby used to work late at night by torchlight (as the landlord turned the lights off at 10 p.m.), in order to take and pass an examination in Aeronautical Engineering.

During the war years much of the work done at the R.A.E. was of a " trouble-shooting " nature with a high security classification and only a small proportion of it was eventually published later. Of Abby's work the first publication [**4**] was in collaboration with two other authors and it is impossible at this stage to determine his particular contribution to it. The paper shows how, by the use of probability concepts, the maxima readings that were being obtained from the simple recorders of velocity and acceleration then in use on aircraft could be interpreted so as to yield a complete picture of the load frequencies in flight. An interesting feature of this paper is that it reveals at an early stage the approach to airworthiness problems based on the use of probability theory and statistics that has characterized British work in this field for many years and which has won international approval and acceptance.

His next paper [5] revealed that he had already mastered the aerodynamic theory of wings, as it was concerned with the loading on wings with endplates. Until then the problem had been solved by means of the assumption of constant induced downwash in any cross-section of the wing wake so that the wing and endplates were taken to shed a vortex sheet of constant cross-sectional shape. The problem then reduced to a simple two dimensional potential flow problem about the trace of the vortex sheet wake in a plane normal to the flow far downstream. Abby recognized that the assumption of constant cross-sectional shape of the vortex sheet was a considerable over-simplification and he was successful in producing a theory that did not use it and could more accurately predict the effects on the loading of geometrical and aerodynamic variations along the wing span and along the endplates.

The next publication on shock transmission in beams [6] was undoubtedly a major piece of work and revealed Abby as the powerful, original and mature scientist that he had rapidly become. Professor Sir Alfred Pugsley recalls that a large flying boat had suffered a main spar failure during a heavy alighting. He asked Abby to examine the failure to see if he could produce an analysis of it. Abby chose to examine the problem not by the conventional method of treating the structure as a dynamic system but instead by studying in detail how stress waves were propagated, reflected and dispersed in the structure in response to imposed impulses or time-dependent loads. His analysis was confined to beams but was otherwise very general and comprehensive, including structural discontinuities. It revealed the basic importance of coupling between longitudinal and shear waves and it clarified the physical factors that could lead to high local stresses and hence to structural failure. Something of Abby's characteristic and unifying approach to mathematics is presaged in a brief appendix in which he showed that the propagation process could be treated in terms of the theory of linear operators, with all the states of a beam constituting an abstract Hilbert space.

In the later years of the war the advent of the jet engine had resulted in a rapid increase of aircraft speeds and the possibilities and problems of supersonic flight became of increasing practical interest. The aerodynamic theory of supersonic flow, when linearized as a problem in small perturbations from a uniform flow, involved a hyperbolic differential equation, the counterpart of the familiar elliptic Laplace equation of hydrodynamics. This offered a happy hunting ground for mathematicians and Abby was transferred to the Aerodynamics Department to work on such problems under the general guidance of H. B. Squire (later Prof. H. B. Squire, F.R.S.). Abby rapidly mastered the existing developments based mainly on the superposition of basic solutions in the form of sources and doublets [9] and he contributed elegantly and with insight to the analytic problems involved. He then directed his attention to a hitherto unsolved problem, the pressure distribution on a flat delta shaped wing for which the leading edge was swept at an angle greater than the Mach angle, so that the leading edge behaved in some respects as if in subsonic flow [10]. To solve this problem Abby devised pseudo-orthogonal co-ordinates, the counterpart of orthogonal co-ordinates for solutions of Laplace's equation, and in particular used hyperboloido-conal co-ordinates which were suited to problems of flow about wings of delta plan form and in terms of which analytic solutions were obtainable in the form of series involving Lamé functions. Not only did this approach enable Abby successfully to deal with the problem he set out to solve but he noted that it opened the way to obtaining analytic solutions for a large and important family of wing shapes of delta plan form, including wings with rounded leading edges. Abby did not pursue this possibility, but later Squire as well as Roper did when the practical importance of such shapes became well recognized.

