EDWARD CHARLES TITCHMARSH

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Edward Charles Titchmarsh was born on 1 June, 1899, at Newbury; he was the son of Edward Harper and Caroline Titchmarsh, and he had an elder sister, and a younger sister and brother. His father was a Congregational minister and an M.A. of London University; his father's people were tradesmen at Royston, never more than fairly prosperous, and on both sides of the family there was a strict religious tradition. Titchmarsh himself wrote an eminently readable account of his family background for his own family; it begins with the derivation of the name from the place Ticcea's marsh, and contains a record going back to the eighteenth and even seventeenth century, and ending with his own schooldays. It is written with the clarity which was characteristic of his mathematical work, and recounts his school days and the somewhat restricted background of his early years with a critical and often humorous detachment. I have used this and the notes which he made for the Royal Society in what follows, in addition to other material supplied by Mrs. Titchmarsh and many mathematical friends, especially A. E. Ingham, J. L. B. Cooper and J. B. McLeod.

His father was chosen later as minister of Nether Chapel, near Sheffield (partly because he was a non-smoker as well as, of course, a teetotaler), and so Titchmarsh was educated at King Edward VII School, Sheffield, from 1908–1917. He wrote that they had far too much homework, and that in the upper part of the school he went on to the classical side, giving up science, and learned "enough Latin to pass Higher Certificate and enough Greek to fail". After that he specialised in mathematics, and did some physics, but experiments always baffled him and he maintained that he knew no chemistry.

In December, 1916, Titchmarsh won the Open Mathematical Scholarship at Balliol College, Oxford; he went up to Balliol for the Michaelmas Term 1917, and was then away for two years on war service. He was a second lieutenant, R.E. (Signals) 1918–19, and was in France from August 1918. He lost a great friend who was a mathematician, and this period seems to have been a nightmare about which he could not speak. He only recorded that "If my superior officers supposed that my special abilities would be particularly useful in this sphere of action, they were mistaken", but he acquired a useful ability to tinker with wireless and other domestic electrical apparatus so as to make them work.

He returned to Oxford in 1919; his tutors were first J. W. Russell, and later J. W. Nicholson. I myself went up to Oxford in October 1919; at Russell's first lecture the room was packed to the doors, and Russell

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said: "Ah, there's my clever pupil Mr. Titchmarsh, he knows it all, he can go away." Russell's methods were extremely regular; he dictated his lectures word by word, and at tutorials selected portions of book work and examples were handed out, and then, if necessary, solutions to examples. Some of Titchmarsh's solutions replaced the official ones. Nicholson, on the other hand, although on rare occasions very illuminating, very seldom saw his pupils, and so it is not surprising that Titchmarsh wrote: "I was however principally influenced by G. H. Hardy. From him I learnt what mathematical analysis is, and at his suggestion I devoted myself to research in pure mathematics."

Titchmarsh won the Junior Mathematical Exhibition in December, 1919, and the Scholarship (jointly with H. O. Newboult) in 1920. He obtained a First Class in Mathematics Moderations in 1920, and in Finals in 1922, taking the B.A. in 1922, and later the M.A. His last year at Oxford was spent in research under the supervision of Hardy nominally working for a D.Phil., but he never completed the residence requirement. He also acted as Hardy's secretary.

In the summer of 1923 he was appointed a senior Lecturer at University College, London, first of all with M. J. M. Hill as professor and then G. B. Jeffery. It was during this period, in the year 1928–29, that he supervised my work for the D.Phil. while Hardy was on leave in the United States, and so far as I remember Titchmarsh was then a Reader. In 1923 he had obtained a Fellowship by examination at Magdalen College, Oxford, and remained a Fellow for seven years, although he only resided occasion, ally. I used to see him at infrequent intervals, sometimes at U.C. and sometimes at Magdalen, and we corresponded a great deal. Meanwhile his father had become minister at Halstead, Essex, and as a result of this he met Kathleen, daughter of Alfred Blomfield, J.P., Secretary and Senior Deacon of the New Congregational Church. They were married on 1 July, 1925, and they eventually had three daughters, all of them now married themselves.

In 1929 Titchmarsh became Professor of Pure Mathematics at Liverpool, where he remained for two years and one term. The administration of the department there seemed to worry him. In 1931 Hardy resigned the Savilian Professorship of Geometry on his election to the Sadleirian Chair at Cambridge, and Titchmarsh was elected to succeed him. Hardy had always lectured on geometry in the strict sense as well as on analysis, but Titchmarsh said in his application that he could not lecture on geometry, and consequently on his election the statute was altered to enable the Professor to lecture on any other branch of pure mathematics instead. Titchmarsh never lectured in Oxford on any subject but analysis, although curiously enough his first published paper was on geometry.

Titchmarsh joined the London Mathematical Society in 1922, and gave it much service: for he was on the Council 1925–29, 1932–36, 1945–48, Vice-President 1928, 1935–37, and President 1945–47. He received the De Morgan Medal in 1953 and the Berwick Prize in 1956. He was elected a Fellow of the Royal Society in 1931, and received the Sylvester Medal in 1955. He was made a honorary Sc.D. of Sheffield in 1953.

