FRANCESCO SEVERI

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I.

Francesco Severi was born at Arezzo on 13 April, 1879. From the age of nine, when he lost his father, until early manhood, he suffered much hardship. Although he had wealthy relatives, they seem to have given him no help whatever, a fact which may well have influenced the development of his character. He entered Turin University, and while he was studying there had to subsist on a meagre scholarship which he eked out by giving private lessons. His first idea was to become an engineer, but the lectures of Corrado Segre attracted him so powerfully that he turned to geometry. Segre was quick to recognise the exceptional talent of his young pupil who, under his master's direction, soon began to produce original work of high quality; in fact Severi's doctoral thesis [5], written in 1900, still retains its interest.

After having acted as assistant to D'Ovidio (Turin), Enriques (Bologna) and Bertini (Pisa), he obtained in 1904 the chair of projective and descriptive geometry at Parma. The interview which led to this appointment must be unique† in Italian university annals, for the selection committee insisted on Severi's giving a trial lesson in descriptive geometry at the blackboard (and this at a time when the candidate had publications which placed him in the front rank). In 1905 he transferred to Padua, where he was Professor for 17 years. His tenure of the chair was interrupted by the first world war, during which he served as a volunteer in the artillery, gaining promotions and decorations. In 1922 he was called to Rome University, over which he presided as Rector for the period 1923–25. In 1939 he became head of the newly formed Istituto di Alta Matematica; in 1950 the Italian Government decided to make him life President of the Institute. He died on 8 December, 1961, after a painful illness, necessitating several operations, which had lasted for over four years.

Severi's scientific work presents several features which, taken together, must make his career a rarity. To begin with, there is the uniformly high level of his very considerable scientific production: as a rule Severi attacks only important questions of general character and usually of great difficulty. Those few works of his which are devoted to the study of particular examples either throw light on such questions or else provide experimental evidence which, to his extraordinary intuitive faculties,

† So Severi himself maintained. After the meeting, Castelnuovo (who was not on the committee) made his way to Rome railway station, sought out Severi, who was sitting in the train for Arezzo, and gave him the good news: again an unusual proceeding.

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suggests significant results. Two illustrations must suffice. First, his examination [11] of the surface representing the pairs of points of a given curve† not only reveals a very interesting type of irregular surface, including the Jacobi surface which was to play an important role in Severi's later researches, but leads naturally to extensions in the theory of higher manifolds. Again, Severi's method of studying correspondences on a curve [14] transports the problem into the theory of surfaces, and the union between the two subjects produces a stupendous offspring: the theory of the base [24].

In the second place, one cannot fail to observe an essential unity of outlook. Severi maintains a balance between geometry and analysis—he has actually made outstanding contributions to function theory. But within his geometrical work itself the same unity is manifest: Severi passes from one topic to another only to turn back at some future time. His production resembles a vast network linking many nodal points, and his thought may pass from one such point to another by seemingly devious paths. A striking example is furnished by [27], the key result of which is that, on any non-singular‡ primal (form) of a space S_4, every surface is a complete intersection with some other primal. To prove this, Severi uses in turn: an inequality of Noether's for the genus of a curve common to two surfaces; two formulae from his own memoir [8] concerning surfaces in S_4; and then the theory of birational geometry on a threefold (linear equivalence, adjoint systems, canonical systems). A second example is the great paper [42] on Grassmannians: beginning with a display of synthetic technique in the manner of Corrado Segre, it runs through the whole gamut of birational and postulation theory, ending with a stiff dose of algebraic calculation.

Thirdly, there is the topicality of Severi's work. Few of his papers have been relegated to the museum; the rest remain part of the living body of geometrical discipline. Aside from their theoretical importance, they follow the best of all recipes for scientific longevity: embedded in many of them is a most interesting problem which still defies all attempts at solution.

Finally, there is the sheer length of Severi's career: it extends to 60 years of continuous activity in which again and again he returns to the problems of his youth, sometimes after an interval of 40 or 50 years, during which his own pupils have carried his work forward. And his pupils, acknowledged or otherwise, are legion: in the last two generations, almost every algebraic geometer (classical or abstract) has mastered his subject and acquired his technique by reading Severi's papers; the topologists, too, have learned much from him. In the following survey, which

† In what follows, unless the contrary is stated, all manifolds mentioned are algebraic and defined in a complex projective space; and all correspondences are likewise algebraic.

‡ A variety is called non-singular if it is irreducible and without multiple points.

merely glances at his achievements, one can mention only a very few of those who are indebted to him; for the truth is, nearly everybody has been to school with Severi.

II.

Severi's earliest publication [**1**] is a 20-page pamphlet, printed at his own expense, dealing with extensions of Pascal's and Brianchon's theorems; he did not know that these results were already familiar. But with his first group [**2–8**] of papers, which treat of enumerative geometry, he at once comes into his own; in them one sees how he develops the germ of an idea into a general theory, whose implications are destined to interest him for a lifetime, and at whose core lies an unsolved problem which will lead him far into other fields.

In the previous generation H. Schubert had laid the foundations of enumerative geometry by observing that conditions imposed on geometrical entities—for example, the conics of ordinary space, or the subspaces of given dimension in a given space—may be represented by symbols which to some extent satisfy algebraic laws. By using this calculus of conditions he was able to perform considerable feats of computation. Severi applies this method in [**2, 4**] to conics which satisfy conditions of incidence or tangency. But the most striking of Schubert's achievements was his formula for the product of two conditions imposed on a system of subspaces; on this idea Severi erected one of his most celebrated works. After a preliminary note [**3**] which clears the ground, he first studies the special case [**6**] of a surface in S_4; a consideration of its manifold of chords shows that such a surface will in general possess a finite number of improper nodes, *i.e.* nodes which leave unaltered the genera of space sections containing them. The great memoir [**8**] extends the preceding notions to any variety V_k $(k > 2)$ of k dimensions immersed in a space S_r $(r > k)$. Severi applies Schubert's calculus to the tangent S_k's of V_k; certain basic conditions imposed on these give rise to the *elementary projective characters*, analogous to the order and rank of a space curve; such characters are additive, and are invariant under projection and section. Given two varieties V_h and V_k which intersect simply in a V_{h+k-r}, one can calculate the elementary projective characters of the latter variety in terms of those of V_h and V_k.

Severi then makes a series of elegant applications of this result. One of the most important concerns the complete intersection of $r-2$ primals of S_r which contain a given surface† V_2; he calculates the characters of the residual intersection in terms of those of V_2, as also those of the curve common to the two surfaces. In the special case $r = 4$, Severi shows that

† The case of $r-1$ primals containing a given curve had already been dealt with by Veronese. It is considered by Severi in [**12**] and [**30**].

any primal containing V_2 will have nodes not only at the improper nodes of V_2 but also (in general) at a finite number of simple points of V_2. He extends this last result to any V_k of S_r for which $2k \geqslant r$. The complications which this phenomenon causes are still making themselves felt today.

