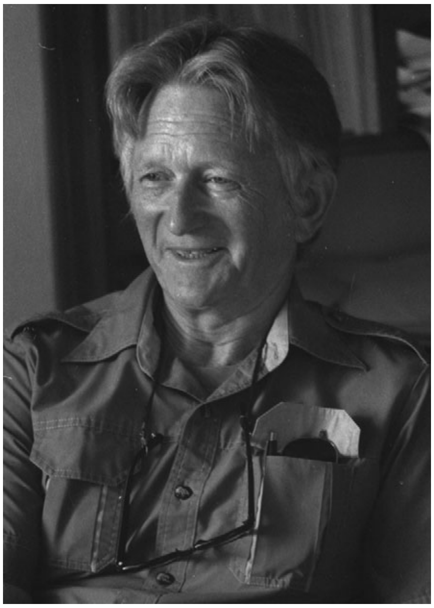
Bull. London Math. Soc. 42 (2010) 949–955
doi:10.1112/blms/bdp138
Published online 12 March 2010

OBITUARY

Atle Selberg



Herman Landshoff (1981)

1. *Selberg's life*

The Norwegian mathematician, Atle Selberg, an honorary member of the London Mathematical Society, died on 6 August 2007, at the age of 90. He was arguably the greatest analytic number theorist of the twentieth century. He was best known for his work on the Riemann zeta-function, on prime numbers and sieves, and on the spectral theory of automorphic forms. He received a Fields Medal at the 1950 International Congress of Mathematicians.

Selberg was born in Lagesund, Norway on 14 June 1917, the youngest of nine children. His father was a mathematician, and two of his brothers, Henrik and Sigmund, went on to become members of the Norwegian Academy of Sciences and Letters. From the age of 13 he read mathematics in his father's library, where he came across Leibniz's series $\pi/4 = 1 - 1/3 + 1/5 - \ldots$. Remembering this occasion, he later said it was 'such a very strange and beautiful relationship that I determined I would read that book in order to find out how this formula came about.' A few years subsequently, at the age of 17, he encountered the collected works of Srinivasa Ramanujan, which he later described as a major influence in directing him into mathematics. Indeed, it was around the same time that he wrote his first paper, 'On some arithmetical identities'. A year later he began studying at the University of Oslo, where he gained his doctorate in 1943. He remained in Oslo throughout the war, working in isolation after the closure of the university during the German occupation.

It was during this period that Selberg produced many of his key papers on the Riemann zeta-function and Dirichlet L-functions. The most famous of these, [**9**] from 1942, proved that a positive proportion of the zeros of $\zeta(s)$ lie on the critical line. For large T, the number of zeros of $\zeta(s)$ in the rectangle $0 < \Re(s) < 1$ and $0 < \Im(s) \leqslant T$ grows like $(T \log T)/2\pi$. It had been shown by Hardy and Littlewood that there is a small constant $c > 0$ such that at least cT of these zeros must actually have $\Re(s) = 1/2$, in accordance with the Riemann hypothesis. However, this still leaves the possibility that 'almost all' zeros lie off the critical line. In contrast, Selberg's result shows that the number of zeros with $\Re(s) = 1/2$ is at least $cT \log T$, this being a positive proportion of the total.

This, and his other results from the same period, made a tremendous impact on the theory. However, it was not until the end of the war that they became known outside Europe. Indeed, it is said that Siegel, who by then was working in Princeton, asked Harald Bohr whether there had been any developments in Europe; and that Bohr replied with the single word 'Selberg!'

In 1947 Selberg moved to the USA, and married Hedvig Liebermann. Around this time he used some of the ideas that had been so successful with the Riemann zeta-function to address elementary questions about primes, by what is now known as the 'Selberg sieve'. The method allows one to give an upper bound for the number of primes in suitable sets. In its simplest form it reduces to a remarkably straightforward and elegant idea that continues to have important applications in prime number theory. Selberg went on to investigate other fundamental problems in sieve theory, and his work is still a source of inspiration, but the subject still has more questions than answers.