For many years he was a dominant figure in Oxford mathematics; in his early years as professor nearly all the research students in pure mathematics were supervised by him, and he continued to attract a flourishing group. He was notably successful in producing problems which would extend, but not defeat, his students. His habits of work were always very regular; I remember him as a supervisor asking me whether I worked regular hours, or sat up late, and he seemed to be comforted by the reply that, unlike Hardy, I worked during the day and never late into the night, because he himself could work only in this way. Since the foundation of the Mathematical Institute about ten years ago, every morning in term and vacation except for short spells when he and his wife were on holiday, he worked at the Institute or at home; in the afternoon he gardened or walked with his wife if there was no meeting, and after tea he worked again, but never later than 8 o'clock, and outside these working hours he never picked up paper and pencil because he had an idea. One colleague writes: "For him mathematics was not a thing which could be discussed and talked about. It could only be done in quiet seclusion with pencil and paper." I must, however, say that as a student I found our rare discussions very valuable, but it is true that he was seldom seen at informal mathematical gatherings such as the British Mathematical Colloquium, although he attended all the International Congresses of Mathematicians from that at Harvard in 1950 onwards, and I remember him at the St. Andrew's Mathematical Colloquium in 1930. He was invited to give a one-hour address at the Amsterdam Congress in 1954, and to give lectures at Utrecht and elsewhere in Holland after it. He also went on lecture tours to Vienna and Liège, and at the time of his death he was looking forward to a fortnight's lecture tour in the U.S.S.R., where his work, especially his recent work on eigen functions, is highly valued.

As senior mathematical professor he became Curator of the Mathematical Institute at Oxford from its creation, and as such was responsible for the general administration. He accepted this situation, leaving most of the details to an extremely able staff, and continued to carry on the job for the remainder of his life. He himself made hardly any use of the modern apparatus of administration; he had no telephone in his room and never dictated a letter, and except under compulsion his manuscripts were submitted for publication in his graceful, clear handwriting. His lectures were admirably clear, although signs of enthusiasm were seldom apparent; at most a quiet satisfaction, the craftsman's pride at something well done. He found most kinds of talking, and lecturing in particular, difficult, and the secretary of the Institute soon learned that it was useless to speak to him just before a lecture. He was shy and diffident in ordinary conversations, although he could make a good speech in public on occasion. One friend writes of "silences which were somehow companionable", and I remember the occasional remark when it came was so often very much in line with my ideas. His contributions to committee discussions were apt to be long delayed, but decisive. It seems as if he could not express himself unless he could do it clearly. He was aware of the drawbacks of his shyness and lack of small talk. He kept in his own hands the job of distributing Institute keys to new research students, perhaps partly because it enabled him to meet them all at least once.

As holder of the Savilian professorship of Geometry Titchmarsh was a Fellow of New College; he was punctilious in his attendance at meetings of the Governing Body and served for 22 years on the Audit and Finance Committee and for 11 years on its Estates Committee as well as on numerous *ad hoc* committees. His annual reports on the College accounts were models of succinct intelligibility, and he filled the socially demanding office of Subwarden with courtesy and benevolence. Like Hardy, he was fond of cricket. He came of a cricketing family, and learned the pleasure of watching good cricket at the county cricket ground, Bramall Lane, when he was at school, but he was not very successful himself, which was a disappointment to his father. However, the annual cricket match at New College of the Senior Common Room against the Choir School gave him an opportunity, and he made the most of it, not only in leading his team, but in fraternising over lunch with the opoosite side. For with children (especially his own grandchildren) he could talk freely and easily.

The regularity of his habits, and the fact that he was seldom away from Oxford for long stretches, was bound up with the fact that his life was very closely linked to that of his family; he worried over their troubles great and small, including domestic details such as frozen pipes, and he enjoyed their activities, especially the music in which they took part, although he did not perform himself. He was always there in times of stress, steady as a rock, even though he felt deeply, as he certainly did in the serious illnesses of his wife and second daughter. Haslam-Jones and his wife were close friends, and Titchmarsh gave regular practical help over a long period during Haslam-Jones' long illness.

Besides the interests already mentioned he read a great deal, especially history and biography, and was interested in politics. He was a great admirer of Churchill. One of his recurring dreams was that he was making an impassioned speech in Parliament.

Titchmarsh died quite suddenly on 18 January, 1963. He had been coping with the usual outdoor chores of the exceptionally cold weather, snow and coal, etc., and apparently came in to rest by the fire, where his wife found him on her return from shopping.

Titchmarsh's mathematical output was prodigious; it included important work on Fourier integrals, integral equations, Fourier series, integral functions, the Riemann zeta-function, and eigenfunctions of second order differential equations. Excluding his first and only paper on geometry, there is an easily recognisable link between all these subjects and the work on Fourier transforms with which he began. He used, as Ferrar put it, to "sign off" with a book on a subject, synthesising his own discoveries and all that he had learnt of other people's in the course of his own research. Titchmarsh certainly told me that he would find it impossible to return to a subject once he had left it. In his obituary notice of Hardy, Titchmarsh recorded that Hardy described himself as a problem-solver, and did not claim to have introduced any new system of ideas, and yet in fact Hardy had a profound influence on the mathematics of his time. According to Mrs. Titchmarsh, her husband said that he did not so much prove new theorems as find better and simpler ways of proving them. Just as in Hardy's case there is a substantial element of truth, but by no no means the whole truth: for results and methods are not independent.

