CHARLES-JOSEPH DE LA VALLÉE POUSSIN

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Charles-Joseph de la Vallée Poussin died on 2 March 1962 in his ninety-sixth year. He was born at Louvain on 14 August, 1866. He came from a family with interests artistic, literary and scientific, his father being for nearly forty years professor of mineralogy and geology in the University of Louvain.

A reader in the University Library at Cambridge looking in the general catalogue for the author de la Vallée Poussin is unlikely to find it before his third attempt. The name is entered under "L", and with reason. The family, of French origin, had the name Lavallée. It was a great-grandfather of the mathematician, an artist of the early 18th century, who, being related to Nicolas Poussin by marriage, added the painter's name to his own.

VP, as I shall call him in this notice, was entered at the Jesuit College at Mons, but he found the instruction in some subjects, notably philosophy, uncongenial. After obtaining the *diplôme d'ingénieur* he turned to the life of a professional mathematician. In 1891 he became a teacher at the University of Louvain as assistant to Louis-Philippe Gilbert, who since VP's boyhood had encouraged his bent towards mathematics. Gilbert, who was reputed to be an inspiring lecturer, wrote an excellent Cours d'Analyse. He died in 1892 and, at the age of 26, VP succeeded to his chair. He remained all his life at Louvain, a city twice submerged by the tide of invasion. After his retirement he continued mathematical work and published papers as late as 1957. In 1961 he fractured a shoulder, and it was the failure of this to heal which led after some months to his death.

There is evidence that his home life was of great happiness. He married the lively and gifted daughter of a Belgian family whom he met on holiday in Norway in 1900. Late in life VP paid her this tribute: "C'est elle qui, collaboratrice sans le savoir, a constamment écarté de ma route les ronces et les épines, et rendu ma tâche plus facile et plus douce, elle qui a gardé toujours vivante la flamme du foyer où j'ai réchauffé mon coeur."

In accordance with custom, celebrations in his honour were held at Louvain in 1928 after 35 years of office and again in 1943. At the former an appreciation of his work and teaching was given by G. Verriest and at the latter one by F. Simonart. In 1928 the King of the Belgians sent a message of congratulation and shortly afterwards conferred the title of Baron on VP. He was presented with a bust by Lagae which was said to portray "la finesse de cette expression si pensive, la mobilité de ces traits dont la légère ironie semble se jouer de toutes les difficultés".

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Among the honours which VP received were membership of the Belgian Academy (1909) and associate membership of the Paris Académie des Sciences (1945). He was a Commander of the Legion of Honour, and honorary president of the International Mathematical Union. He had been an honorary member of our society since 1952.

VP's earliest researches were on topics of analysis which could have arisen directly out of his teaching. He aimed at stating in general form, free of unaesthetic restrictions, theorems about integrals depending on a parameter, multiple integrals and solutions of differential equations. A memoir on this last topic was *couronné* by the Belgian Academy in 1892. But he quickly showed his power in a more spectacular way by his work on the distribution of primes. (**96a, 96b, 99a**)

The "prime number theorem" may be stated in various equivalent forms, such as

$$\pi(x) \sim \frac{x}{\log x}, \quad \psi(x) \equiv \sum_{n \leqslant x} \Lambda(n) \sim x, \tag{1}$$

as $x \to \infty$, where $\pi(x)$ is the number of primes $p \leqslant x$, and $\Lambda(n)$ is $\log p$ if n is a prime power p^m $(m \geqslant 1)$ and 0 otherwise. After much speculation and preliminary work, extending over nearly 100 years, this theorem was proved independently by Hadamard and by VP in 1896. Among preliminary results the most significant were those of Chebyshev, who proved (in 1848) that if the ratio $\pi(x)\big/(x/\log x)$ tends to a limit as $x \to \infty$ the limit must be 1, and (in 1850) that this ratio lies between two positive constants a and A for all sufficiently large x; but it seemed unlikely (as observed by Sylvester) that the interval (a, A) could be narrowed indefinitely by a direct refinement of his methods. In 1859 Riemann transformed the subject by demonstrating the relevance of the function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \quad (p \text{ prime}) \tag{2}$$

