BERTRAND RUSSELL, as Mathematician

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Bertrand Russell is now and always will be chiefly remembered by mathematicians for his paradox concerning the class of all classes which are not members of themselves and for his creation of the theory of types. He will be remembered by historians of mathematics as co-author of *Principia Mathematica*, as a persuasive advocate for mathematical logic and logicism, and as a superb expositor and populariser of mathematical philosophy.

Russell had a brilliant, original and fertile mind. He devoted ten years (1900–1910) primarily to work in mathematics and its philosophy. So his actual and lasting contribution appears a poor return on capital invested; all the more so when we recall that Russell himself said ([**20**; p. 228]) of his work on the paradoxes and their solution " What made it more annoying was that the contradictions were trivial and that my time was spent in considering matters that seemed unworthy of serious attention ". Why was there such a gap between promise and achievement? One reason is accidental: in his discovery of logicism and his development of mathematical logic he had been anticipated by Frege. (And in many respects Frege's treatment is sharper and more lucid than Russell's). But the main reason lies, I believe, in the *framework* of his philosophy. Although his philosophical beliefs changed substantially through the years, this framework remained the same from 1898 to his death. And the effect of maintaining it was continually to divert his attention from problems of mathematics and mathematical philosophy to more general philosophical questions. So this framework will be discussed first, then his contributions to mathematical philosophy and finally his contributions to mathematics.

1. *Philosophical Framework*

In [**19**] he describes his brief period (1894–1898) as an idealist. During it he wrote " An essay on the foundations of geometry ", a study in the Kantian tradition in which it is shown that geometry is only " possible " in a space of constant curvature. In 1898 he (together with G. E. Moore) abandoned idealism, and became convinced that there is an objective real world of which we can have exact knowledge. In what follows I shall refer to this simply as " the world ". Initially his view was platonistic: the world contained abstract objects, such as numbers and points of space-time. But he moved, at first rapidly, and then with a gentler drift *towards* (not *to*) nominalism. The following tenets represent more or less constant features of his philosophy from 1900 to his death.

(1) There is a real world independent of human thought.

(2) By philosophical analysis we can, at least in principle, discover how the world may be described in terms of a certain minimum of irreducible constituents.

[BULL. LONDON MATH. SOC., 5 (1973), 342–348]

(3) The description will take the form of the assertion or the denial of atomic propositions. Each such proposition expresses that a particular primitive relation holds between certain particulars. Russell's views on the sort of things which will be counted as primitive relations or as particulars have varied greatly.

(4) By means of logical constructions we build up complex propositions and complex objects. This process covers not only the whole of logic, mathematics and physics, but also everyday objects.

(5) Russell's views on what should count as legitimate logical constructions also varied. But, except for the brief period during which he considered the " no-classes " theory (see below), the *minimum* apparatus considered was that provided by the ramified type theory of *Principia*.

An absolutely fundamental feature of this framework is that although the logical constructs need not themselves belong to the world,† they are built up from the constituents of the world and the atomic propositions which describe it. When Russell wrote " The fact that all Mathematics is Symbolic Logic is one of the greatest discoveries of our age " ([**3**; p. 5]), he had in mind, I believe, not only the fact that the abstract objects of mathematics could be reduced to logical constructions, but also that the reduction answered, almost trivially, the question: why does mathematics apply to the real world? Indeed, Russell writes that " The principles of mathematics " grew from a question in the philosophy of dynamics ([**3**; p. xvi]). This is a strength of Russell's logicism; but, as we shall see, it is this anchoring of mathematics to "the world" which helped to make his work so unproductive for mathematics.

2. *Philosophy of Mathematics*

(a) *The principle of abstraction*

Russell was proud of his discovery (also due to Frege) of, as we should now say, the notion of equivalence class. It is obviously an advance on the vague " *the* property which equivalent things have in common " (used, for example, by Peano). And unlike Cantor's method of postulation, the method does not call for new abstract objects other than classes. Russell worked it out in some detail in [**2**], and expounded its advantages in [**3**]. After the discovery of the contradiction, and the adoption of the theory of types it had to be reworked. This is done briefly in [**7**], and in great detail (for cardinal, ordinal, order-type and relation-number) in *Principia*—of which it forms indeed one of the main themes.