This point is well illustrated by his early work on Fourier transforms. In 1924 [**5**] he extended Plancherel's theorem for L^2 to L^p, $p > 1$, a result corresponding to the Hausdorff-Young theorem for Fourier series which now appears as a rather routine extension. However, this was a far from routine matter at the time, for Plancherel's original paper was not primarily concerned with Fourier transforms, and perhaps the new proof of the L^2 case in the paper on Hankel transforms is the more important achievement. It was done by using a Riemann type approximation for the integrals and thus reducing the integral case to the series case. In his introduction to this paper Titchmarsh describes Plancherel's theory briefly as follows: "We have two sequences of orthogonal functions

$$\phi_1(x),\ \phi_2(x),\ \ldots,\ \phi_n(x),\ \ldots$$
$$\psi_1(x),\ \psi_2(x),\ \ldots,\ \psi_n(x)\ \ldots.$$

We form the 'Fourier series' with respect to the first sequence, of a function of integrable square $f(x)$

$$f(x) \sim a_1\phi_1(x)+a_2\phi_2(x)+\ldots,$$

then the transform of $f(x)$ in this system is the function $F(x)$ which has the same Fourier coefficients with respect to the second sequence.

$$F(x) \sim a_1\psi_1(x)+a_2\psi_2(x)+\ldots.$$

"Plancherel's theory is of a very general character, but its application to the ordinary Fourier transform is by no means immediate. It depends on the expressions of $\cos xy$ in the form

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \cos xy = \sum_{n=0}^{\infty} \phi_n(x)\,\psi_n(y),$$

where the ϕ's and ψ's form systems of orthogonal functions in $(0, \infty)$;

or at any rate, it is necessary that the result obtained by integrating term by term with respect to x and y should be true. The functions found are

$$\phi_n(x) = \frac{2^{\frac{1}{2}} e^x}{n!} \frac{d^n}{dx^n} (e^{-2x} x^n),$$

$$\psi_n(y) = \frac{2}{\pi^{\frac{1}{2}}} \frac{2}{(1+y^2)}^{n+1} \sum_{k=0}^{\infty} (-1)^{n+k} \binom{2n+1}{2k} y^{2k}.$$

"The possibility of being expressed in this way is, from Plancherel's point of view, the fundamental property of $\cos xy$ which gives rise to Fourier transforms."

I have quoted this at length because it shows the enormous gulf between the theory as it then was and as it was when Titchmarsh's *Theory of the Fourier integral* appeared. This book, which has been translated into a number of foreign languages, synthesised many of Titchmarsh's earlier papers on Fourier transforms and series with other work, and remains the clearest and most thorough exposition of the classical theory. Among these papers the most striking are probably those on Hilbert transforms [1926, **2**] and conjugate functions [1929, **2**].

In the first he gave a new account from first principles for the theory of Hilbert transforms of L^p functions, which had been developed by M. Riesz using complex variable methods, by showing that the transformation

$$b_n = \sum_{m=-\infty}^{\infty} \frac{a_m}{m+n+\frac{1}{2}}$$

is involutory and continuous on L^p to itself. Then overcoming considerable analytical difficulties, he treated the continuous integral as the limit of the discrete case.

In [1929, **2**] he studied the properties of the conjugate function

$$g(x) = \frac{1}{2\pi} P \int_{-\pi}^{\pi} f(t) \cot \tfrac{1}{2}(t-x)\, dt$$

for general integrable f, giving new and simpler proofs of the existence almost everywhere of g and of certain other properties established by Kolmogoroff and Besicovitch. The new method led to an improvement of Kolmogoroff's result, and in a further paper he showed that $g(x)$ is integrable provided that

$$f(x) \log\left(2+|f(x)|\right)$$

is integrable, a result which had been obtained by Zygmund in the meantime. However, it is Titchmarsh's proof rather than Zygmund's which is the basis of most modern treatments. Titchmarsh also showed that the allied Fourier series of f has as coefficients the Fourier coefficients of g, provided that these coefficients are calculated as principal values in a certain series—as the limit of integrals over suitable increasing sequences of sets whose measure tends to 2π.

Titchmarsh's papers on integral equations were all written jointly with Hardy, and concerned special types of integral equation not amenable to the general Fredholm treatment. The mere fact that Hardy published a joint paper with Titchmarsh as early as 1924 may be taken to indicate that in Hardy's opinion Titchmarsh's position in the world of mathematical research was by then already established. Most of these joint papers were discussed by Smithies in connection with Hardy's obituary, but it is perhaps worth mentioning [1933, **2**] in addition. This deals with functions which are Fourier k kernels, that is for which a formula

$$f(x) = \int_0^\infty k(xu)\,du \int_0^\infty k(uv) f(v)\,dv$$

subsists.