(already familiar for real $s > 1$) as a function of the complex variable $s = \sigma + ti$, defined by (2) for $\sigma > 1$ and extended by analytic continuation. Inspired by the challenge of this special function, Hadamard developed the general theory of integral functions to the point where he was able (in 1893) to confirm some of Riemann's conjectures by proving that $(s-1)\zeta(s)$ is an integral function of order 1 having an infinity of complex zeros $\rho = \beta + \gamma i$ with $0 \leqslant \beta \leqslant 1$, in addition to its real ("trivial") zeros at $s = -2, -4, -6, \ldots$, and to deduce the form of the partial fraction expansion of the meromorphic function $Z(s) = -\zeta'(s)/\zeta(s)$.

The Euler product in (2) shows that $\zeta(s)$ has no zero in the half-plane $\sigma > 1$. The first step in the proof of the prime number theorem was to prove that $\zeta(s)$ has no zero on the line $\sigma = 1$. Hadamard's proof of this was the simpler, and later work has been based on his idea. Since VP's original argument is not easily accessible, it may be of interest to outline a simplified version of it.

Suppose that $\zeta(s)$ has a zero $\rho_1 = 1+\gamma_1 i$ $(\gamma_1 \lessgtr 0)$, necessarily simple since

$$|Z(\sigma+\gamma_1 i)| \leqslant Z(\sigma) \sim (\sigma-1)^{-1} \text{ as } \sigma \to 1+.$$

Modifying VP's formulae slightly (to avoid later complications), we start from an identity, or "explicit formula",

$$\sum_{n \leqslant x} \Lambda(n)\left(\frac{1}{n}+\frac{1}{n^{\rho_1}}\right)\left(1-\frac{n}{x}\right) = L - \Sigma'' \frac{x^{\rho-1}}{(\rho-1)\rho} - \Sigma' \frac{x^{\rho-\rho_1}}{(\rho-\rho_1)(\rho-\rho_1+1)}, \quad (3)$$

where $x \geqslant 1$, L denotes generally a function of x (not always the same) that tends to a finite limit when $x \to \infty$, the sum Σ' is over all ρ except $\rho = \rho_1$, $\rho_1 - 1$, and Σ'' over all ρ except $\rho = \bar{\rho}_1 = 2-\rho_1$.

[This identity could now be proved without the theory of integral functions by integrating the function $\{Z(s+1)+Z(s+\rho_1)\}x^s/s(s+1)$ round the rectangle $(2 \pm Ti, -T \pm Ti)$ and making $T \to \infty$ through a suitable sequence. But VP proved his identity (or rather a general identity from which his special identity, and ours, may be derived) by expanding $Z(s)$ in partial fractions, multiplying by a factor $x^s/(s-u)(s-v)$, integrating along a line $s = a+ti$ $(-b \leqslant t \leqslant b)$, and making $b \to \infty$, justifying all term-by-term operations by uniform convergence.]

On the right-hand side of (3), the terms $\rho = \beta + \gamma i$ with $\beta < 1$ form two absolutely-uniformly convergent series in which each term tends to 0 as $x \to \infty$ since $|x^{\rho-1}| = |x^{\rho-\rho_1}| = x^{\beta-1}$; the sums themselves therefore tend to 0 and may be thrown into L. Thus the right-hand side of (3) may be rewritten with Σ replaced by $\mathscr{S}$ to indicate summations over the terms with $\beta = 1$; and this in turn may be reduced to $L+P$, where P is a sum

$$\sum_n \{a_n \cos(\lambda_n \xi) + b_n \sin(\lambda_n \xi)\} \quad (\xi = \log x), \quad (4)$$

in which the λ_n are distinct positive numbers without finite limit point and the sums $\Sigma|a_n|$ and $\Sigma|b_n|$ have finite values. On the left-hand side of (3), the real part of the summand is a positive increasing function of x (in the wide senses) for fixed n; the real part of the sum therefore increases with x. But this real part is bounded since the right-hand side of (3) is bounded; it therefore tends to a finite limit as $x \to \infty$ and may be transposed and thrown into L. Taking real parts in (3) we thus obtain an identity $0 = L' + P'$, where $P' = \Re P$. By a uniqueness theorem (based on a consideration of mean values with respect to ξ) this implies that L' and P' are 0 identically, P' in the formal sense that its coefficients a_n' and b_n' are all 0. But such a formal identity persists after certain formal operations involving differentiation (regardless of any question of convergence). Thus we may replace P in $\Re P = 0$ by $-xD_x^2(xP)$ and obtain the formal identity

$$\mathscr{S}'' \cos \gamma\xi + \mathscr{S}' \cos(\gamma - \gamma_1)\xi = 0. \tag{5}$$