(b) *The contradictions and their solution*

Russell discovered his paradox in 1901 (" Never glad confident morning again " quoted Whitehead). From then until 1906 he searched for a solution; the miseries,

† In 1960 in conversation Russell recalled: " When I met Professor Gödel I discovered that he was a complete platonist. I asked him if he believed that the real world contained the operation of negation. He said ' yes ', and, do you know, he gave a very good account of himself ".

both intellectual and personal, of this period are movingly described in the *Autobiography* ([**20**; Chapter VI]). He published many discussions of the paradoxes and possible solutions; the most important are in [3] (especially Appendix B), [**4**], [**6**] and [7]. Of these, [**4**] is the most profound and the most interesting mathematically. (It is the only paper which he published in the *Proceedings* of this Society.) In it, for once, the requirements of the mathematical imagination take precedence over those of the philosophical framework. (For a quotation, see below.) He describes very clearly three solutions.

(A) The Zigzag Theory: (the name is misleading). The idea is that only straightforward, non-tricky propositional functions may determine classes. The difficulty is how to define " straightforward ". This was overcome by Quine in *New Foundations* ([**Q1**]). But as Russell foresaw the restrictions are not *intrinsically* plausible; they would look artificial to anyone ignorant of the paradoxes.

(B) The Theory of Limitation of Size: a propositional function shall only determine a class (i.e. a set in current terminology) if its extent is not too large. This is now the accepted solution. It was first suggested (but not published) by Cantor in [**C1**]. It is formalized in Zermelo-Fraenkel set-theory. Russell's objection to it has point. Namely, axioms are required to determine what is *not* too large. The theory is essentially an *open* one: from time to time new and more powerful axioms of infinity may be proposed and accepted. For this reason, and also because it introduces objects which are neither logical constructions nor represented (for non-platonists) in the world, Russell never accepted this theory.

(C) The No Classes Theory: this has little connection with Russell's later view that classes are logical fictions. It says, essentially, that propositional functions, and variables for them shall not appear as arguments, but only through their values. Russell soon abandoned it.

His final solution was the theory of types (first described in [7], elaborated in [**8**] and revised in [**14**]). Propositions and propositional functions (relations in intension) are to be classified through the expressions which describe them into a hierarchy of types so that the type of a function (or a proposition for (b)) is: (a) above the type of any of its arguments and (b) above all the types over which quantified variables in its expression range. The Axiom of Reducibility says, in effect, that every function has the same extension as one which can be expressed by using quantifiers whose ranges are not of higher type than its arguments. Condition (a) is satisfied by ordinary mathematical practice, where variables always have well-defined ranges. But Russell, because of his concern with the philosophic framework, never appeals to this fact. Condition (b) is only required in two extreme cases:

(1) Platonist: if the world or the logical constructions contain *absolute* semantic notions (such as " is true " or " defines "), then (b) is necessary to avoid the

semantic paradoxes. On this view (which seems to be that of the first edition of *Principia*) the axiom of reducibility is plausible. Russell later accepted the suggestion of Ramsey **[R1]**, where (b) is dropped, and semantic notions are not absolute, but are assigned various linguistic levels.

(2) Nominalist: if propositional functions arise *only* by the finitary logical constructions of quantification theory then (b) is needed to avoid genuinely *vicious* circularities. In this case the axiom of reducibility is totally implausible. This is the position taken in the second edition of *Principia*. It is not adequate for classical analysis. Indeed, in **[G1]**, which contains the most penetrating discussion of all these matters, Gödel casts doubt on its adequacy for number-theory†.

Finally, it should be remarked that if only finitely generated types are used, then the theory is inadequate for the theory of cardinals not less than $\aleph_\omega$. Russell, as far as I know, never publicly discussed this limitation.

(c) *Criticism*

In **[4]** Russell expresses a moderate platonist viewpoint:

> " When a new entity is introduced Dr. Hobson regards the entity as created by the activity of the mind, while I regard it as merely discerned; but this difference of interpretation can hardly affect the question whether the introduction of the entity is legitimate or not, which is the only question with which mathematics as opposed to philosophy is concerned."

But (because of his increasing concern with philosophical problems?) Russell gave up these admirable sentiments. In *Principia* the comment on the axiom of infinity is:

> It seems plain that there is nothing in logic to necessitate its truth or falsehood, and that it can only be legitimately believed or disbelieved on empirical grounds ([**8**; vol. II, p. 183]).