The intersection problems solved in [**8**] are essentially questions concerning simultaneous polynomial equations; but no algebra yet developed could even now begin to grapple with them. All the same, it is clear that the various procedures of enumerative geometry need some justification before they can be accepted. Most of Schubert's own work is founded on skilful specialisation of geometrical figures, for which a principle known as "the constancy of the number" is invoked. The vindication of this principle was one of the problems proposed by Hilbert in 1900. In [**38**] Severi gives the essentials of the solution. This he achieves by showing that any condition on geometrical entities can be regarded as a correspondence between manifolds; this correspondence in its turn can be mapped on a variety and hence be classified as irreducible or reducible (either pure or impure). The specialisation principle applies to the first two categories of correspondence, but not to the third.

But this proof in its turn rests on general intersection theory on a given manifold V_d, as Severi points out in a later work [**43**] on systems of conics. And here a second and more difficult question presents itself—the theory of the base for hypersurfaces V_{d-1} of V_d. The former problem is attacked in [**59, 71**] by considerations of continuity; but the latter, which Severi had already solved in 1906, is only the first step towards a general theory of enumerative geometry, to which the memoir [**61**] is devoted: this requires the theorem of the base for any V_k's of V_d ($1 \leqslant k \leqslant d-1$), a subject to which we return in IV. Even the detailed discussion of [**61**] leaves various matters unsettled; Severi himself appears to think that there can be no finality in posing problems of enumerative geometry.

Another tantalising problem, raised in the thesis [**5**], is the justification of the so-called "functional method" of Cayley. Suppose, for example, that we wish to determine the number (assumed finite) of quadrisecants of a non-singular curve C in S_3. Replacing this curve by two disconnected curves C, C', both non-singular, we observe that the quadrisecants to the composite curve $C+C'$ consist of quadrisecants to C or C'; trisecants to C (or C') which meet C' (or C); and common chords to C, C'. Assuming that the required number depends only on the orders n, n' and ranks r, r' of C, C', we can write down a functional equation for the number in question; since the characters n and r are additive, this takes the form

$$f(n+n', r+r') = f(n, r)+f(n', r')+g(n, n', r, r') \tag{1}$$

where g is a known polynomial. Now it is impossible to justify this proceeding by continuity arguments, since an irreducible curve cannot tend continuously to a form consisting of two disconnected components. And even if this difficulty were disposed of, there seems to be no *a priori* reason why the required number should depend on n and r alone—as in fact it does.

As a foundation for the results in [**5**] which concern secant spaces to curves, Severi obtains the general solution of the equation analogous to (1), in any number of variables. In 1948 Severi returns again to these topics in the lectures [**73**]; they represent a mere fragment of his project, which he hoped to complete by justifying Cayley's method. Incidentally, one may remark that this method has been successfully applied to many enumerative problems concerning not only curves but surfaces; a rigorous treatment has still to be given.

Yet another important problem occurs implicitly in [**4**] and [**7**]. Here, in addition to Schubert's calculus, Severi employs a degeneration method which consists in replacing a curve by a connected polygon of lines. Prompted by the publication of [**4**], the Danish Academy of Sciences offered in 1901 a prize for a work justifying such methods; but there were no competitors. In 1915, Severi himself took up the problem in a work [**41**] dealing with families of curves, which goes far beyond the limits of the prize question. His basic entity is the family of plane curves of given order whose only singularities are nodes (to this any family of non-singular curves in higher space can be reduced by projection). The elegant method employed is that of the *falde analitiche*, which are a generalisation of an analytic curve branch. Among the many results (which include propositions on Riemann's existence theorem and the moduli of curves) of this investigation, we may quote the following as typical:

(i) The family of curves of order n and genus p in S_r, where $n \geqslant p+r$, contains all possible connected n-gons† of S_r with virtual genus p.

(ii) For every value of $d \leqslant \frac{1}{2}(n-1)(n-2)$, there exist irreducible plane curves of order n with d nodes.

(iii) The manifold of the classes of birationally distinct curves of given genus p is irreducible.

A full account of this work will be found in the treatise [**44**]. We may add that various attempts have been made to apply Severi's methods to systems of curves possessing other types of singularities; however, even where these are only cusps, the difficulties encountered are great.

An interesting sequel to these researches comes in 1941. Since a ruled surface of given order and genus is rather like a curve with analogous characters, it might be expected that classification problems for scrolls

† It is to be understood that the vertices of these polygons are all simple intersections; the virtual genus is computed by regarding $\frac{1}{2}(n-1)(n-2)-p$ of these vertices as nodes, and by ignoring the remainder.

could be treated in a manner similar to the above. This is what Severi does in [**62**], which he composed as a sort of object lesson in research for his students at the Istituto.

III.

When Severi entered the field of birational geometry of surfaces, Castelnuovo and Enriques had been working at the subject for about ten years. Although they had laid much of the foundation, a great deal remained to be done. Severi's own contributions were numerous and invaluable. To begin with, he improved existing demonstrations [**15, 22, 26**], giving among other things the first 'modern' proof of the Riemann-Roch theorem for surfaces (*cf.* VI). He then introduced new concepts, with the aid of which he carried the whole theory rapidly forward. His work [**10**] is the first of a long chain of results, lasting through most of his life, in which he continued to give new and fruitful definitions† of the (linear) canonical system on a surface or a variety. In [**13**] he deals with the problem of correspondence between surfaces by a procedure which is a model for all such discussions, including the correspondence problem for manifolds: he first breaks down the question into one of $(1, n)$ correspondence and then considers—in all but name—the behaviour of the invariant series and systems of equivalence under rational transformation. The analogous problem for threefolds was first dealt with under wide hypotheses by B. Segre in 1936.

In [**26**] and [**36**] he gives criteria for linear equivalence on a surface or a variety which have been used ever since. A fundamental advance in the theory of linear systems is made in [**20**], where the notion of virtual curve first occurs explicitly in the literature; in their previous work Castelnuovo and Enriques had considered only the case of surfaces of geometric genus zero but with some plurigenera greater than zero. Severi now introduces the virtual curve as the difference between two effective curves, thereby completing the arithmetic of linear equivalence.

An interesting application of this idea occurs in one of Severi's last papers‡ [**79**]. Suppose that the surface F has geometric genus zero but that the virtual canonical system, reversed in sign, is effective of positive order: we then say that F possesses an *anticanonical system* $|A|$. It turns out that the only surfaces which can have such systems are rational or (possibly) elliptic scrollar; however, if the *antigenus* $(=\dim|A|+1)$ exceeds 2, the latter possibility is excluded. The maximum antigenus is 10, and is attained on, for example, the Del Pezzo novenic surface. Since the antigenus is clearly a relative invariant of F, we are confronted with the problem of separating all non-singular surfaces with anticanonical

† In recent developments in the theory of canonical varieties of any dimension, these have proved capable of wide generalisation.