Continuing his work on elementary methods in prime number theory, in 1948 Selberg produced his famous 'elementary' proof [**21**] of the prime number theorem, stating that

$$\#\{p \leqslant x : p \text{ prime}\} \sim \frac{x}{\log x} \quad (x \longrightarrow \infty).$$

The original proof, by Hadamard and by de la Vallée Poussin in 1896, was one of the greatest triumphs of nineteenth-century mathematics, and required the use of complex analysis. Many other results in prime number theory had been proved by more combinatorial methods, and, in particular, without the use of calculus. Thus it was natural to look for a proof of the prime number theorem itself by such 'elementary' techniques. However, there appeared to be an inherent obstacle to such a proof, and it came to be believed that no elementary argument was possible. Thus Selberg's work came as a shock to the mathematical community.

Controversy has always surrounded the elementary proof. Selberg first found a recursive estimate, which appeared to give some hope, and certainly produced some new partial results. However, it was not at all clear that the recursive estimate was sufficient for a proof of the prime number theorem. Erdős learned of Selberg's unpublished work from Turán, and within a few days had completed the proof. Around the same time Selberg himself found the necessary Tauberian argument. A bitter priority dispute ensued, over which one still hears arguments.

Selberg became a permanent member of the Institute for Advanced Study in Princeton in 1949, and remained there until his death. In 1950 he was awarded a Fields Medal at the International Congress of Mathematicians at Harvard. The citation mentions his work on sieve methods and on the zeros of the Riemann zeta-function. Over the next decade he turned his attention to the application of spectral theory to automorphic forms. His 1956 paper [**27**], in the *Journal of the Indian Mathematical Society*, has been described as one of the most influential mathematical papers of the twentieth century. It establishes what has come to be known as the Selberg trace formula, and lays the foundations for the modern theory of automorphic forms.

Although Selberg published relatively little in his later years, he continued to work on his research, and to lecture, into his eighties. He was rumoured to have a large collection of unpublished material, and many mathematicians feared they would announce their latest research only for Selberg to tell them he had 'discovered it back in the 1940s, but did not publish it.' In fact, Selberg's collected papers, published in two volumes in 1989 and 1991, included much that had not been fully aired before.

Selberg was very modest, even about his most significant achievements, as is exemplified when he said, in 1990, 'I think the things I have done ... although sometimes there were technical details, and sometimes even a lot of calculation, in some of my early work ... the basic ideas were rather simple always, and could be explained in rather simple terms ... in some ways, I probably have a rather simplistic mind, so that these are the only kind of ideas I can work with. I don't think that other people have had grave difficulties in understanding my work.' Those who knew him will recognize not just the sentiment, but also his characteristic turn of phrase.

Selberg received many distinctions in addition to his 1950 Fields Medal. He won the 1986 Wolf Prize in Mathematics, and was awarded the Abel Bicentennial Anniversary Prize in 2002; he was an honorary member of the London Mathematical Society; he was elected to the national academies of Norway, Sweden, Denmark, the USA, and India; and in 1987 he became a Knight Commander with Star of the Royal Order of Saint Olaf.

Selberg's first wife, Hedvig, died in 1995. He is survived by his second wife, Betty Compton Selberg, and by his two children and two step-children.

2. *Selberg's mathematics*

It is, of course, impossible to discuss Selberg's work in any real depth here, but it is hoped that the following may give an idea of some of his most beautiful ideas. The breadth of his work is indicated by the ubiquity of Selberg's name. Examples include the Selberg sieve, the Selberg trace formula, the Selberg zeta-function, the Selberg integral, the Selberg class, and the Rankin–Selberg convolution.

2.1. *Mollifiers*

As mentioned above, one of Selberg's earliest major achievements was to prove that a positive proportion of the zeros of the Riemann zeta-function lie on the critical line. His argument uses the same underlying method as that of Hardy and Littlewood, but adds one important extra ingredient, a 'mollifier'. Mollifiers are extremely useful in handling questions concerning the zeros of Dirichlet series. Although they were originally introduced in 1914 by Bohr and Landau, the version used by Selberg is far more accurate, and has led to many further developments. It is worth looking more closely at this.