But this is impossible; for $\mathscr{S}''$ contains a term with $\gamma = \gamma_1$ and the left-hand side of (5) will contain, on reduction to the form (4), a term with $a_n \geqslant 1$.

This proof looks very different from Hadamard's, but the two have something in common. Hadamard argued roughly as follows. For $\sigma > 1$ and t real,

$$\Re \log \zeta(\sigma + ti) = \sum_{p, m} m^{-1} p^{-\sigma m} \cos(tm \log p) = f(\sigma, t),$$

say, where p runs over primes and m over positive integers. In $f(\sigma, 0)$ all cosines are $+1$; and $f(\sigma, 0) \sim -\log(\sigma - 1)$ as $\sigma \to 1+$ since $\zeta(s)$ has a simple pole at $s = 1$. If $\zeta(s)$ has a zero at $1 + \gamma_1 i$, then $f(\sigma, \gamma_1) \sim -f(\sigma, 0)$, so that (in some sense) nearly all cosines in $f(\sigma, \gamma_1)$ must be nearly -1. But, if $\cos\theta$ is nearly -1, then $\cos 2\theta$ is nearly $+1$. So nearly all cosines in $f(\sigma, 2\gamma_1)$ are nearly $+1$, and $1 + 2\gamma_1 i$ must be a pole of $\zeta(s)$; which is impossible. Hadamard made this idea precise by dissecting each sum $f(\sigma, r\gamma_1)$ $(r = 0, 1, 2)$ into two parts according as the angle $\gamma_1 m \log p$ was or was not within a given distance α of an odd multiple of π. In VP's proof, as outlined above, the special fact that $\zeta(s)$ has no pole at $1 + 2\gamma_1 i$ does not seem to play the same vital part, but this is because it has been subsumed under the general fact of regularity on the line $\sigma = 1$ except for the pole at $s = 1$. If we allowed that $\zeta(s)$ might have other poles on this line, we should have a modified form of (5) with coefficients $+1$ for zeros and -1 for poles, and the term of $\mathscr{S}''$ with $\gamma = \gamma_1$ would have to be balanced by a term of $\mathscr{S}'$ with $\gamma = 2\gamma_1$ and coefficient -1, corresponding to a pole at $1 + 2\gamma_1 i$.

The next step was to prove the formula

$$\sum_{n \leqslant y} \frac{\Lambda(n)}{n} \left(1 - \frac{n}{y}\right) = \log y + c + o(1) \quad \text{as } y \to \infty, \tag{6}$$

where c is a constant. This again was based on an "explicit formula", combined with a uniform convergence argument like the one applied to the sums on the right of (3), but applicable now to a sum over all ρ since $\beta < 1$ in each term. By integration of (6), combined with (6) itself, it was then proved that

$$\int_0^x \frac{\psi(y)}{y}\, dy = x + o(x) \quad \text{as } x \to \infty,$$

from which (1) was deduced by a differencing argument. This last step was an early instance of a "Tauberian" argument—an inference from an average to the function averaged, subject to some special condition, in this case the monotonic character of $\psi(y)$.

VP extended his researches to the distribution of primes in arithmetical progressions, and of primes representable by binary quadratic forms. These investigations included an elegant functiontheoretical proof of the non-vanishing of the Dirichlet L-functions $L(s, \chi)$ for $s = 1$ and real non-principal characters χ. He also made an advance of the first importance in the original prime number theorem by giving it the more precise expression

$$\pi(x) = \operatorname{li} x + O(xe^{-a\sqrt{(\log x)}}) \quad \text{as } x \to \infty, \tag{7}$$