‡ It actually occurs in [**55**], but other authors had employed the concept long before.

systems into classes such that, within each class, all members are birationally equivalent without exceptions. This problem is by no means simple. The notion of anticanonical system, applied to varieties, has proved useful in other work, both before and since the publication of Severi's paper.

Besides completing the theory of linear equivalence, Severi has the credit for introducing the concept of algebraic equivalence into geometry. One day in 1904 Severi announced to his astonished superior Enriques that, given any continuous system $\{C\}$ of curves on a surface F, one can define on the generic member C_0 a linear series—the "characteristic series"—which consists of the sets of intersections of C_0 with the proximate curves in $\{C\}$. After some reflection, Enriques not only accepted the idea but produced a demonstration that, for any *complete* system $\{C\}$ (*i.e.* one not contained in an ampler system of the same order) the characteristic series on C_0 is complete. A different proof of this, the "fundamental theorem", was soon after proposed by Severi; and a third demonstration which, unlike the preceding, was transcendental in character, was given in 1910 by Poincaré, but this depended on ideas which were not available in the period immediately following 1904. The first two proofs were generally accepted until 1921, when Severi himself perceived them to be inconclusive; the discovery [**45**] of this circumstance was made by applying the same method of *falde analitiche* that we have mentioned in II. In the meantime, the new concept and the seemingly unexceptionable result had proved their worth in various applications, such as the important theorem [**31, 70**] that the *adjoint* system† of a sufficiently general system of curves on a surface, is always regular, *i.e.* the virtual dimension of the system, as given by the Riemann-Roch theorem, is also the effective dimension. In 1954 Kodaira extended this result to manifolds of any dimension whatever. Recently Demaria has made other extensions, using purely classical methods. As history shows, it was fortunate that the fundamental theorem was not called into question earlier; it was indispensable to—among other things—the development of the transcendental theory of 1905 (*cf.* IV).

During the last thirty years various attempts have been made to obtain an algebro-geometric proof of the theorem in all its generality; for a detailed account [**85**] may be consulted. Examples constructed by Zappa [**85**] have revealed that the theorem does not hold for all complete continuous systems. The best result to date is Severi's own [**67**], of 1944, which establishes the fundamental theorem for any *semi-regular* curve C of a surface F, *i.e.* one with the property that the canonical system of F cuts on C a complete series. Recent work by Kodaira and Spencer on complex analytic manifolds has shown that the notion of semi-regularity, and also the fundamental theorem, extend naturally to such manifolds; indeed the notion turns out to be just right for the authors' purpose.

† The system adjoint to a given linear system of curves is characterised by the fact that it cuts sets of the canonical series on each curve of the system.

IV.

It was in 1884 that Picard, deliberately imitating Abel's procedure for curves, began his study of the integrals attached to algebraic surfaces from which has sprung so much of modern mathematics. However, apart from a few significant applications by G. Humbert, the French school showed little desire to exploit Picard's results. Just 20 years later, the Italians came to the theory, with immediate and spectacular consequences. In this field Severi, with his great analytical skill, was to produce his most celebrated work; it falls roughly into six categories.

(i) *Abel's theorem for surfaces.* A special case of Abel's theorem for curves, particularly useful in algebraic geometry, is the following. Given a curve of genus p, suppose that u_i $(i = 1, 2, \ldots, p)$ are the p linearly independent integrals of the first kind attached to it: then a necessary and sufficient condition for the linear equivalence of two point-sets $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$ on the curve is the existence of the relations

$$u_i(x_1)+u_i(x_2)+\ldots+u_i(x_n) \equiv u_i(y_1)+u_i(y_2)+\ldots+u_i(y_n), \tag{2}$$

modulo the periods $(i = 1, 2, \ldots, p)$.

In [**21**] Severi gives two analogues of this theorem, each expressing necessary and sufficient conditions for the linear equivalence of two curves on a surface F; these are in terms of the q linearly independent simple integrals of the first kind attached to F: the character q is the *irregularity* of F. Further results are given in [**25**]. Among applications, Severi establishes the following analogue† of a classical property of curves: a surface of irregularity q cannot contain a continuous system of irregular involutions (of point-sets) unless it carries a pencil, of genus q, of curves. In 1942 Severi returns to this topic in [**64**], where he determines those types of surface which contain involutions of maximum irregularity q.

(ii) *The main theorem.* Historically, the irregularity q of a surface F was first defined as the difference $p_g - p_a$, where p_g, p_a are respectively the geometric and arithmetic genera of F (see VI). In reality the definition given above in (i) is half of the following theorem: *A surface of irregularity q possesses exactly q linearly independent simple integrals of the first kind, and exactly $2q$ linearly independent simple integrals of the second kind.*

The fascinating story of how this result was guessed at, and then established, by the combined efforts of Picard, Humbert, Enriques, Castelnuovo and Severi, is too long to be told here; fortunately it has been recounted in detail, with ample references, by Severi himself [**88**]. We mention briefly Severi's own contributions to the result. In the memoir [**18**] he introduces the concept of residue function for a simple integral of the second kind, and proves that a surface which admits simple integrals of the second (or first) kind is necessarily irregular. In [**18**] he also estab-

† This result in its turn can be extended to varieties of any dimension.

lishes the inequality $i-i_1 \leqslant q$, where i, i_1 are respectively the numbers of integrals of the second and first kinds. Then in [**19**] he proves that $i_1 \leqslant q$, $i \leqslant 2q$. For the final stage, as well as for other reasons, we must turn to a work† by Castelnuovo, also published in 1905, in which it is shown that $i_1 \geqslant q$. Whence the above theorem. It should be noted, however, that in his demonstration Castelnuovo invokes the fundamental theorem for continuous systems, a use which Poincaré's work post-legitimised. To complete the interlocking, he also needs a lemma of Severi's from [**21**].

(iii) *The fundamental theorem.* The series [**45**] of notes in which Severi incidentally reveals the flaw in the classical proofs of this theorem was actually written with another aim, namely that of re-elaborating and simplifying Poincaré's own demonstration: this proves that any *arithmetically effective* curve C on F (*i.e.* one belonging to a linear system whose virtual dimension is non-negative) defines a unique complete continuous system $\{C\}$ consisting of ∞^q distinct linear systems. Poincaré constructs a continuous system the members of which are linearly isolated—that is, the complete linear systems defined by them all have zero dimension; from this system one can deduce the existence of $\{C\}$ by using Severi's results on the Riemann-Roch theorem.

Severi now defines *algebraic equivalence* on F as follows: two curves A, B are algebraically equivalent if there exists a third curve C (effective or virtual) such that $A+C$, $B+C$ belong to the same complete continuous system Σ. In that case we write

$$A+C \equiv B+C. \tag{3}$$

The curves A and B may well be reducible, as indeed may be all the curves in Σ. Again, the behaviour of particular linear systems in a system Σ, as opposed to the generic linear system, may be very odd; instructive examples will be found in [**85**].