A key observation is that, if $\theta(t)$ is defined to be real and continuous, with

$$\theta(t) = \arg\left(\pi^{it/2}\Gamma\left(\frac{1}{4}+i\frac{t}{2}\right)\right),$$

then $Z(t) := e^{i\theta(t)}\zeta(1/2+it)$ will be real. In proving their theorem, Hardy and Littlewood considered, in effect, the integrals

$$f_1(T) = \int_T^{T+h} |Z(t)|\,dt \quad \text{and} \quad f_2(T) = \left|\int_T^{T+h} Z(t)\,dt\right|.$$

Any value for which $f_1(T) \neq f_2(T)$ must correspond to at least one sign change of $Z(t)$ on $(T, T+h)$, and therefore to at least one zero of $\zeta(s)$ on the critical line. It transpires that it is better to multiply $Z(t)$ by a function $M_X(1/2+it)$, say, which dampens the oscillations of $\zeta(1/2+it)$. A natural choice might be

$$M_X(s) = \sum_{n \leqslant X} \mu(n) n^{-s},$$

since $\zeta(s)^{-1} = \sum_1^\infty \mu(n)n^{-s}$. For technical reasons, it is only possible to use a sum for $n \leqslant X$, with X being a suitable power of T. Selberg's innovation was not merely to introduce a mollifier

$M_X(s)$ into the method of Hardy and Littlewood, but also to make a more subtle choice. Since Selberg wanted his mollifier to be a square, it might seem natural to use

$$M_X(s) = \left(\sum_{n \leqslant Y} \nu(s) n^{-s} \right)^2,$$

where

$$\zeta(s)^{-1/2} = \sum_{n=1}^{\infty} \nu(s) n^{-s}.$$

However, this results in the loss of some crucial logarithmic factors. Instead, Selberg used

$$M_X(s) = \left(\sum_{n \leqslant Y} \nu(s) n^{-s} \left\{ 1 - \frac{\log n}{\log Y} \right\} \right)^2,$$

where the smoothing effect of the factor $1 - (\log n)/(\log Y)$ results in the removal of any extraneous logarithmic factors.

Selberg took this idea further in a second problem, in which he had to consider a mean value of the shape

$$I(T) := \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) M_X\left(\frac{1}{2} + it\right) \right|^2 dt.$$

As before, one wants to use $M_X(s)$ to dampen the oscillations of $\zeta(s)$. Here, however, Selberg took

$$M_X(s) = \sum_{n \leqslant X} \alpha(n) n^{-s}$$

with $\alpha(1) = 1$, and arbitrary real coefficients $\alpha(n)$ for $2 \leqslant n \leqslant X$. Instead of choosing the unknown coefficients as $\mu(n)$, he merely regarded $I(T)$ as a quadratic form in the $\alpha(n)$, and solved the corresponding optimization problem. The resulting values are close to the previous function $\mu(n)(1 - (\log n)/(\log X))$, but are, of course, slightly better.

2.2. *The Selberg sieve*

Selberg took his ideas on mollifiers in a quite different direction in a paper in 1947 that saw the birth of the 'Selberg sieve'. Let

$$\chi_n : \{n \in \mathbb{N} : N < n \leqslant 2N\} \longrightarrow \{0, 1\}$$

be the characteristic function of the primes. If P is the product of all primes $p \leqslant (2N)^{1/2}$, then we may write χ_n, using the Möbius function, as

$$\chi_n = \sum_{d|P,\, d|n} \mu(d).$$

It follows that, if one has a set $A \subseteq \{n \in \mathbb{N} : N < n \leqslant 2N\}$ containing $\pi(A)$ primes, then

$$\pi(A) = \sum_{d|P} \mu(d) \# A_d,$$

where

$$A_d := \{n \in A : d \mid n\}$$

and $\#A_d$ is the cardinality of A_d. Unfortunately, this formula is useless in practice, since it involves too many terms. Sieve theory tries to replace the equality by an inequality, involving fewer values of d. Selberg's method is based on the observation that

$$\chi_n \leqslant \left(\sum_{d|P, d|n} \alpha(d) \right)^2$$

for any real function $\alpha(d)$ with $\alpha(1) = 1$. By taking the function to be supported on the integers up to X, one can restrict the available values of d at will. This leads to an upper bound

$$\pi(A) \leqslant \sum_{d_1,\, d_2 \leqslant X} \alpha(d_1)\alpha(d_2)\#A_d.$$

In order to get as sharp a bound as possible, one considers the right-hand side as a quadratic form in the $\alpha(d)$ and optimizes, just as with the mollifier problem. This simple but elegant device produces best possible answers in many cases.