where $\operatorname{li} x$ is the logarithmic integral and a is a positive numerical constant. This result remained the best of its kind (apart from the value of a) for over 20 years. In this later work VP took over the idea of Hadamard's proof that $\zeta(1+ti) \neq 0$, but avoided Hadamard's dissection by using the relations

$$2(1-\cos\theta)(1+\cos\theta) = 1-\cos 2\theta, \quad 0 \leqslant 1-\cos\theta \leqslant 2, \tag{8}$$

which embody the basic idea in the form: If $1+\cos\theta$ is small, so is $1-\cos 2\theta$. In this way, working with $Z(s)$ rather than $\log \zeta(s)$, he strengthened the inequality $\beta < 1$ to $(1-\beta)\log|\gamma| > A$ with a positive numerical A, and so prepared the way for his proof of (7), which again was based on an "explicit formula". Combination of the relations (8) yields the inequality

$$4(1+\cos\theta) \geqslant 1-\cos 2\theta, \quad \text{or} \quad 3+4\cos\theta+\cos 2\theta \geqslant 0, \tag{9}$$

which forms the basis of modern presentations. This inequality thus arose out of the work of the two authors jointly. Hadamard supplied the idea that led to it; VP, adopting this idea in preference to his original method, translated it into symbols in the form (8). The inequality (9) was used directly (in its first form) by Mertens in 1898.

VP left the further development of his ideas on primes to others, but he returned to the zeta-function in two short notes in 1916. Riemann had conjectured that the complex zeros of $\zeta(s)$ all lie on the line $\sigma = \frac{1}{2}$. This, the now famous "Riemann hypothesis", is still undecided, but Hardy proved in 1914 that an infinity of zeros lie on this line, in other words that, if $N_0(T)$ is the number of zeros $\rho = \beta + \gamma i$ with $\beta = \frac{1}{2}$, $0 < \gamma \leqslant T$, then $N_0(T) \to \infty$ as $T \to \infty$. VP replaced this by explicit lower estimates of $N_0(T)$ and of $N_0(T+T^a) - N_0(T)$ ($\frac{3}{4} < a \leqslant 1$). His results are now mainly of historical interest, for they were superseded by stronger estimates found by Hardy and Littlewood in 1918 and 1921, and by the still stronger ones obtained by A. Selberg in 1942.

If the proof of the prime number theorem is VP's highest achievement, the contribution to mathematical literature for which he is most widely known is his *Cours d'Analyse*. It is likely that the most significant of all the *Cours d'Analyse* was Jordan's, of which the first volume appeared in

1882. It was this book which, as is recorded by Hardy and other mathematicians of his generation, opened their eyes to what Analysis really was. If Jordan's is the most noble of the *Cours d'Analyse* and perhaps Goursat's (helped by its translation by Hedrick) the most widely read, it can hardly be doubted that VP's is the most elegant and lucid. After half a century it is still put before the more able undergraduates as a model of style, and there are parts of it which no other writer has presented with anything like the same economy and clarity.

The *Cours* already existed in 1899 in the form of autographed notes of VP's lectures. The first edition of Volume I was printed in 1903 and of Volume II in 1906. Volume I dealt with the differential calculus of functions of one or more variables and the integration of $f(x)$, and Volume II with multiple integrals, differential equations, differential geometry. Two sizes of type were used, the larger for a course complete in itself for *débutants* including engineering students, and the smaller for supplementary matter addressed to mathematical specialists. Mansion, reviewing the first edition, wrote: " On y rencontre une foule d'innovations pédagogiques excellentes; au point de vue de la précision et de la rigueur dans les questions difficiles, il est supérieur à tous les ouvrages analogues."

It is convenient to follow the *Cours* through its successive editions. No book can have been more liable to drastic change from one edition to the next. The first edition was just too early for the Lebesgue integral, a topic which must have immediately appealed to VP for exposition and research. In the second edition (Vol. I 1909, Vol. II 1912), VP greatly expanded the part in small type to include the theory of sets (including the Schröder-Bernstein theorem), a stylish account of measure and the Lebesgue integral, functions of bounded variation, properties of curves (including Jordan's theorem), polynomial approximation, trigonometric series up to Parseval's theorem, uniqueness and Fejér's $C1$ theorem.