(iv) *Theorem of the base.* Perhaps the most famous of all Severi's works is the paper [**24**] in which he establishes the theorem of the base: *on any surface F there exist ρ algebraically independent curves $C_1, C_2, \ldots, C_\rho$ such that any other curve C on F satisfies a relation of the form*

$$\lambda C \equiv \lambda_1 C_1 + \lambda_2 C_2 + \ldots + \lambda_\rho C_\rho \tag{4}$$

where the λ and λ_i's are integers.

At the time (1905) of this discovery, it was by no means an obvious deduction from Picard's work; in fact no one but Severi seems to have imagined that such a result might be true. Coming as it did years before the topological theory of the base, it was a stroke of genius. Severi has related how the theorem suddenly dawned on him one winter night as he was walking in the streets of Padua; as we have already remarked, the

† Castelnuovo, *Rend. Acc. Lincei* (5), 14 $(1905)_1$ (three notes).

hint came from his work on correspondence theory of curves. Max Noether, who wrote to him requesting his memoir [**24**] for the *Mathematische Annalen*, commented: "You have shed a great light on geometry."

The consequences of this theory were to interest Severi for the rest of his life (*cf.* [**77, 82**]); in all he made some 20 contributions to the subject. In one elegant minor work [**35**] he shows how the birational self-transformations of a surface can be deduced from the automorphisms of a certain quadratic form; this paper is the forerunner of a considerable literature.

Another interesting deduction from it is an inequality, first given in [**52**]. After reading a simplified proof by Kähler of Hodge's theorem concerning double integrals attached to a surface, Severi noticed that one consequence of this theorem is that $2p_g \leqslant \rho_0$, where ρ_0 is the number of transcendental 2-cycles of the surface. This inequality has since proved to be the key to various problems† in the theory of surfaces.

But perhaps the most interesting application of the theorem is to the theory of division (as it was then called) or of torsion (as it now is). A given linear system $|C|$ of F may possess submultiples, *i.e.* there may exist disequivalent linear systems $|C_1|$, $|C_2|$ such that $C \equiv \lambda C_1 \equiv \lambda C_2$. In other words, F may possess zero divisors with respect to algebraic equivalence. It follows from the theory of the base that every zero divisor on F must be algebraically equivalent to one of a certain finite number ($\sigma - 1$, say) of such zero divisors. The character σ is an absolute invariant, and is called the Severi divisor; it is the order of the torsion group of F. These matters are expounded at length in the memoir [**32**], where applications will also be found.

Both the theory of the base and the phenomenon of torsion, which Severi considered in the classical case alone, have a natural extension in abstract algebraic geometry, as recent literature shows. But they are best presented from the point of view of topology, as Lefschetz was the first to reveal. In fact, in his last account [**85**] of the subject, Severi follows Albanese's simplified treatment of Lefschetz's work.

It is clear that all the above results can be transported, with slight verbal changes, into the theory of hypersurfaces V_{d-1} of a variety V_d ($d \geqslant 3$). But it is far from obvious that an analogous theory must hold for all subvarieties V_k of V_d. In [**60**] Severi attacks this question with, however, only partial success: he establishes the existence of a base

† Here are some examples:—

The only non-singular surfaces with pure canonical *series* of order zero are the elliptic and the Picard surfaces (Dantoni).

The only non-singular surfaces for which the class is less than the order are the plane and the Veronese surface and its projections (Marchionna).

The only non-singular surfaces for which the class is equal to the order are the ruled surfaces (Gallarati).

for V_k's of any dimension on V_d, but subject to the hypothesis that for these varieties algebraic equivalence is the same as numerical equivalence (in the sense of intersection theory). Such a result suffices for the applications to enumerative geometry [61] which he has in mind.

(v) *Applications of the Picard variety.* The general Picard surface first appeared in 1889 as the surface characterised by the fact that it admits an ∞^2 continuous group of automorphisms which is permutable and completely transitive over the surface. The special case of the Jacobi surface, which maps the pairs of points of a curve of genus 2, was already well known. The Picard variety V_q ($q \geqslant 3$), defined in an analogous way, was introduced in 1895; the work of Picard and Painlevé established that it possesses exactly q simple integrals of the first kind. Simultaneous inversion of these integrals furnishes a parametric representation of V_q by means of Abelian functions of rank 1: that is, there exists a (1, 1) correspondence between points of V_q and those of the fundamental period polytope.

The first application of V_q to the theory of surfaces was made by Castelnuovo, in the work quoted in (ii). Suppose that a surface F of irregularity q contains a continuous system $\{C\}$ composed of ∞^q distinct linear systems $|C|$; if we map these systems by the points of a manifold, it can readily be shown that the latter has the property characteristic of V_q. Severi proves in [37] and [39] that, provided F does not contain a pencil of genus q, it can thus be mapped, simply or multiply, upon a surface G, likewise of irregularity q, lying on V_q.

In [40] Severi obtains a different mapping for F which is in effect the inversion theorem for surfaces. We first construct the Picard variety V_q' whose period matrix is that of the q simple integrals of the first kind attached to F; then, subject to the same proviso as above, F is mapped, simply or multiply, on a surface G', of irregularity q, lying on V_q'. The two Picardians† V_q and V_q' are both considered in the memoir [64]. Ever since 1911 these mappings have been applied to problems concerning irregular surfaces and varieties, first in classical and, recently, in abstract algebraic geometry; and we may expect the method to yield much more.

The generalisation to higher varieties is immediate. Suppose that V_d ($d \geqslant 3$) is a manifold of *superficial irregularity* q, *i.e.* endowed with exactly q simple integrals of the first kind; then in general we may map V_d, simply or multiply, on a d-dimensional manifold, also of superficial irregularity q, immersed in a Picard variety V_q or V_q' constructed as above: V_q arises from a continuous system of hypersurfaces on V_d, V_q' from the period matrix of the q integrals attached to V_d. It may be proved [39, 64] that either map exists if and only if V_d does not carry a *congruence* (system of index 1) of subvarieties V_k ($1 \leqslant k \leqslant d-1$) such that the congruence has superficial irregularity q.

† These are in general distinct; the precise relation between them was first established by Andreotti.

In [37] Severi uses V_q to generalise a property of surfaces due to Castelnuovo: he proves that any V_d with superficial irregularity q and geometric genus zero must contain some congruence of superficial irregularity q, the fact being that, under these hypotheses, the mapping process must fail. This is one of those rare occasions where Severi might have taken a shorter cut.

(vi) *General theory of the irregularities.* The first extensions of the previous results concerning surfaces are to threefolds; these occur in the great memoir [33] of 1909, which lays the foundation for most later work on higher varieties. For a threefold V_3, Severi defines two characters: the superficial irregularity q_2 and the tridimensional irregularity

$$q_3 = P_g(V_3) - P_a(V_3) \quad (cf.\ \text{VI}).$$

Let $|C|$ be any complete linear system of surfaces, without base points, on V_3, and let $|C'|$ be its complete adjoint system; then Severi proves (a) The deficiency of the characteristic system $|C^2|$ on C cannot exceed q_2; and there exist systems for which the maximum is attained.