This view is reiterated (and applied also to the axiom of choice) in 1937 ([**16**; p. viii]). It is a consequence of his philosophical framework, and " empirical " must surely mean that abstract objects are not to be counted as existing in the world. If that world is finite, then mathematical objects, which are constructed from the constituents of the world, are also finite. But this is absurd. Suppose it were discovered that Eddington was right and that there are only $2.136.2^{256}$ fundamental particles in the universe. Would mathematicians immediately abandon classical number theory and analysis? Surely not; it would be the philosophical interpretation that might change. Indeed *Principia* itself exhibits the mathematician's indifference to empirical matters: by using typical ambiguity the authors ensure that their theorems do not depend, if the world *is* finite, on its exact size.

† J. Myhill (in an article to appear) has shown that with any reasonable definition of natural number in the ramified theory of types there will be instances of induction that cannot be proved. Gödel's suggestion is that this might be the case even for purely *arithmetic* propositions.

Russell's view is not only absurd, but it is also stultifying. This is because it denies the significance of the mathematical imagination and the value of mathematical experience. The mathematical ideas and results (not just the philosophical opinions) of, for example, Cantor, Hilbert, Brouwer, Zermelo, Skolem, Gödel, Tarski have totally transformed the philosophy of mathematics and our understanding of abstract and infinite objects. To be fair to Russell, he was not as dogmatic as the above quotation suggests. Also in [**16**] he writes:

> " It seems to me that these axioms either do or do not, have the characteristic of formal truth which characterises logic ([**16**; p. xii])."

He then suggests that the notion of " logical form " points to a solution. If this is true, then new logical notions will be required (e.g. infinitary constructions). But these (as also the types in the hierarchy) are abstract objects and, in general, infinite. Further investigations of them will lean heavily on *purely mathematical* results concerning infinite sets, or infinite constructions.

To sum up: Russell realized that for deciding the truth of certain crucial axioms the logic he had developed was inadequate. But he remained blind to the moral of this failure: an adequate logic must be based on *mathematical* imagination and knowledge.

3. *Mathematical Content of Russell's Work*

Almost all of Russell's mathematical work is represented in *Principia*. He felt that its mathematical content had been unduly ignored by other mathematicians (see Chapter 8 of [**19**]). It is not, of course, easy to use the book for rapid reference; though complete mastery of the elaborate notation is not in fact necessary for an understanding of the summaries which precede each section. But does it contain interesting *results* not available eslewhere? By and large the answer is: no. In particular, Hausdorff's book [**H1**] (1914) not only contains almost all the mathematically important theorems about cardinals, ordinals, and the order-types of linear orderings which appear in *Principia*, but much more—e.g. the theory of exponentiation for ordinals. The reason for this limitation is plain: the richness and rigidity of the notation and the superficial complexities with which the formal proofs have to cope, were obstacles for the authors, just as much as they are for the reader. For example, they refer to their proof of Cantor's theorem concerning the isomorphism of two countable open dense linear orderings as " long and complicated " (vol. III, p. 299). It takes five closely packed pages as compared with the two leisurely pages of Cantor's original ([**C2**; p. 505–506]) on which it is modelled. Complications arise because types have to be adjusted, disjointness of certain sets has to be secured, and (with a view to applications to the real world) the two orderings are not given enumerations, but are merely allotted orderings of type ω. The devices necessary to deal with these complications have been frequently used earlier in the book; at this stage they are purely routine. Interestingly, the authors do *not* introduce the genuine simplification of a back *and* forth argument (which is often, erroneously, attributed to Cantor).

Why didn't Russell and Whitehead content themselves with some sample complete formal proofs, while leaving much of the development informal? Besides the natural inertia which a large-scale project has, reasons are to be found in the mathematical climate of Cambridge in the 1890s. In ([**19**; p. 38]) Russell writes, "The proofs that were offered of mathematical theorems were an insult to the logical intelligence. Indeed the whole subject of mathematics was presented as a set of clever tricks by which to pile up marks for the Tripos. The effect of all this upon me was to make me think mathematics disgusting "†. The metamathematics in Principia is not always quite precise. But every theorem is given a formal proof from the axioms; use of the axioms of infinity and of choice is always made explicit; great care is taken not to use unnecessary premises. In these respects it *is* a pioneering work—much more rigorous than was Peano, and covering a much wider range of subject matter than did Frege (as well as being based on a consistent class-logic). It records a triumph of logic. Above all, it is a monument to the fortitude and perseverance of its authors.

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† Russell still felt resentment against his teachers in 1960. In conversation he recalled that though they only proved the binomial for a positive integral exponent, they stated and used it with a real exponent. " I asked them how the general case could be proved. ' We prove it ' they replied, ' by the principle of permanence of form '! ".

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