The third edition of Volume I (1914) introduced the Stolz-Fréchet definition of differentiability of $f(x, y)$. A translation into German was announced as being in preparation. The third edition of Volume II was burned when the German army over-ran Louvain. It was intended to contain further discussion of the Lebesgue integral. VP, invited to Harvard and to Paris in 1915 and 1916, expanded this work into a Borel tract.

The editions of the *Cours* after 1919 reverted to forming a course for *débutants* and the small type was omitted. In the *avertissement* to the 4th edition of Volume II (1922), VP said that a third volume might follow. It did not, and it is to the Borel tract that we must turn for the relevant matter.

This tract (1916) is composed of three parts, *Intégrales de Lebesgue, fonctions d'ensemble, classes de Baire*. Lebesgue's own *Leçons sur l'intégration* had appeared in the Borel series in 1904. VP's account of the integral bears the marks of successive refinements of treatment. He deals with functions of sets, getting all the results he can out of the sole assumption of additivity and then imposing the condition of absolute continuity. To differentiate functions of sets he used networks (and conjugate networks, thereby avoiding Vitali's theorem). The third part embodies a number of simplifications of Baire's original classification.

The second edition (1934) of the tract was considerably expanded mainly by a long appendix on analytic sets (Lusin, Souslin) and one on the Stieltjes integral.

We now pick up the threads of VP's activity after his proof of the prime number theorem in so far as they are not discernible in the *Cours d'Analyse* or the Borel tract on Integration which was hived off from it. In the decade from 1908 his main work was in approximation to functions by polynomials, algebraic and trigonometric. These studies culminated in another Borel tract, *Leçons sur l'approximation des fonctions d'une variable réelle* (1919), which contained many theorems of his own. The first long paper **(08b)** is based on the algebraic and trigonometric polynomials defined by the respective singular integrals

$$P_n = \tfrac{1}{2}k_n \int_0^1 f(u)\{1-(u-x)^2\}^n \, du,$$

$$I_n = \tfrac{1}{2}h_n \int_{-\pi}^{\pi} f(u)\{\cos\tfrac{1}{2}(u-x)\}^{2n} \, du,$$

where $$k_n = \frac{3.5\ldots(2n+1)}{2.4\ldots 2n}, \quad h_n = \frac{1}{\pi}\,\frac{2.4\ldots 2n}{1.3\ldots(2n-1)}.$$

The former of these was investigated at the same time by Landau. VP showed that these polynomials provide approximations to $f(x)$ and that, if $f(x)$ has derivatives up to a given order, the corresponding derivatives of P_n and I_n provide approximations to them. VP was led to define generalized derivatives, which now carry his name, the derivatives a_{2r+1} of odd order being defined by

$$\frac{f(x+h)-f(x-h)}{2} = a_1 h + \ldots + a_{2r-1}\frac{h^{2r-1}}{(2r-1)!} + (a_{2r+1}+\eta)\frac{h^{2r+1}}{(2r+1)!},$$

where $\eta \to 0$ as $h \to 0$, with a corresponding formula for derivatives of even order.

VP proved that, if $f(x)$ has a bounded derivative, then the goodness of the approximation of P_n to f is of order $1/\sqrt{n}$. Shortly afterwards he proved **(10b)** that a function whose graph consisted of segments of straight lines could be approximated by a polynomial of degree n to order $1/n$.

In a second long paper (**08c**) VP investigated his interpolation formula

$$F(x)=\frac{\sin mx}{m}\sum_a^b(-1)^k\frac{f(\alpha_k)}{x-\alpha_k},$$

summed over $\alpha_k = k\pi/m$ lying in (a, b) where $m = n$ or $m = n+\frac{1}{2}$ and k is an integer. Borel and Runge had given examples showing that the polynomial of interpolation of Newton-Lagrange may fail to converge to its generating function as the number of nodes tends to infinity. VP's function $F(x)$ does converge, as $m \to \infty$, to the function $f(x)$.

The work of Bernstein and Jackson on polynomial approximation, combined with his own, ultimately led VP to the comprehensive theorem (Borel tract, 57):

If $f(x)$ has an r-th derivative satisfying a Lipschitz condition of order α $(0 < \alpha < 1)$, then there is a polynomial of degree n approximating to $f(x)$ within $O(1/n^{r+\alpha})$, and the converse is true.