(b) The deficiency of the (canonical) system $|(C'\,C)|$ cut by $|C'|$ on C cannot exceed q_2+q_3; and there exist systems for which the maximum is attained.

Theorem (a) was proved for any V_d by Kodaira in 1954, using modern techniques. The generalisation of (b) was attempted by Severi in one of his last papers [89]. This also has been proved by Kodaira.

In the same 1909 memoir, on extremely slender evidence, Severi conjectures the formula, for any variety V_d,

$$P_a(V_d) = i_d - i_{d-1} + \ldots - (-1)^d i_1 \qquad (5)$$

where i_h denotes the number of linearly independent h-fold integrals of the first kind attached to V_d. Moreover he predicts that the proof of this result will be fraught with difficulties. A proof was given in 1954 by Kodaira, using practically the entire array of known transcendental and topological methods. These contrast strangely with the equipment then at Severi's disposal. As he himself remarked: "*We* had only the old army rifle: *they* have the atomic bomb."

With (5) now established, Severi returns in 1956 to complete the theory of the irregularities of a (non-singular) variety V_d [84]; in this work he uses differential forms instead of integrals. First, introducing the "last irregularity" $q_d = P_g - P_a$, we may write (5) as

$$q_d = i_{d-1} - i_{d-2} + \ldots - (-1)^{d-1} i_1. \qquad (6)$$

Next, applying (6) to the general prime section V_{d-1} of V_d, we obtain a similar equation for q_{d-1}; we then apply (6) to the general prime section of V_{d-1}; and so on. We thus deduce a set of characters $q_d, q_{d-1}, \ldots, q_2$, which are obviously absolute invariants of V_d, in virtue of the absolute invariance of $i_d, i_{d-1}, \ldots$. These are the irregularities of V_d. An "ordinary variety" of V_d is any non-singular V_h $(2 \leqslant h \leqslant d-1)$ whose irregularities are all equal respectively to the corresponding irregularities of V_d. An open question is that of determining best possible conditions for a sub-variety V_h $(h \geqslant 3)$ to be ordinary; the case $h = 2, d = 3$ was settled by Castelnuovo and Enriques, using topological arguments, in 1906.

In [**83**] Severi gives interesting applications of differential forms of the first kind. Using the exterior differential calculus, he establishes conditions under which V_d will carry a superficially irregular congruence of sub-varieties. This work generalises a theorem of Castelnuovo and Comessatti, according to which any V_d for which $P_g \leqslant d(q_2-d)$ must carry a superficially irregular congruence. In the abstract algebraic geometry much of the work described above has been carried over by substituting differential forms for integrals.

V.

The knowledge of Picard varieties required for the above applications —important as these are—merely goes skin deep. The memoir [**34**] by Enriques and Severi on hyperelliptic surfaces, which was awarded the Prix Bordin of the French Academy for 1907, is based on a searching investigation of Picard surfaces from several points of view.

For present purposes we may define a hyperelliptic surface as a surface which is representable parametrically by Abelian functions of genus 2 with the same period matrix, and which is neither rational nor (elliptic) scrollar. The work [**34**], which is in two big sections, is an imposing display of both geometrical and transcendental virtuosity, providing the authors with an opportunity to apply the theory acquired during the preceding years. Part I gives a detailed account of the curve systems on a Picard surface; of the various possible types of automorphism of the surface; and a classification of the irregular hyperelliptic surfaces—these are all Picardian or elliptic $(p_g = 0,\ p_a = -1)$. Part II deals with the regular types; although the classification does not claim to be complete, the study combines synthetic and algebraic geometry with group theory in a remarkable fashion.

Competing for the same prize was another pair of collaborators, G. Bagnera and M. De Franchis, but they were unable to finish their work in time (however, the following year they wrote a second paper on the subject, and this was successful). Their first memoir, which is entirely transcendental and group-theoretic in character, aims at the complete classification and effective construction of the hyperelliptic surfaces. Now these may all be mapped by involutions on Picard surfaces. Accordingly, one needs the theorem: *any involution of order n on a Picard surface V_2 which maps a hyperelliptic surface which is not itself Picardian, is generable by a* *group, of order n, of automorphisms of* V_2. The proof put forward by Bagnera and De Franchis admittedly contained a lacuna, for they did not see how to exclude one awkward possibility. However, in 1936 De Franchis showed quite simply that this possibility cannot arise; the classification was therefore *a posteriori* complete.

Enriques and Severi, who also needed this theorem (although not so desperately) gave what appeared to be an ingenious synthetic proof of the result. But for reasons that are still obscure, their argument is unsound: when applied to other surfaces, it can lead to erroneous conclusions. However, this was revealed only many years later.

Severi's interest in Picard varieties and Abelian functions was life-long. His first note [**28**] of 1907 on this topic was followed 40 years later by the treatise [**69**] on quasi-Abelian functions. This subject originated with Weierstrass: when he had terminated his lectures on elliptic functions he announced, without proof, that the only meromorphic functions of two or more complex variables which possess an algebraic addition theorem are the Abelian functions and their degenerate cases. In 1903 Painlevé established the truth of this assertion for the case of two variables, stating that the general result could be proved similarly.

Severi's approach to the question is geometrical. If in the defining property of the Picard variety V_q we replace the completely transitive group by one that is only *generally* transitive, we obtain what Severi calls a quasi-Abelian variety; this admits a parametric representation by functions of the kind specified by Weierstrass. The main theorem is that any such variety W_q of superficial irregularity p ($\geqslant 0$) is birationally equivalent to the product of a Picard variety V_p and a linear space S_{q-p}. Severi's proof uses a working hypothesis which in a later note [**78**] is shown to be superfluous. Severi's treatise illustrates how the pioneer work of Abel and Riemann on curves can be adapted to yield new results in similar fields; in particular, in connection with the quasi-Abelian functions, it reopens the whole question of Riemann matrices; these were investigated by F. Conforto in the quasi-Abelian case.

VI.

Severi's work on postulation theory, to which we now turn, grows out of one simple idea, Noether's "$Af+B\phi$" theorem for plane curves. Noether himself used the three-dimensional extension of the theorem in his researches on space curves: Severi was to develop the concept into an important branch of geometry. The title of his note [**9**] of 1902, which forms the basis for this study, describes the subject very aptly. The following year, in [**12**], he gives the first striking application of the method. Let F be a non-singular surface of S_r ($r \geqslant 5$) or a surface of S_4 with only a finite number of improper nodes; any $r-2$ primals drawn through F will meet again in a surface F'. Assuming this to be irreducible, one can apply the formulae of [8] (*cf.* II) to deduce the elementary projective characters of F' and of the curve common to F, F'. Severi then uses these results to calculate the *postulation* of F for primals of sufficiently high order l, *i.e.* the number of conditions—which are all linear—imposed on such primals if they are to contain F.