The book on the Fourier integral also contains a number of applications of Fourier transforms to partial differential equations of various kinds, and includes in particular the dual integral equations developed later by Dr. Busbridge. The solution of the intial value problem for

$$L(u) \equiv \left(q(x) - \frac{\partial^2}{\partial x^2}\right) u = \frac{\partial u}{\partial t} \text{ and } L(u) \equiv \frac{\partial^2 u}{\partial t^2}$$

leads to the eigen value problem for

$$L(u) = \lambda u$$

if the Fourier transforms with respect to t are carried out. This seems to have been Titchmarsh's first approach to the subject which occupied most of his later years, although he went over almost immediately to the discussion of the behaviour of the solution for complex λ.

His first full-sized book was *The theory of functions*; it was based on lectures given in London and Liverpool, and Hardy worked through the notes in connection with his lectures at Oxford. The earlier chapters form a serviceable textbook for an ordinary undergraduate course, but it also made available to anyone with an ordinary grounding in analysis many aspects of the theory of functions of a real, as well as a complex, variable which were otherwise almost inaccessible in English; in particular those aspects relevant to applications in the theory of numbers, the theory of Fourier series and the theory of summability of series were singled out and beautifully done. It has been said that he steadfastly refused to operate in any capacity in areas of mathematics where he could not feel absolute confidence, and he was certainly very reluctant to express an opinion outside a very limited field. As we have seen, he considered geometry outside his field, and when it came to the part of the theory of functions of a complex variable depending on the theory of sets of points or analysis situs, as it was then called, he refused to tackle it, and referred the reader to Watson's *Complex integration* and Cauchy's theorem. The major part of the modern developments of the theory of functions of a complex variable have been in the geometric theory, and the reference to Watson, although probably the best book in English then available, did not go far enough to provide the general background needed in parts of the chapters on Analytic Continuation and Conformal Representation. On the other hand, the later chapters and particularly the examples on them contain much valuable material in both real and complex variable theory in a convenient form. For instance, there is a theorem of a Tauberian type, proved in 1914 by Valiron in a more general form, and rediscovered by Titchmarsh in 1927, on functions with real negative zeros. Since its rediscovery and inclusion in *The theory of functions* it has been widely quoted.

His paper on the zeros of certain integral functions, 1926, and problems arising out of it suggested by him in 1928 were the basis of a good deal of my own work on the Phragmén-Lindelöf function $h(\theta)$, and Titchmarsh's results have found a recent application to difference-integral equations. However, the importance of the paper to the main development of pure mathematics probably lies in the so-called Convolution theorem: if $\phi(t)$ and $\psi(t)$ are integrable functions such that

$$\int_0^x \phi(t)\psi(x-t)\,dt = 0 \tag{1}$$

almost everywhere in the interval $0 < x < \kappa$, then $\phi(t) = 0$ almost everywhere in $(0, \lambda)$ and $\psi(t) = 0$ almost everywhere in $(0, \mu)$, where $\lambda+\mu \geqslant \kappa$. The expression on the left of (1) is sometimes called the resultant of ϕ and ψ, the result or a generalisation of it will always have an honoured place in functional analysis, in particular in the theory of distributions in the form of the statement that a certain algebra has no zero divisors, and it is crucial in Mikusinski's *Operational theory.* (In Mikusinski's book a real variable proof due to Ryll Nardzewski is given using a moment theorem.) In fact it seems possible that Titchmarsh's name will be remembered more for this result in regions of mathematics of which he would have denied all knowledge than in the fields which he regarded as his own. The resultant, or *faltung*, as (1) was sometimes called, was a familiar expression in the literature, and appears in the *Fourier integral* in connection with the uniqueness of the solution $f(x)$ of

$$g(x) = \int_0^x k(x-y)f(y)\,dy,$$

but I think that he told me that the motivation for the paper was to try out methods for

$$\int_a^b e^{zt} f(t)\, dt$$

which might afterwards be used for certain functions represented by infinite integrals and their zeros, and thus applied to the problem of the Riemann zeta-function.

Titchmarsh published a large number of papers on the Riemann zeta-function, a tract in 1930 and a book in 1951: the last two will be referred to as **z** and **Z** respectively in what follows. As Titchmarsh stated in his preface, **z** and the companion to it by Ingham made use of a manuscript of considerable size by Bohr and Littlewood which was intended for a tract but became out-of-date largely due to the subsequent researches of Bohr and Littlewood themselves, in particular Chapter IV, "The general distribution of the values of $\zeta(s)$", was taken from the manuscript with but slight changes. Titchmarsh also made use of Littlewood's lecture notes.

Of Titchmarsh's work on the zeta-function Ingham writes:

"Titchmarsh began to publish papers on the Riemann zeta-function $\zeta(s)$ $(s = \sigma+it)$ and related topics in 1927, but there is reason to believe that his interest had been aroused some years earlier while he was supervising an advanced student at University College, London. Whatever its origin, his interest in this subject was closely linked with his earlier interest in Fourier transforms, and much of his success came from the use of identities that sprang naturally from Parseval's formula but would not have been so readily suggested by the more classical methods of the calculus of residues. Thus in his work on mean values he used this method to set up relationships between integrals of the types

$$I(\delta) = \int_0^\infty |\zeta(\sigma+it)|^{2k} e^{-2\delta t}\, dt, \quad J(\delta) = \int_0^\infty |\phi_k(xie^{-i\delta})|^2\, dx$$