In 1946 the College of Aeronautics was founded at Cranfield and Professor W. J. Duncan, who had been appointed Head of the Department of Aerodynamics, persuaded Abby to join the staff as Senior Lecturer in charge of the teaching of Mathematics. At Cranfield his interests in the aerodynamic theory of wings, both in subsonic and supersonic flow, broadened and became increasingly comprehensive. A paper **[12]** written in collaboration with me showed that although at transonic speeds linearized theory could not be regarded as strictly valid, nevertheless over that speed range it had a welcome degree of intrinsic consistency and did not in general lead to obviously unacceptable results; for slender wings and bodies this was a result of some practical importance. In a later paper Abby with one of his students **[13]** made a study of the effects of sweepback and maximum thickness position on the cruise performance of an aircraft with delta wings at supersonic speeds. With another colleague (Dr. S. Kirkby) Abby examined the interference on the aerodynamic characteristics of an unswept wing due to a conical shaped body at supersonic speeds **[14]**.

Meanwhile, Abby had given considerable thought to the nature of the singularities that introduced special difficulties in the use of source and doublet (or vortex) distributions in solving problems of linearized supersonic flow. In a fundamental and characteristically brilliant paper **[17]** he adapted Hadamard's concept of the finite part of an infinite integral and showed that a logical structure free of these difficulties could be built up which was analogous to that of incompressible flow and which preserved the basic physical relations expressed by the divergence theorem and Stokes theorem. The analysis was subsequently applied in another paper **[15]** written in collaboration with a student (later Air Marshal Sir John Hunter-Tod) to determine the bound and trailing vortex system of a wing in supersonic flow and in particular the downwash distribution in the wake behind a delta wing. A subsequent paper with the same collaborator **[19]** presented the results of calculations of the aerodynamic derivatives with respect to sideslip for a delta wing at supersonic speeds, for which the method of conical flows due to Stewart was used but its equivalence to Abby's method using hyperboloido-conal co-ordinates was also demonstrated.

At that time Abby also turned his attention to the problems of unsteady supersonic theory. In a paper that has yet to receive the attention that it deserves **[16]** he first demonstrated the analogy between three dimensional steady flow and two dimensional unsteady flow and used the work that had already been done on the former problem to solve the latter problem, and he was able to demonstrate the order of approximation involved in using the assumption of quasi-steady flow. In the same paper he then went on to solve the problem of the oscillating delta wing with " subsonic " leading edges in supersonic flow and for this he used his method of pseudo-orthogonal co-ordinates.

Incredible as it may seem, during these immediate post-war years when he was establishing himself as an applied mathematician of international stature and an acknowledged expert on wing theory, he was simultaneously working on a thesis on " The Metamathematics of Algebraic Systems " under the supervision of Professor

P. Dienes of Birkbeck College, London University, for which he was awarded the Ph.D. degree in 1949. This work was subsequently published in 1957 by the North Holland Publishing Company [I], and immediately established Abby as a leading exponent in the field of mathematical logic. A further discussion of this work and its implications follow in the next section of this Notice but it was felt that a reference to it here is necessary to illustrate the extraordinary level of scientific creativity that Abby had reached by the time he was thirty.

A cursory study of Abby's publications in the years 1949–1951 might suggest that his growing interest in symbolic logic was at that stage causing his interest in aerodynamics to wane, but this was not in fact the case. It was indeed at that stage that he undertook the major task of writing a book on wing theory in collaboration with J. A. Laurmann who had then just graduated from the College of Aeronautics [III]. The book took some three years to prepare and it developed into an impressive work of comprehensive scholarship and authority. It covered subsonic and supersonic wings in both steady and unsteady flow, and the subject was presented with that heightened sense of structure and unity that was a characteristic of Abby's work. The work of preparation was effectively finished by about 1953 and the book was published in 1956.

In 1950 Abby developed an interest in the problem of the interaction of boundary layers and shock waves that was then beginning to exercise aerodynamicists. Early attempts by other workers at dealing with this had involved very simple models, e.g. small disturbances in a uniform supersonic stream bounded by a parallel subsonic stream with or without a wall and without any attempt to represent the boundary layer velocity profile realistically. Abby therefore tackled and successfully solved the problem in which the velocity profile has the characteristics of a boundary layer **[26]**. With typical honesty Abby noted that his results did not agree with available experimental results for shock and boundary layer interactions and pointed out that like other workers at that time he had not allowed for the effects of viscosity and had treated the perturbation as small. We now know that these factors, particularly the latter, are important if the interaction between a shock and boundary layer is to be fully modelled; but for very weak disturbance waves traversing a defined shear layer Abby's analysis was in essentials correct and it revealed in an illuminating way the process of reflection and refraction that the disturbance undergoes.