Quite recently, Marchionna has extended these results to varieties of any dimension. His work has interesting applications to the arithmetically normal varieties mentioned below.

The general concept of postulation, already used in [9], had been formulated by Hilbert in 1890; this is the foundation for much of [33] and also of [76], which amplifies and clarifies the 1909 work. Suppose that V_d is a variety of S_r, possibly reducible but pure, and free from multiple components; then the postulation of V_d for all primals of sufficiently high order l is given by a formula

$$\phi(l;\, V_d) = \sum_{i=0}^{d} k_i \binom{l+d-1}{d-1}, \tag{7}$$

where the k_i are certain integers depending on V_d, and satisfying the condition that $k_0, k_1, \ldots, k_{d-1}$ are the analogous coefficients in the postulation formula for the section of V_d by a generic prime S_{r-1}.

Supposing now that W_d is a second variety, satisfying similar conditions, and intersecting V_d simply in a pure variety T which may possibly be empty; then we have, for all $l \geqslant l_0$,

$$\phi(l;\, V+W) = \phi(l;\, V)+\phi(l;\, W)+\phi(l,\, T). \tag{8}$$

Within this order of ideas alone there is a notable literature, including work by Dubreil and Gaeta on varieties of finite residual†.

The passage from these projective notions to birational geometry is simple. Assuming now that V_d is non-singular, let $r-d$ primals of orders n_i $(i = 1, 2, \ldots, r-d)$ be drawn through V_d to meet residually in a variety W_d; then it is easily shown that, whenever the canonical system of V_d is effective, it is cut on V_d, residually to the variety $T = V_d \cdot W_d$, by the primals of order $\Sigma n_i - r - 1$ which contain W_d. The dimension of the canonical system is denoted by $P_g(V_d)-1$, and P_g is called the *geometric genus* of V_d.

The *virtual arithmetic genus* of V_d is defined, with reference to (7), by the formula

$$p_a(V_d) = (-1)^d\,(k_0+k_1+\ldots+k_d-1). \tag{9}$$

For the purposes of this definition, the previous wider hypotheses concerning V_d may be made.

† Such varieties form a chain: first, the complete intersections; next, those varieties which are residual to complete intersections; and so on.

In the abstract case, Muhly and Zariski have investigated the arithmetic genera of *arithmetically normal* varieties; such a manifold has the property that the primals of any given order cut on it a complete linear system.

Returning to the classical case, we observe that pioneer work on curves and surfaces suggests a second definition for the arithmetic genus. Suppose that V_d is non-singular and of order m, and that we make a general projection on to a space S_{d+1}, thereby obtaining a primal V_d', also of order m. This will have a double hypersurface D, arising from the chords of V_d which meet the vertex of the projection. Severi now defines the second arithmetic genus by the formula

$$P_a(V_d) = \binom{m-1}{d+1} - \phi(m-d-2; D). \tag{10}$$

The question whether $p_a(V_d) = P_a(V_d)$ turns out to be of great importance in the theory. For $d = 1, 2$, the result is simple. For $d = 3$, it was proved by Severi in [**33**], not without some difficulty; later, the case $d = 4$ was treated by Albanese. The general result was established recently by Kodaira and Spencer, using the theory of complex analytic manifolds. Severi himself made several attempts at an algebro-geometric demonstration, the last of them in his monograph [**86**] on the Riemann-Roch theorem.

This theorem has a remarkable history, in which Severi's methods and results are conspicuous. By analogy with the classical case $d = 1$, geometers were led to seek a formula for the dimension r of the complete linear system $|D|$ defined by a given hypersurface D of V_d; such a formula would naturally be in terms of the invariantive characters of $|D|$ and of V_d. Work on the case $d = 2$ began in earnest in 1893 (there had been a previous attempt by Noether); but the successive efforts of Enriques, Castelnuovo and Severi yielded only inequalities for r. The same must be said of the results of Severi [**33**] and B. Segre concerning the case $d = 3$.

A true equation for r, valid for all d and under ample hypotheses, was established by F. Hirzebruch† in purely topological terms. In 1955 Hodge translated this result into algebro-geometric language by introducing into the problem $d-1$ auxiliary linear systems of suitable character; it then appears that the formula for r involves not only the invariantive characters mentioned above but also certain others which depend on the intersections of $|D|$ with these auxiliary systems. In 1958 Marchionna gave a purely algebro-geometric proof of the Riemann-Roch theorem, in the same form as Hodge's equation, and containing $d-1$ auxiliary systems of nearly the same significance. The hypotheses here are quite general: it is merely assumed that the system $|D|$, whether effective or virtual, is virtually

† This followed previous contributions by Kodaira, Spencer and Serre.

free from base points. A full account will be found in [88]. It is interesting to note that the method of proof follows the lines laid down in Severi's work of 1903; it is based on the use of the composite variety V_d+W_d described above.

Severi's results concerning the moduli of forms and postulation theory find elegant applications to the study of Grassmannians [42]. We have already alluded to the note [27] in which it is proved that, on any non-singular primal V_d ($d \geqslant 3$), any hypersurface is the complete intersection of V_d with another primal. In [68] Severi shows that this property persists even if V_d possesses double points, provided they are not too numerous. Now any Grassmannian G is a non-singular variety with the above property of the non-singular primal; further than that, the section of G by a generic linear space of sufficiently high dimension likewise has this property.

In [42] Severi obtains the postulation formula for the particular case of a Grassmannian of lines in any space, and indicates how to compute the analogous formula for any variety G. This work was completed by Hodge, who then used the formula to find the arithmetic genera of the various space sections of G.

VII.

(i) *Rational equivalence*. Linear equivalence on a variety V_d is entirely concerned with hypersurfaces and their mutual intersections. For many years Severi considered the possibility of constructing an analogous theory for subvarieties of any dimension on V_d, but the attendant difficulties seemed insuperable. The solution finally arrived at is an equivalence based on *rational*†, as distinct from linear, systems of varieties.

The first definitive step towards the theory was taken in 1932, and concerned surfaces only [54]. But the general concept was developed at great length in two papers [55, 57] which soon followed. Here, however, is a case where last thoughts are best. In 1955, as the result of a correspondence with van der Waerden, Severi [80, 81] propounds a simple definition of rational equivalence which is exactly analogous to that of algebraic equivalence on a surface (*cf.* equation (3) above). Thus, two subvarieties A_h, B_h ($1 \leqslant h \leqslant d-2$) of V_d are rationally equivalent if there exists a third variety C_h (effective or virtual) such that A_h+C_h and B_h+C_h belong to the same rational system. At this point we may mention the illuminating idea of J. A. Todd, according to which all types of equivalence on a V_d may be regarded from a group-theoretic standpoint. The notion was exploited by Severi in [72]. Clearly rational equivalence is invariant under birational transformation; Todd's presentation guarantees that it will have the other properties obviously required of it.