In general, the 'information input' for Selberg's sieve will be an approximation to $\#A_d$. For example, if A is the entire interval $(N, 2N]$, then $\#A_d = N/d + O(1)$. In this case the optimal bound available from Selberg's method turns out to be too large by a factor of 2. However, Selberg produced an alternative set, in which $\#A_d$ is also approximated by N/d, but for which his upper bound gives the true asymptotic behaviour. Examples of this type reveal the limitations of sieve methods — the so-called 'parity problem' — and are fundamental to modern thinking on questions involving primes.

2.3. *The elementary proof of the prime number theorem*

Selberg's recursive estimate can be phrased in various ways. If one defines

$$\theta(x) := \sum_{p \leqslant x} \log p,$$

then it is not hard to prove that the prime number theorem is equivalent to the statement that $\theta(x) \sim x$. One version of Selberg's formula then says that

$$\theta(x)\log x + \sum_{p \leqslant x} \theta(x/p)\log p \sim 2x \log x.$$

It is apparent that, if one inputs asymptotic information about $\theta(x/p)$ into the sum on the left, then one can extract corresponding asymptotic information about the term $\theta(x)\log x$ on the right.

Instead, Selberg produced a Tauberian argument, based on the formula above, to deduce that $\theta(x) \sim x$. Such arguments require positivity information, which is provided here by the fact that $\log p \geqslant 0$.

Although the original arguments of Selberg and Erdős only gave the qualitative statement that $\theta \sim x$, later workers showed that one could extract quantitative bounds, proving, for example, that

$$\theta(x) = x + O\left(\frac{x}{(\log x)^A} \right)$$

for any constant exponent A. These works require improvements to Selberg's formula, but it is the Tauberian argument that is technically the most involved step.

2.4. *The Selberg trace formula*

The trace formula can be viewed as a non-commutative version of the Poisson summation formula. In its classical form the latter states that, for suitable functions f, one has

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n),$$

where $\hat{f}$ denotes the Fourier transform. One can generalize this by replacing $\mathbb{Z}^+$ on the left by any closed subgroup, say H, of a separable locally compact commutative group G. The corresponding sum (integral) on the right will then be over characters of G that are trivial on H.

Selberg examined a version of this in which H is a discrete subgroup of a semi-simple Lie group, and produced a formula in which the sum on the left is now over eigenvalues for the Laplacian, and so is a 'trace', while on the right one has a sum over primitive hyperbolic classes. The form of the relationship is strikingly similar to the 'explicit formulae' occurring in prime number theory. If $1/2 + i\gamma$ runs over the non-trivial zeros of the Riemann zeta-function, then one has

$$\sum_{\gamma} h(\gamma) = h(i/2) + h(-i/2) - \tilde{h}(0)\log\pi + \frac{1}{2\pi}\int_{-\infty}^{\infty} h(t)\frac{\Gamma'(1/4+it/2)}{\Gamma(1/4+it/2)}\,dt$$
$$- 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\tilde{h}(\log n)$$

for suitable functions h, where $\tilde{h}$ is a certain integral transform of h. Here $\Lambda(n)$ is the von Mangoldt function, and essentially counts primes. It is not necessary to assume that the numbers γ are real. However, in the trace formula the eigenvalues of the Laplacian, say r_j, will be real. Selberg was able to construct a zeta-function with zeros at $1/2 + ir_j$, and that therefore satisfied the Riemann hypothesis. This function is now known as the Selberg zeta-function.

Acknowledgement. The photograph has been reproduced with kind permission by The Shelby White and Leon Levy Archives Center, Institute for Advanced Study, Princeton, NJ, USA.

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