or simple modifications of them, and obtained information about the behaviour of $I(\delta)$ when $\delta \to 0$ by a study of $J(\delta)$. The function $\phi_k(z) = \phi_k(\sigma, z)$ took different forms in different versions of the method. In one version it was expressed in terms of a series involving $d_k(n)$, the number of ways of representing n as a product of k factors, and in cases $k = 1$ and $k = 2$ the behaviour of ϕ_k could be determined with sufficient accuracy to provide an asymptotic formula for $J(\delta)$, and therefore $I(\delta)$, over the whole range $\frac{1}{2} \leqslant \sigma < 1$. For larger k, however, there is difficulty with the small values of x in $J(\delta)$ and no such asymptotic formulae are known. But Titchmarsh noted that, since the integrand is non-negative, lower estimates can be obtained by omitting the range $0 < x \leqslant 1$. By taking k large in this and in somewhat similar arguments he proved (and in one case refined) Ω—results that had been obtained by Littlewood only on the assumption of the Riemann hypothesis that all the complex zeros of $\zeta(s)$ lie on the line $\sigma = \frac{1}{2}$ [1928, **1**; 1929, **1**; 1933, **1**; 1937; **Z**, ch. VII, VIII]. He also used the transform method to simplify the proofs of known results, for example the lower estimates of Hardy and Littlewood and of A. Selberg for the number $N_0(T)$ of zeros $\rho = \beta + i\gamma$ with $\beta = \frac{1}{2}$, $0 < \gamma \leqslant T$ [z, §§ 3.41–3.44; 1947; **Z**, §§ 10.1–10.22].

"Titchmarsh also rendered valuable service in other contexts by presenting simplified versions of work initiated by others, and in the process he made substantial contributions of his own. Particularly noteworthy in this connection is his series of papers on van der Corput's method of estimating trigonometric sums with special reference to zeta-functions and lattice point problems. This culminated in an extension of the method to double sums, with applications to improved estimates of the error in the problem of the lattice points in a circle and of the order of $\zeta(s)$ and of a particular Epstein zeta-function on the critical line $\sigma = \frac{1}{2}$. [1931 **3**, **4**; 1932 **3**; 1934 **2**, **3**; 1934, **1**; 1935 **1**, **2**; 1942 **2**]. He also gave an exposition of an early version of Vinogradov's method of estimating trigonometric sums, and applied the main result to the estimation of $\zeta(s)$ near $\sigma = 1$ and of the error in the prime number theorem [1938 **1**.]

"Titchmarsh was naturally interested in the distribution of the zeros of $\zeta(s)$. He made a close study, on the Riemann hypothesis, of the remainder in the formula of Riemann and von Mangoldt for the number $N(T)$ of $\rho = \beta + i\gamma$ with $0 < \gamma \leqslant T$ and of the related properties of $\log \zeta(s)$, simplifying the proofs of some of Littlewood's results and adding new results of his own [1927 **6**; 1928 **2**; 1943; **Z**, §§ 14.10–14.19.] In the same order of ideas he considered the error in the analogous formula concerning the aggregate of non-trivial zeros of all Dirichlet L-functions for a given prime modulus k, regarded as a function of T and k. [1931 **2**.] He also improved the then known estimates of $N(\sigma, T)$, the number of ρ with $\beta \geqslant \sigma$, $0 < \gamma \leqslant T$ [1930 **2**]. One of his papers on van der Corput's method was devoted to a study of zeros on the line $\sigma = \frac{1}{2}$ by the method of estimating the frequency of occurrence of Gram's law—the statement that $\zeta(\frac{1}{2} + it)$ has one zero between consecutive maxima and minima of a certain function $\cos \vartheta(t)$. [1934 **2**; **Z**, § 10.6.] From this he passed on to a numerical investigation of the Gram phenomenon, and with the co-operation of L. J. Comrie proved that, in spite of several departures from Gram's law, all ρ with $0 < \gamma \leqslant 1468$ are simple and have $\beta = \frac{1}{2}$. In this work he made systematic use of the recently revealed Riemann-Siegel asymptotic formula for $\zeta(s)$. [1935 **5**; 1936 **2**; **Z**, ch. XV.] The numerical computations have been greatly extended since the advent of electronic computers. In collaboration with H. S. A. Potter he also considered the zeros of a class of Epstein zeta-functions, extending Hardy's theorem on the existence of an infinity of zeros on the critical line, even in cases where there is no Euler product to confine the complex zeros to a 'critical strip'. [1935 **4**; **Z**, § 10.25.]

"Among miscellaneous contributions to zeta-function theory may be mentioned a study of the mean value of $|\zeta(\sigma+it)|^{2k}$ for a range of non-integral values of k [1929 **5**], a simplified account and further development of work of Littlewood on the gaps between the ordinates of successive zeros of $\zeta(s)$ [1932 **4**; **Z**, §§ 9.11–9.14], and a proof of the Hardy-Littlewood approximate functional equation for $\zeta^2(s)$ [1938 **2**].