† A rational system is one whose members can be mapped biunivocally by points of a rational manifold.

The train of thought behind [**54**], showing how Severi was led by means of transcendental theory to rational equivalence and then to invariant point-series on a surface, is too intricate to be described here; in any case, accounts of it have been published elsewhere†.

(ii) *Transcendental-topological developments*. In the first place, the topological view of the theory gives immediate significance to the idea of effective or virtual variety on V_d, corresponding to the orientation (positive or negative) of the associated Riemann manifold. It also clarifies the notion of pseudo-equivalence which, as will appear, plays a fundamental role. We say that an irreducible system Σ of subvarieties on V_d has null (or pseudo-null) circulation with regard to a (real) dimension n if every n-cycle of Σ on the Riemannian of V_d is homologous (or weakly homologous) to zero. We also say that Σ has n-dimensional algebraic circulation if every n-cycle on this Riemannian is algebraic. Severi [**56**] proves that every system of equivalence has null circulation with regard to the odd dimensions, and algebraic circulation and zero torsion with regard to the even dimensions. But it is still not known whether these properties are characteristic of systems of equivalence. A difficulty arises from the possible presence of systems of pseudo-equivalence, *i.e.* irreducible systems of V_h's ($h < d-1$) such that, for a fixed m, all the varieties mV_h are rationally equivalent while the V_h's themselves are not; this cannot happen in the case of linear equivalence ($h = d-1$).

For *series* (of points) of pseudo-equivalence on a V_d, Severi [**56**] has established a property similar to the direct form of Abel's theorem [IV (i)]: the sum of the values taken by every differential form of the first kind and of degree n ($n = 1, 2, \ldots, d-1$) at a set of points which varies in a series of pseudo-equivalence, is zero. A converse theorem is known only for the case $d = 2$. All the above results are expounded in [**88**].

Severi has established a Riemann-Roch theorem for point-sets on an irregular surface ([**65**], reproduced in [**88**]). The parallel with the transcendental theory for curves is remarkable: a point-set is special or non-special according as it is or is not contained in the canonical series (see below) of the surface. The results are different in the two cases.

In the series [**56**] of notes from which the above theorems are taken, Severi introduces the notion of correspondence with valency on a surface. He returned to the subject in later work. However, various points in the theory still require clarification.

(iii) *The canonical systems of a V_d*. Of these developments—in which, for the case $d > 2$, Severi had no part—there are excellent accounts by B. Segre‡ and J. A. Todd§. A few remarks will therefore suffice.

† *E.g.* Zariski, *Algebraic surfaces* (1948); Severi [**63, 85**].

‡ B. Segre, *Proc. Int. Math. Congress* (Amsterdam, 1954), III, 497.

§ J. A. Todd, *Bol. Soc. Mat. Mexicana* (1957), 26.

Briefly, the theory of the canonical systems has been built up in four different ways. Following Severi, the initial step was taken by B. Segre in 1934 when, on any (non-singular) V_3, he established the existence of a new invariant curve-system. About two years later Todd, starting from this result and using algebro-geometric methods, produced a set of d invariant systems of equivalence $\{X_h(V_d)\}$ $(h = 0, 1, \ldots, d-1)$ of h-dimensional varieties X_h, the "canonical varieties" of V_d. This work appeared during the years 1937–1939.

In 1936–1937 M. Eger, following up Severi's transcendental approach, produced the same set $\{X_h(V_d)\}$ of invariant systems for any superficially irregular variety V_d of suitably general character, and then converted his theory into one applicable to any (non-singular) variety whatever. A full account of Eger's work was not published until 1943.

A novel algebro-geometric formulation of the theory was given by B. Segre in 1953, based on the concept of "covariant succession" of varieties. The following year, using similar techniques, Segre proved that the canonical varieties are topological invariants.

Lastly, there is the purely topological development of the subject, due to Chern and Hirzebruch, and dating from about 1953 onwards.

All the foregoing theory has interesting connections with Severi's own thought, some traceable back to the work of 1909. Thus, in [**33**] Severi was led to the hypothesis that the arithmetic genus $P_a(V_d)$ [VI, equation (10)] is an enumerative character of V_d, *i.e.* expressible in terms of the elementary projective characters of V_d (II). In 1937 Todd showed that, on the same hypothesis, P_a is expressible as a linear function, with constant coefficients, of the various intersection numbers of the systems $\{X_h(V_d)\}$, a fact already established for $d = 1, 2, 3$. In 1953 Hirzebruch gave a topological proof of this result, free from any such assumption.

Again, if we start from either the transcendental or the topological definition of $X_h(V_d)$, it is easy to show that, for any Picard variety V_q, all the canonical varieties are the null varieties of the relative systems of equivalence. In 1941 Severi made the conjecture that such a property characterises V_q. This is certainly true for $q = 2$, though even here the proof is not immediate. The general question still remains open.

A bolder conjecture, at which Severi seems to have hinted, concerns the pseudo-Abelian varieties, *i.e.* those varieties which admit continuous groups of automorphisms whose trajectories are Picard varieties. In the simplest case, that of the elliptic surfaces, the canonical series is the null series and, as we have remarked [IV(iv)], this property characterises the elliptic and the Picard surfaces. Suppose then that, for a manifold V_d, all the varieties $X_h(V_d)$ $(h = 0, 1, \ldots, r-1)$ are null, while X_r is not. Is V_d necessarily pseudo-Abelian, with trajectories of dimension r?

VIII.

Severi was always interested in function theory—for a good many years he was Professor of analysis in Rome—and particularly in functions of several complex variables. From 1930 onwards he made outstanding contributions, such as [**48**], to this subject; among them we may also point out his studies on biharmonic functions and the solution of the associated Dirichlet's problem [**49–51**]. In [**53**] he completes a theorem, originating with Weierstrass and Hurwitz, according to which any function which is everywhere meromorphic on an algebraic variety is necessarily rational; this result is essential to the developments of IV(v). An attractive account of Severi's work on functions of complex variables will be found in [**87**]; this, his last book in order of composition, includes chapters by other writers.

Severi's long experience of teaching analysis bore fruit in three volumes [**58, 66, 75**], the last two written in collaboration with Scorza Dragoni; these are partly based on previous lithographed editions. They deserve to be more widely known; for one thing, in addition to the main text, which includes a good deal of algebra and analytical theory of differential equations, they are crammed with historical notes, examples and critical comments which provide a survey of most branches of pure mathematics, with geometry of course well to the fore. These notes reveal Severi's great erudition and breadth of outlook. To such gifts his various original articles on differential geometry, relativity theory and foundations of mathematics bear further witness.