In the same year Abby returned to wing theory to determine the aerodynamic characteristics of swallow-tail wings of small aspect ratio **[28]**. These shapes were of some practical interest at the time as possible shapes for future tailless designs. Abby used the slender wing theory of R. T. Jones, and produced an exact solution within the limits of that theory.

In 1951, Abby, who by then was Deputy Head of the Department of Aerodynamics at the College of Aeronautics, left to join the Department of Applied Mathematics in Toronto University as Associate Professor and later Professor. The Department in Toronto had been sadly depleted by the departures of Professor Stevenson and Professor Infeld the year before and to Abby fell the major part of the task of rebuilding the strength of the Department. His teaching was largely in traditional subjects of courses in Applied Mathematics (Differential Equations, Fluid Mechanics, Aerodynamics) but he was able to devote increasing time to his interests in logic and mathematical foundations, and he was able to attract a few students to these latter fields. Over the next few years the incidence of his papers in these fields increased, whilst that of papers in applied mathematics decreased. In 1953 a paper of his on non-uniform supersonic flow which was finished shortly after he joined Toronto University was published [30]. In this paper he considered the problem of the characteristics of an aerofoil in a flow which is non-uniform but has a plane of symmetry in which the aerofoil is placed. His interest in this problem was stimulated by earlier papers by other workers concerned with aerofoils in non-uniform incompressible flow.

He also presented a paper at the Symposium on High Speed Aerodynamics, held at the National Aeronautical Establishment, Ottawa, 1953, and another at the Second Canadian Symposium on Aerodynamics, held in Toronto, 1954. The first paper [31] was concerned with the flow around compound lifting units and the second [33] on some problems of unsteady wing theory. In 1956 he published a paper on the motion of small particles in a potential field of flow [45], of relevance to the problem of aircraft icing as well as to silting up processes. Later in 1956 we find Abby showing his continued interest in wave propagation in a paper in which he considered a wave possessing a velocity discontinuity and its propagation in an elastic medium with variable properties [47]. He demonstrated that longitudinal waves do not transform into transverse waves and vice versa. In a subsequent paper in 1957 he went on to examine the transient stresses in a beam of variable characteristics subjected to an impulsive or concentrated load and investigated the associated variation in the discontinuity in bending moment that can occur across the front of a shear wave [48].

In 1957 Abby returned to his Alma Mater, the Hebrew University of Jerusalem, taking up the Chair of Mathematics that his old teacher Professor Fraenkel had held. From then on he devoted himself almost entirely to pure mathematics. We find in his list of subsequent publications only two papers which are clearly of an applied nature. He was invited by the Editor, Professor Flugge, to write the chapter on Aerofoil Theory for the McGraw–Hill Handbook of Engineering Mechanics [69] which appeared in 1962. His chapter is a remarkable demonstration of concentration without loss of clarity; in the 24 pages allotted to him the essentials are all to be found. His last publication in applied mathematics appeared in 1968 and was on flexural wave propagation in non-homogeneous elastic plates, an evident sequel to his earlier work on beams [94].

The fact that his creative energies in the last seventeen years of his life were almost entirely devoted to his pure mathematical interests need not be taken as evidence that he had by then completely lost interest in the applied mathematical fields in which he had earlier worked with such distinction. Those of us who had worked with him in those fields know how closely he continued to follow developments and read all important publications and he took a keen interest in the careers and work of his former students and colleagues. One felt that the applied mathematician in Abby was never far away, and that some exciting development offering the challenge of a structural unity to be uncovered could yet again bring the applied mathematician into action.

The Pure Mathematician
On Abraham Robinson's Work in Mathematical Logic

by Simon Kochen

Abraham Robinson was one of the dominant figures in the world of mathematical logic, and pre-eminent in his own specialty of model theory. However, his total output of over 100 research papers and nine books covered an extraordinarily broad range of topics. Aside from the many papers in aeronautics, already referred to, and in mathematical logic itself, he wrote articles in algebra, analysis, and number theory, in mathematical economics, and quantum mechanics.