"In the course of these multifarious activities Titchmarsh made incidental contacts with analytic number theory, some of which have already been noted. As a natural development of his study of mean values by the transform method he established mutual relationships between the mean order of $|\zeta(\sigma+it)|^{2k}$ in t and the mean order of $\{\Delta_k(x)\}^2$ in x, where $\Delta_k(x)$ is the error in the Piltz divisor problem, *i.e.* the function defined by

$$\sum_{n\leqslant x} d_k(n) = xP_k(\log x)+\Delta_k(x),$$

where $xP_k(\log x)$ is the residue of $\zeta^k(s)x^s/s$ at $s=1$. [1938 **3**; **Z**, §§ 12.5–12.8.] In a somewhat similar way he related the properties of $1/\zeta(\sigma+it)$ and the mean square of its modulus to those of $M(x)$ and its mean square, where $M(x)$ is the sum-function $\sum_{n\leqslant x}\mu(n)$ of the Möbius μ-function; but this situation is complicated by our imperfect knowledge of the zeros of $\zeta(s)$. [1943; **Z**, §§ 14.25–14.31.] Excursions into number theory for its own sake were rare, but important and provocative. A particularly instructive example, involving a variety of analytical and arithmetical techniques, was the problem of the asymptotic behaviour as $x\to\infty$ of

$$\sum_{l<p\leqslant x} d(p-1),$$

where p runs over primes, l is a fixed integer, and $d(n)=d_2(n)$ is the number of divisors of n. Upper and lower estimates were obtained, but an asymptotic formula was achieved only on the assumption of the extended Riemann hypothesis (that the L-functions have no zeros in the half-plane $\sigma<\frac{1}{2}$), and even so the analytical argument had to be supplemented by the elementary sieve method of Brun. [1930 **6**, **7**.] Another important problem was that of the asymptotic behaviour as $\delta\to 0+$ of

$$\sum_{n=1}^{\infty} d_k(n)\,d_l(n+r)\,e^{-2n\delta}$$

for fixed integers k, $l\geqslant 2$ and a fixed integer $r\geqslant 0$. When $k=l=2$ an asymptotic formula is known for each r, but the passage to higher values of k or l introduces formidable difficulties, and Titchmarsh's main object was to show that a heuristic application of the Hardy-Ramanujan-Littlewood 'circle method' can lead to results that are demonstrably false. This is made possible by the fact that a rigorous treatment by the use of Dirichlet series can be given when $r=0$. Titchmarsh's calculations show that, in this case, the heuristically suggested formula is correct when $k = 2$, $l \leqslant 3$, but incorrect by a factor $\frac{255}{256}$ when $k = l = 3$. [1938 **5**; 1942 **3**; 1948 **1**.] The inconclusive nature of the two investigations just described acted as a challenge to other workers, and important progress has been made since Titchmarsh wrote; but there are many problems still outstanding.

"The flow of papers on the zeta-function and related topics ceased almost as abruptly as it began. Titchmarsh wrote a connected account of the subject in his book [**Z**] published in 1951, but by that time he had become interested in other things and he never returned to the zeta-function."

For the last twenty-five or so years of his life, except for completing his work on the zeta function, Titchmarsh devoted himself almost entirely to eigen functions; he wrote of Hardy: "I worked on the theory of Fourier integrals under his guidance for a good many years before I discovered that this theory has applications in applied mathematics, if the solution of certain differential equations can be called applied," and he explained in his presidential address to the London Mathematical Society how he began to read books on "the other side of the library", especially Dirac's Principles of Quantum Mechanics, but he seems to have had little or no personal contacts or discussions with the physicists from whose books he took his themes. In his preface to *Eigenfunction expansions*, Part II, he wrote:—

> "The whole work is the result of an attempt by an 'analyst' to understand those parts of quantum mechanics which can be regarded as exercises in analysis. The subject is, however, pursued without much regard to the interests of theoretical physicists. It seems that physicists do not object to rigorous proofs provided that they are rather short and simple. I have much sympathy with this point of view. Unfortunately it has not always been possible to provide proofs of this kind."

The work is relevant to problems of wave propagation in continuous media, radiation and semi-infinite boundary value problems; it contains methods for physicists with adequate mathematics to solve problems, but perhaps mainly rather formal problems, which may not be of much importance in practice, but need to be dealt with.

Eigenfunctions Part I attracted much attention in the Soviet Union, where the rest of Titchmarsh's work on the subject has been followed with great interest, and Professor Levitan has written the following account of it.

"The earliest work of Titchmarsh on the theory of differential operators was published in 1939–40. In this work he studied the eigenfunction resolution for the operator

$$y''+\{\lambda-q(x)\}y=0 \quad (0\leqslant x<\infty) \tag{1}$$

$$y'(0)\cos\alpha+y(0)\sin\alpha=0 \tag{2}$$

where $q(x)$ is a real continuous function and α a real number. From that time Titchmarsh's work on the spectral theory of differential operators flowed in an ever-increasing stream. Altogether he published about 40 papers on these problems, and this work has enriched our knowledge by many beautiful and fundamental results. His basic results up to the year 1955 are in his fundamental work *Eigenfunction expansions associated with second order differential equations* (Vol. 1, 1946; Vol. 2, 1958). Apparently Titchmarsh's interest in the spectral theory of differential operators arose at first from his concern with the classical Fourier integral which can be considered as an eigenfunction decomposition with respect to the simple operator

$$y''+\lambda y=0.$$

Later the connection with the problems of quantum mechanics played an essential part. The subject matter of much of his work was prompted by, or borrowed directly from, quantum mechanics.