However, as may be imagined, it was as a teacher of geometry that Severi excelled. His lectures on his own work were unforgettable; the style was beautifully simple—in public speaking he rarely descended to mere oratory—and the presentation masterly. He was greatly interested in teaching for its own sake, and his didactic skill found an outlet in a whole stream of books; if one counts school texts, written mostly in collaboration, as well as the variants represented by successive editions, they amount to over 40 volumes. But it must be admitted that, while his treatises have many merits, they do not quite fulfil the promise of his spoken lectures; one sometimes feels that clarity and comprehensiveness have been sacrified to stylistic brilliance.

Severi's first book [**16**], which was lithographed, later developed into an excellent treatise on projective geometry. A striking testimonial to his devotion to teaching is the biggish book [**23**] of examples in projective geometry, which he produced in the very thick of his research of 1905–1906. The preface opens with a piece of advice quoted from Cremona: "Never write a treatise!" Fortunately, Severi continued to disregard it†.

Two years after, we have in [**29**] the first modern account of algebraic curves, combining the typically Italian approach with the transcendental; this book, which is now rare, was later expanded into the well-known

† Cremona goes on: "It will cost you more blood and sweat than any amount of original work, and all that you will get in return is a list of the book's shortcomings."

German version [44]. The algebro-geometric aspect of the theory is further elaborated in the volume [46] of 1926, the most polished of Severi's writing on the subject. It was announced as the first of a series which would eventually embrace the whole of algebraic geometry; but hardly any of the projected works ever came to birth.

Very many years ago, Severi prophesied that the future of mathematics lay with functional analysis and topology. In 1927–1929 he gave a course of lectures [47], which were compiled by B. Segre, and in which the elements of topology are expounded; this was a pioneer work in Italy, but it came just too soon to feel the benefit of modern improvements in theory and presentation. Incidentally, it contains the first printed account of Severi's important construction for an *algebraic* Riemannian of an algebric variety, which indeed he had taught in early days at Padua.

It is a strange thing that, all this time and also for years to come, his books remained on the fringe of his own original work. It was not until 1942 that a substantial portion [63] of it was put together from his lecture courses. The resulting book, dealing mainly with rational equivalence, is however disconnected and diffuse; it unwittingly reveals the weakness of the theory—even that of point-series on a surface—namely, that practically nothing can be done with it†. The author tries to recast the classical theory of linear systems so as to hide the awkward fact, but to no avail.

Meanwhile the world was still waiting for a definitive account of Severi's contributions to the transcendental and invariantive theory of algebraic surfaces and varieties. In point of fact it only just missed getting none at all. For about 12 years after the last war Severi lectured on the theory of integrals attached to algebraic surfaces and actually, for a good part of the time, from proof sheets, which however he would not pass for press. At last, in 1957, the book [85] appeared. It is very readable, and includes accounts of the topological aspects of the theory. One can easily guess why it was so long delayed: evidently Severi had hoped to find an algebro-geometric proof of the fundamental theorem for curve systems on a surface. In the event, he gives no complete proof of this result; there is the attempt at a synthetic demonstration, followed by a sketch of Poincaré's treatment, but the latter is too indigestible to be presented in entirety.

The sequel [88] to the above volume was even luckier in getting published—in the author's eightieth year. Most of it is a collection of Severi's later papers, supplementary to [74], with just a few retouches, but one is glad to have it all the same. Both books should prove useful for a long time to come; as sources of information they are, and may well remain, unique. A curious feature of [85] is the apologetic tone of the preface, which all

† Certainly not for lack of effort: Severi's pupils were often set the task of finding uses for it. (This criticism does not apply to the topological developments of the theory.)

but concedes that classical Italian geometry has had its day. But when the author penned these words, his fatal illness was already upon him.

IX.

In the foregoing account—which perforce dismisses in a few lines works that merit pages of description—some of Severi's activities have been casually mentioned. But these give only a pallid notion of what Severi was and did. As he approached middle age, mathematics came to occupy less and less of his time; it had to compete with a host of other occupations. For Severi by then was (among other things) President of an Arezzo bank, head of the engineering faculty at Padua, an expert agriculturalist who managed his own estate. After his transfer to Rome he became a sort of unofficial deputy for his home town, to whom local people looked to remove grievances and pull wires—all of which entailed a heavy correspondence. Later still he administered personally the Istituto di Alta Matematica, down to the smallest detail; he would not delegate authority.

Although after 1915 he made very important discoveries—such as the solution of Dirichlet's problem and the theory of rational equivalence—and continued to publish voluminously, none of this work attains the quality of what had gone before. The wonder is, however, that it got done at all. Somehow time was found for it; but whereas in his young days Severi had taken immense pains in preparing his manuscripts, he now dictated a few pages (occasionally by telephone) whenever he felt in the mood, and was inclined to leave proof sheets to shift for themselves.

Any description of Severi must certainly introduce the forceful personality of his wife: Signora Rosanna, from the Val d'Arno, the dominant influence throughout his career. Very shrewd and humorous, she shared to the full his intense zest for living; she also encouraged his belief—firm enough in any case—that the world at large failed to treat him with due consideration. For, incredible as it may seem, although during the whole period of his maturity honours† were showered upon him and invitations poured in, yet he remained forever unsatiated. Intellectually, materially and socially, he had nearly everything a man could hope for; he possessed a towering presence, with a leonine head, he was a superb talker‡ and writer, a connoisseur of art and the humanities in general, a world traveller —but despite all this he seemed more or less permanently aggrieved. He was fond of appearing as a martyr, a part that he played with conviction.

It may be that his early hardships had left such a deep impression that he failed to realise that he had long since arrived. But whatever the explanation, the struggle for further recognition was waged without a

† Severi was elected corresponding member of the French Academy in 1957, and honorary member of the London Mathematical Society in 1959.

‡ Severi's conversation was inconceivably fascinating; one could listen to him for hours—and one frequently did.

truce. Personal relationships with Severi, however complicated in appearance, were always reducible to two basically simple situations: either he had just taken offence or else he was in process of giving it—and quite often genuinely unaware that he was doing so. Paradoxically, endowed as he was with even more wit than most of his fellow Tuscans, he showed a childlike incapacity either for self-criticism or for cool judgment. Thus he meddled in politics, whereas it would have been far better had he left them alone. Moreover, with his passionate desire to remain young, he insisted on being regarded simultaneously as revered master and as youthful rival, a circumstance which made it terribly easy for both colleagues and pupils to be caught on the wrong foot.

In 1957 the struggle changed its character; it became bitter and tragic. He fought his adversary to the very end. During the last four years of his life he continued to keep all his affairs in a progressively enfeebled grasp. He was still publishing [**89**, **90**] when he was no longer in a condition to recollect his own previous results. He repeatedly expressed the belief that his health was improving, and that he would yet address another Congress.

Once, and once only, he seemed to glimpse reality, and then he went further than the truth. On the last occasion that the writer saw him, just before the onset of his illness, he suddenly exclaimed: "Do you know what they are all thinking? 'If only this Severi were dead!'" But here he did his compatriots an injustice: as they would all regretfully affirm, without him Italian mathematics will never be the same again.

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