How is one to give a detailed assessment of such a wealth of ideas? Certainly, I cannot attempt it here. But I believe that within this diversity, Abraham Robinson exhibited a viewpoint of mathematics which remained remarkably consistent throughout his life. It was a viewpoint which was capable of organic growth, so that, at the time of his death, he was at the height of his mathematical powers and full of new and provocative ideas. I personally felt this most keenly when I began working with Robinson during his visit to the Institute for Advanced Study in 1973. Sadly, this collaboration was cut short by his untimely death. An idea of the new directions of his thoughts may be gleaned from the list of the dozen or so problems that formed his retiring presidential address to the Association for Symbolic Logic. These problems will be a touchstone for much of the future progress in logic.

I would like here to suggest the possible key to understanding Robinson's attitude and insights to his subject. Almost one-half of his papers and one book are devoted to aerodynamics and structures and his last paper on the latter subject was written as recently as 1968 [**94**]. I think, in assessing Robinson's mathematical outlook, one would ignore such a large and integral part of his work at one's peril.

I believe that the thread that runs through all his work lies precisely, in fact, in this aspect: that also as a mathematical logician, his viewpoint was that of an applied mathematician in the original and best sense of that phrase; that is, in the sense of the 18th and 19th century mathematicians, who used the problems and insights of the real world (that is, physics) to develop mathematical ideas. To logicians, it is the world of mathematics which is the real world. Robinson was the first logician to have systematically taken concepts from mathematics, principally algebra, and used them as the basis of a general theory of models. Before Robinson, logic was mainly devoted to studying the foundations of mathematics. Robinson, via model theory, wedded logic to the mainstreams of mathematics. When he entered the field, there existed some scattered results stretching back to the original theorem of Loewenheim and Skolem 50 years ago, the completeness theorem of Gödel, the decidability of the elementary theory of real numbers of Tarski. At present, principally because of the work of Abraham Robinson, model theory is just that : a full-fledged theory with manifold interrelations with the rest of mathematics. Model theory has come of age so that a harvest of applications back into algebra and analysis has been reaped from the seeds sown by Robinson.

This program, of which I have been speaking, is already quite explicitly outlined in Robinson's doctoral thesis. It found its best expression in his early books, " On the metamathematics of algebra " [I] and " Théorie Metamathématique des ideaux " [II]. Algebraic concepts such as algebraic variety and ideal are put into a natural model-theoretic setting. I would like briefly to outline the evolution of another idea. Robinson starts with an algebraic notion of closure. For fields, this is the concept of algebraic closure; for ordered fields it is the concept of real closure. Robinson now defines a general model theoretic notion of model companion which subsumes the original algebraic concepts.

It is important that this concept refers not to individual structures, but to theories —and it is here that logic enters, since the theories consist of statements lying within the Predicate calculus. A theory L is a model companion of a theory K if every model of K is imbeddable in a model of L, and every model of L which is an extension of a model of L is an elementary extension.

Robinson proved many interesting results about model companions. In particular, he obtained a simple but effective test for the existence of model companions. The fruitfulness of this concept outside of model theory appeared when Robinson proved by this test that the theory of differential fields has a model companion. This led him to the notion of a differentially closed field, which has proved to be the correct notion of closure within that domain. Nevertheless, many theories do not have model companions. In the late 1960's, he adapted Paul Cohen's notion of forcing in set theory to general model theory. The result was the concept of a forcing companion which exists for all inductive theories and coincides with the model companion when that exists. This innovative idea has led to many new lines of research. This example illustrates how Robinson started with an algebraic concept, that of algebraic closure, and by wedding it to the logical concepts of elementary extension and forcing was able to deepen and transform it into a model-theoretic instrument.

In 1955 there appeared a paper by Robinson which marks a watershed in the development of model theory. This paper [**38**] did not introduce any new concepts or even prove any new theorems on the subject. Nevertheless, the results forced one to look at the nascent model theory with a new seriousness. This paper gave a model theoretic proof of Hilbert's 17th problem: that a positive definite real rational function is a sum of squares of rational functions. This had been proved by Artin in 1927, but the new proof had the advantage of giving uniform effective bounds on the number of squares and their degrees (thus answering a question which Artin had raised). However, the main interest of the model theoretic proof lay in its extreme elegance and simplicity.

In a second paper of that year [**37**], Robinson (with Gilmore) gave a similarly elegant model theoretic proof of Hilbert's irreducibility theorem. Here for the first time in these two papers, quite non-trivial algebraic results had been obtained by these new methods. The possibility now seriously arose that model theory could be a new method of attack on purely algebraic questions. This hope has been amply borne out in a number of cases where the only proof of algebraic results to date is the model-theoretic one.