"The spectral theory of the operator (1) in the case of a finite interval $[a, b]$ and a continuous function $q(x)$ constitutes the classical problem of Sturm-Liouville theory. In the case of a half-line $[0, \infty]$, or of the whole real axis, it was considered by H. Weyl about 1910 together with the related problem of D. Hilbert on the spectral theory of bounded symmetric operators.

"After H. Weyl the efforts of leading mathematicians, viz. F. Riesz, T. Carleman, J. von Neumann and M. H. Stone, were directed mainly towards creating and developing a general theory of linear self adjoint operators in Hilbert space. In particular the proof of the basic spectral formula played a fundamental role. According to this formula every self adjoint operator A in a Hilbert space can be expressed in the form

$$A=\int_{-\infty}^{\infty}\lambda\, dE_\lambda,$$

where E_λ, the so-called decomposition of unity, is the family of projective operators satisfying the following conditions:—

(i) continuity on the left, (iii) $E_{+\infty}=E$,
(ii) $E_{-\infty}=0$, (iv) $E_\lambda E_\mu=E_{\min(\lambda,\mu)}$.

It is not possible to obtain much from the general theory of linear operators, apart from the above properties of E_λ, and clearly this is insufficient for applications; we need to know much more. For instance, when considering the operator (1) we have to know how to construct the operators E_λ from the eigenfunction solutions, and a problem of (1) and (2) taken together is how E_λ behaves for large λ and large x, and there are many others. The answer to some of these problems is contained, not always sufficiently fully, in the work of Weyl just mentioned. Many important results were obtained in the period between 1925–40 by theoretical physicists. These results though correct often had no adequate theoretical basis. Thus when Titchmarsh in about 1939 began to consider the spectral theory of differential operators there was a very wide field of activity in front of him which, in spite of the great achievements in the abstract theory of linear operators, had been little explored.

"For some time Titchmarsh was almost the only mathematician working in this field. In 1946 the first part of Titchmarsh's monograph on the spectral theory of the operator (1)-(2) appeared. Besides a new treatment of the theory of H. Weyl the monograph contains many important new results,† including examples of particular decompositions and general theorems about the nature of the spectrum of the operator (1)-(2).

"Titchmarsh's monograph attracted the attention of mathematicians in the Soviet Union (Yu. M. Berezanskii, I. M. Gel'fand, A. C. Kostyuchenko, M. G. Krein, B. M. Levitan, V. A. Marchenko, A. M. Molchanov, M. A. Naimark, A. Ya. Povsner, etc.), in the U.S.A. (A. Wintner, P. Hartman, E. A. Coddington, N. Levinson, R. E. Langer, F. I. Mautner, D. Ray, C. Putnam, etc.), and also of mathematicians in other countries, to the problems of the spectral theory of differential operators. Certainly the investigations of these mathematicians were initiated for different reasons, for instance the inadequacies of the abstract theory of operators which were mentioned above and the connection between problems of the spectral theory of differential operators and problems of quantum mechanics, but the influence of Titchmarsh's monograph in this domain can hardly be over-estimated.

"It is impossible in a short article to give even a brief description of all his work in this field. We shall therefore confine ourselves to what appear to us to be the most important results.

"One of the fundamental questions in the theory of operators of type (1)-(2) is that of the nature of the spectrum. Roughly speaking, this is the question of which solutions of the problems (1)-(2) are involved in decompositions of arbitrary functions of $L_2(0, \infty)$. Particular results in this direction had been obtained by H. Weyl, and then by theoretical physicists. For the case $q(x) \to -\infty$, Titchmarsh obtained a conclusive answer. He showed that under certain natural conditions of smoothness and regularity of growth of $q(x)$ the spectrum is continuous and fills the entire axis if

$$\int^{\infty} \frac{dx}{\sqrt{|q(x)|}} = \infty,$$

† Some had been published earlier in separate papers. (M.L.C.)

and is discrete, and unbounded above and below, if this integral is finite. The proof of this remarkable theorem is based on new asymptotic formulae for eigenfunctions which were also obtained by Titchmarsh.

"Another remarkable result of Titchmarsh is related to this one; the theorem according to which the operator (1)-(2) is self-adjoint, *i.e.* has a unique Green's function, if

$$q(x) \geqslant -Ax^2 - B$$

where $A \geqslant 0$, $B \geqslant 0$.

"Titchmarsh showed also that analogous results hold for partial differential operators. These results can be compared for significance and conclusiveness with the criteria of uniqueness of the classical moment problem. The investigations of Titchmarsh on the uniqueness of the Green's function were continued and in a certain sense completed by his pupil Sears.

"If $q(x) \to +\infty$, then the spectrum of (1)-(2) is discrete and tends to $+\infty$. This result was established by Weyl. In this case the resolution into eigenfunctions takes the form of a series. Many of Titchmarsh's papers are devoted to the behaviour of these series, in particular to the proof of convergence under Fourier conditions. In the course of these investigations he obtained remarkable asymptotic formulae for the eigenvalues and eigenfunctions which are important in themselves and have wider significance than the questions from which they arose.

"In about 1949 Titchmarsh began to concern himself with the spectral theory of partial differential operators, of the basic form

$$L(u) = \Delta u - q(x_1, x_2, \ldots, x_n)\, u \tag{3}$$

over both finite and infinite domains of n-dimensional Euclidean space R_n. The first fundamental question which arises here is that of the concrete form of the decomposition of the identity E_λ for partial differential operators. It appears that for the most general elliptic operators of arbitrary order the decomposition of the identity E_λ is an integral operator of the form

$$E_\lambda f = \int_{-\infty}^{\infty} \theta(x, y; \lambda) f(y)\, dy \tag{4}$$

where x and y are points of R_n and λ a real number. The function $\theta(x, y; \lambda)$ can conveniently be called the spectral function of the operator. The formula (4) for operators of form (3) for the case $n = 3$ was first found in 1934 by T. Carleman.

"In the first of his fundamental articles on spectral theory for partial differential operators Titchmarsh deduced the formula (4) for operators (3), and for somewhat more general forms by an original method, for the case of finite and also for infinite domains. This result is set out in the chapters XI–XII of the second part of his monograph.

"In later work he studied the spectral function in more detail, and in particular its asymptotic behaviour as $\lambda \to +\infty$. At the same time L. Gårding, B. N. Levitan and other mathematicians studied this problem. Just as is the case for ordinary differential equations, if

$$q(x_1, x_2, \ldots, x_n) \to +\infty \text{ as } r = (x_1^2 + x_2^2 + \ldots + x_n^2)^{\frac{1}{2}} \to \infty,$$

the spectrum of the operator (3) is discrete and the eigenvalues λ_n tend to $+\infty$. $N(\lambda)$ denotes the number of λ_n which are $< \lambda$. The study of the asymptotic behaviour of the function $N(\lambda)$ for a finite domain was the subject of classical works of Weyl, Courant and Carleman. In the case of an infinite domain a deep study of the function $N(\lambda)$ was carried out by Titchmarsh. (Chapter XVII of the monograph.) This work was continued by B. M. Levitan and by the American mathematician D. Ray.

"A large number of articles by Titchmarsh are devoted to perturbation problems for differential operators, as are Chapters XIX, XX and XXI of his monograph. He was able to give a rigorous deduction of results which were obtained earlier without rigorous proof. Of special interest is his study of the case in which the unperturbed operator has a discrete spectrum, and the perturbed operator has a continuous spectrum. He studied the character of the degeneration of a continuous into a discrete spectrum and the asymptotics of this process (Chapter XX of the monograph). In order to study this difficult question Titchmarsh created a new method depending on the study of the complex poles of the resolvent of the perturbed operator. In physics an analogous situation had been encountered earlier in connection with so-called 'weak quantisation'.

"These investigations of Titchmarsh are interesting in that for the first time methods typical for the study of non self-adjoint operators are applied to self-adjoint operators. I consider that these investigations of Titchmarsh and also his results on the perturbation of the Schrödinger operator with periodic potential (Chapter XXI, 21.11–21.13 of his monograph) will receive their full appreciation only in the future.

"In the last period of his life Titchmarsh treated spectral theory of systems of the form

$$(L-\lambda)\psi = 0, \quad \psi = (\psi_1, \psi_2)$$

$$L = \begin{pmatrix} p(x), & q(x) + \dfrac{d}{dx} \\ q(x) - \dfrac{d}{dx}, & r(x) \end{pmatrix}$$

where $p(x)$, $q(x)$, $r(x)$ are real functions defined on $(0, \infty)$ or $(-\infty, \infty)$. This system is connected with Dirac's equations in relativistic quantum mechanics. As in his other investigations, Titchmarsh did not content himself with establishing general theorems but carried out a deep investigation of the problem, studying in particular the spectrum of L. He established under very wide assumptions the fact, known to theoretical physicists, that the relativistic equations of quantum mechanics can be regarded as a perturbation of the non-relativistic Schrödinger equation. In this connection he again met a situation similar to that discussed in Chapter XX of his monograph.

"If we put Titchmarsh's researches on the theory of eigenvalues in chronological order we see that each year he went deeper and deeper into the most difficult problems of the theory. Unfortunately, his premature death has terminated this process and we do not know and possibly will not know for a long time many important parts of this theory.

"Although Titchmarsh's studies in the theory of eigenvalues represent an outstanding contribution to functional analysis, he used completely classical methods. As an analyst he had no equal in this field. His work shows an extraordinary insight and a complete disregard for analytical difficulties. It creates the impression that analytical difficulties did not exist for him."

I might add that Kodaira obtained the asymptotic formulae and the formula for the density matrix which is the key to the differentiation between a continuous and a discrete spectrum when he was cut off from western mathematics during and after the war. His proofs which were based on the general theory of linear operators in Hilbert space were not published in the west until he had seen Titchmarsh's work.

Functional analysis and abstract methods were just becoming popular as Titchmarsh was putting a final polish on the methods of classical analysis. He himself never used even the terminology of linear spaces in his proofs though their use would have shortened his work, but in *Eigenfunctions* Part II he translated some of his results into operator form. It seems that his antipathy to geometry prevented him from using certain methods which would have led to the kind of simplification at which he aimed. His own simplifications paved the way for others to achieve further improvements, but it may be that, after all, it will be for his results that he will be remembered.

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