I turn now to the subject for which Robinson is certainly most famous, at least to the general mathematical community—non-standard analysis. I want to emphasize that non-standard analysis was not a sudden tangential direction in which Robinson moved, quite separate from his earlier work. Rather, it was the systematic application of the same viewpoint which he earlier applied to algebra to the study of analysis. Algebra and analysis were not the only fields of application of non-standard methods; Robinson became increasingly interested in the use of such methods in the theory of numbers and his contribution to this is described in the following note by Professor Roquette. Non-standard analysis is nothing but the detailed study of the non-standard models of R and their application to specific problems in analysis via the transfer principle. This is precisely the idea in Robinson's proof of Hilbert's 17th problem, and in fact, that proof can be easily translated into the language of non-standard analysis.

One thing that immediately emerged is that non-standard analysis is not simply a consistent way of adding infinitesimals to R in order to make non-rigorous engineering proofs work. Historically, many such attempts were made. For instance in the

1890's, Levi-Cività considered a field of formal power series as a model. The new element of Robinson's approach lies in the intimate relationship between R and his non-standard model $^*R$ of R which is expressed by the logical transfer principle that $^*R$ is an elementary extension of R. This, as Robinson himself emphasized, may be considered a formal interpretation of Leibniz's belief that infinitesimals behave like ordinary real numbers. Such a model was simply impossible to construct before recent times. Even in reproving old theorems of analysis such as the Riemann mapping theorem, the new proofs and insights are strikingly different from the old ones. This new insight was beautifully illustrated in the proof by Robinson and Bernstein of the conjecture by Halmos and Smith that polynomically compact operators have an invariant subspace [**83**]. Oversimplifying, we may say that the main point was that the theorem is easy for finite dimensional Hilbert space. It is carried over by the transfer principle to a non-standard Hilbert space of non-standard integer dimension. (Note that this integer is really infinite in the real world.) This space has a standard infinite-dimensional Hilbert space sitting inside it. It now needs some analysis to show that by restriction we obtain an invariant subspace of the operator acting on this space.

Whatever the achievements of non-standard analysis, and I believe they have been considerable for the short period of time it has existed, I think that the potential of the subject as an alternative approach to analysis is far greater. In 1973, when he was visiting the Institute for Advanced Study, Abby Robinson gave a lecture on non-standard analysis. At the end of the lecture, Kurt Gödel gave his view that non-standard analysis will in the future be not simply an alternative but *the* correct way of viewing problems of analysis.

More recently Gödel told me that he felt that Robinson was the logician who did most to follow the original ideal of mathematical logic, as stated by Leibniz and Peano, to find systematic solutions to mathematical problems.

Of very few scientists can it be said that they have done fundamental work in more than one subject. Abraham Robinson was one of this select company. In particular, he will always be known as the father of the twin subjects of model theory and non-standard analysis.

Many years ago at a conference in Cornell, I was sitting with Abby in the cafeteria, which had the insignia of various universities on the wall. He had just enlightened me on a mathematical point. I asked him the meaning of the Hebrew words אורים ותומים on the Yale emblem. He interpreted the words as "light and truth". For me, these words define the man. Abraham Robinson dedicated himself to revealing and interpreting light and truth in the realm of mathematics.

Abraham Robinson's contribution to Arithmetic†

by Peter Roquette

As has already been remarked, Abby became increasingly interested in arithmetic as a field for application of non-standard methods. He devoted a considerable amount of study to arithmetical questions, and his contributions to class field theory [**88**], [**95**] and to diophantine geometry [**11**], [**121**] are to be regarded as fundamental.

† This note was kindly contributed by Professor Roquette as a postscript to the rest of the Notice.

The following remarks are not intended to be an exhaustive survey of his work, but rather to serve to illustrate his viewpoint and, at the same time, to explain the importance of his ideas for number theory.

Robinson's main idea was the introduction of the structure of " enlargement " (of an algebraic number field) as an object of study. This concept, namely enlarging the field of reference in order to discuss arithmetical problems, is not unknown to number theory. In fact, since Hensel introduced the p-adic fields, these have become standard tools in the hand of number theorists; the so-called p-adic method consists of discussing arithmetical problems via the structure of the p-adic completions. It has turned out to be necessary to consider all p-adic completions at once, simultaneously for all primes p. This leads to the introduction of adèles and idèles in the sense of Chevalley, which play a fundamental role, for example, in class field theory. Now, Robinson has pointed out that this notion of enlargement can be regarded in a sense as most universal " completion ", in as much as *every* concurrent relation has a bound in the enlargement—not only those which are connected with p-adic or idèlic convergence. The p-adic completions and also the idèles are naturally contained in the structure of enlargement, and their relevant properties can be obtained via the general transfer principle. Thus we see that Robinson's idea of enlargement can be regarded as a consequent continuation of the classical ideas of Kronecker, Hensel, Hasse, Chevalley, and others. Moreover, enlargements are not just some generalization of known concepts, but they seem to be most universal in the sense of mathematical logic. This explains their importance and usefulness in arithmetic, in accordance with the statement of Gödel.

On Abraham Robinson's Philosophy of Mathematics

by Stephan Körner

By the nature and scope of his mathematical inquiries, which ranged from aerodynamics to mathematical logic, Abraham Robinson soon became aware of the interaction of philosophical and mathematical ideas in his own thinking as well as in the thinking of earlier mathematicians who had made fundamental contributions to their subject. This awareness explains his abiding interest in the history of mathematics, in particular the history of the infinitesimal calculus. It also inspired his philosophy of mathematics, especially his discussions of the notion of infinite totalities and of the problem of their existence. His work in this field is best understood if one sees him as a successor of Leibniz and Hilbert. He found their general approach congenial, and he developed it further in the light of recent mathematical discoveries, including his own. A fundamental problem confronting these thinkers—and indeed most philosophers of mathematics—is the nature of the relation between mathematical theorizing, which assumes the existence of infinitely great totalities and (sometimes) of infinitely small quantities, and ordinary experience (including scientific observations and experiments) which has no acquaintance with either the infinitely great or the infinitely small.

Robinson's main position is succinctly summarized in a paper called " Formalism 64 " [**80**]. It is based on two main principles: (i) " Infinite totalities do not exist in any sense of the word " so that " any mention or purported mention of infinite totalities is, literally, *meaningless* ". Yet (ii) one should nevertheless act " *as if* infinite totalities really existed ". This position is restated in one of his last papers [**129**] where the assertion of meaninglessness is explained as the absence of " any detailed meaning, *i.e.* reference " of " mathematical theories which, allegedly, deal with infinite totalities "; and where the exhortation to continue the construction of such theories is further justified in the light of recent and prospective future developments.

Hilbert's famous attempt at implementing and justifying the formalist programme —the method of ideal adjunction, as it might be called—is a descendant of Kant's attempt at proving the logical compatibility of natural necessity with moral freedom. It consists in demarcating finitist mathematics whose concepts are abstracted from and applicable to experience; in adjoining infinitist mathematics which involves ideal notions or (as Hilbert sometimes put it) Kantian ideas, which are neither abstracted from nor applicable to experience; in formalizing the so combined partly finitist and partly infinitist mathematics; and in trying to prove the consistency of this formal system by means of finitist mathematics only. A consistency proof of the formal system containing the formal correlates of the finitist *and infinitist* concepts, propositions and operations would then be regarded as the justification of the infinitist mathematical theory. That, as was shown by Gödel, for the known infinitist mathematical theories such a proof cannot be given, has led many philosophers of mathematics to reject formalism. It led Robinson to search for a different, non-Hilbertian kind of formalism, some of whose roots are found in Leibniz's mathematics and philosophy.

The following remarks on them are based on *Non-Standard Analysis* [VII]. (Amsterdam 1966 and 1970), Chapter X, which is devoted to the history of the calculus.

Leibniz's attempt at implementing and justifying his formalist conception of infinite totalities and infinitesimals is an instance of his general theory of well founded fictions. His approach, which might be described as the restricted identification of discernibles, can be characterized as follows: Leibniz does not believe that there are truly infinite quantities, or truly infinitesimal ones, but that the corresponding notions are useful fictions of a general kind, exemplified by treating the earth as a point when compared to its distance from the fixed stars or by treating a ball which we handle as a point when compared with the radius of the earth. Just as any mention of these identifications can be translated into an idiom in which they are not mentioned, so the identifications of the infinitesimal calculus can be expressed in the " style of Archimedes " which does not refer to them. Yet as Berkeley and others showed, Leibniz's approach did lead to serious contradictions.

Quite apart from the significance of Robinson's non-standard analysis for pure and applied mathematics, its main philosophical importance lies in its clarification and justification of Leibniz's notion and use of infinitesimal quantities. Thus the dangerously vague Leibnizian identification of x with $x+dx$ is replaced by a well defined equivalence $x+dx \simeq x$ so that it is perfectly clear and precisely statable in which relations the equivalent terms are inter-substitutable and in which they are not. Again, the extent to which finite and non-finite quantities have the same properties, which is left unclear by Leibniz, is precisely expressed by the statement that the standard (Archimedean) model R of Analysis and its non-standard (non-Archimedean) model R^* satisfy the same set of sentences K of a language L—where L is the language of a type-theory of order ω and K the set of all sentences of L which hold in the field of real numbers. (For details see " The Metaphysics of the Calculus " in *Problems in the Philosophy of Mathematics* ed. Lakatos (Amsterdam 1967) **[90]** and *Non-Standard Analysis* [VII].)

The philosophical corollaries of non-standard analysis are only one example of Robinson's use of model theory in the light and in support of his own characteristic formalist conception of mathematics. Another important philosophical application of model theory is his construction of a modal logic which allows one to define the semantics of an infinite structure by means of a concept of potential truth for a set of finite structures. It leads in particular to a precise definition of arithmetical truth in terms of potential truth in initial segments of natural numbers. (See " Formalism 64 " **[80]** esp. Appendix.) His last philosophical paper, which is to be published in a volume devoted to Russell's philosophy, proposes a new solution of some of the philosophical problems connected with names and descriptions by using a generalized model-theoretical notion of satisfaction. (See " On Constrained Denotation ", to appear in a volume devoted to Russell's philosophy.)

It is characteristic of Robinson's philosophical method and personality that in his defence of formalism he never fails to give a fair hearing to its competitors or to emphasize that a conclusive justification of formalism would depend on the answers to some open questions. Two among them seemed to him particularly important and worthy of thorough inquiry. One is the problem of explaining more fully the applicability of infinitist mathematics, i.e. its yielding " results which can be used in material thought and for empirical purposes ". (See **[80]** p. 241 and **[130]** *passim.*) The other is the problem of precisely delimiting the " basic forms of thought—in logic, Arithmetic and perhaps Set Theory—which are prior to the arbitrary choice of mathematical axioms ". (See *e.g.* his comment on a paper by Mostowski in *Problems in the Philosophy of Mathematics* ed. Lakatos (Amsterdam, 1967) p. 104.) Abraham Robinson's premature death has deprived us of the light which his thought could have thrown on the problems of the applicability and of the common core of mathematics and on many other fundamental philosophical and mathematical issues.

The Man

Up to this point we have been mainly concerned with Abby the mathematician and scientist. To some extent this has revealed him as a person; his work and his personality were of a piece. He was not only a great mathematician, one of the greatest of this century, he was a great person in the sense that everyone who met him felt enriched by him. He made many friends all over the world as he was constantly in demand to give lectures and attend conferences, and all of those friends know that something they treasured has been lost to them with his untimely death. He had the humility and the kindness of the truly great, he was interested in people and found it easy to like them and he patronised no-one. He was deeply concerned with most forms of human culture and creativity, and on all he could converse with the fascinating combination of logic, insight and knowledge that characterized his mathematics.

Tributes have come from all over the world on the occasion of his death and in all the same words occur again and again—warm, gentle, kind, sociable, wonderful sense of humour, highly cultivated, strong sense of humanity.

As a summary of the man in all his aspects the following is taken from the tribute paid by one of us (Stephan Körner) at the Memorial Service for Abby that was held on 15th September 1974 at Yale University:—

" When one considers the wealth, profundity and diversity of his interests and the continuous interplay in his thinking of pure mathematics, applied mathematics, logic and philosophy one is constantly reminded of Leibniz to whom he felt a natural affinity and for whom he had the deepest admiration. The one Leibnizian idea in which he could see little merit was Leibniz's *principe de meilleur* according to which this world is the best of all possible worlds. I remember him asking me more than once in his gently ironic way whether I could make any sense of this principle. Today I should like to offer a partial answer: It cannot be a wholly bad world in which an Abraham Robinson could live and think; in which his wife and friends are able to cherish his memory; and in which his life's work will be remembered as long as logic, mathematics and philosophy matter to mankind ".

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