VP went on to consider functions having derivatives of all orders and proved theorems about analytic and quasi-analytic functions. A good account of this work appears in (**25a**), the condensed text of lectures given in 1924 in the U.S.A.

VP made a number of contributions to the theory of trigonometric series. I state the three most striking.

(1) *His test for convergence* (**11b**). The Fourier series of $f(x)$ converges if

$$\frac{1}{t}\int_0^t\{f(x+t)+f(x-t)\}\,dt$$

has bounded variation in some interval $(0, h)$.

(2) *His uniqueness theorem* (**12a**). If a trigonometric series converges everywhere to an integrable function, it is a Fourier series.

(3) *His method of summation* (**08b** and Hardy, *Divergent series*, 88). Take as a kernel $t_m(\theta) = k_m(1+\cos\theta)^m$, where k_m is a normalizing factor and use

$$(1+\cos\theta)^m = 2^{1-m}\frac{(2m)!}{(m!)^2}\left\{\tfrac{1}{2}+\frac{m}{m+1}\cos\theta+\frac{m(m-1)}{(m+1)(m+2)}\cos 2\theta+\ldots\right\}.$$

This led VP to define the sum of Σa_m as

$$\lim_{m\to\infty}\left\{a_0+\frac{m}{m+1}a_1+\frac{m(m-1)}{(m+1)(m+2)}a_2+\ldots\right\}.$$

VP applied his method to the summation of the successive derived series of Fourier series. The method is stronger than all the Cesàro methods.

For the first quarter of this century VP's interests were dominated by the Borel-Lebesgue revolution and were centred on the real variable. (It is not commonly realised that he was nine years older than Lebesgue and five years older than Borel.) Incidentally, his is the one *Cours d'Analyse* which does not touch complex function-theory. After 1925 he turned again to the complex variable, in particular to potential theory and conformal representation.

VP gave a unified presentation of his researches in potential theory in his book *Le Potentiel Logarithmique* published in 1949. The printing of this had suffered delay and, apart from references in footnotes, no account was taken of work of other mathematicians published since 1939, and advances beyond VP's standpoint were already being made, notably by the French school (Brelot, Cartan, Choquet, Deny).

Although VP's book treats only the logarithmic potential, the methods are largely applicable to the Newtonian potential. The first four chapters deal with the basic notions of capacity, balayage, and some new results about the energy integral. Chapters V–VII contain some of VP's characteristic contributions to the theory of capacity of Borel sets; he defines the stability of a point for a set E and he calls a set complete if it is the locus of points which are stable for it. The set of stable points for any Borel set E is its complete adjoint and has the same capacity as E. VP then deals with the problem of balayage on complete sets. In later chapters he leads up to the conformal representation on the unit circle of a simply-connected schlicht domain; the correspondence between the boundaries is expressed by means of the orthogonal trajectories of the level curves of the Green's function.

I have made use of the address of Verriest at the 1928 Louvain celebrations (*Annales de la Société Scientifique de Bruxelles*, 48A) and of Montel's notice for the Académie des Sciences (*C.R.*, 2 April 1962). Professors Papy and Bouckaert kindly lent me printed matter which I could not otherwise have seen. Mr. A. E. Ingham wrote the piece about the prime number theorem. The reader of his paragraphs, exacting and greatly rewarding, will share my debt to him for unfolding the story of VP's greatest work.

Bibliography.

BOOKS

Cours d'Analyse Infinitésimale, (Louvain, Uystpruyst-Dieudonné and Paris, Gauthier-Villars). 1st. Ed., I (1903), 372 p., II (1906), 402 p.; 2nd Ed., I (1909), 423 p., II (1912), 464 p.; 3rd Ed., I (1914), 452 p.; 4th Ed., I (1921), 436 p., II (1922), 478 p. Fewer changes in succeeding editions: 8th edition I (1938), 460 p., 7th II (1938), 524 p.

Intégrales de Lebesgue, fonctions d'ensemble, classes de Baire, (Paris, Gauthier-Villars) 1st Ed. (1916), 154 p.; 2nd Ed. (1934) 193 p.

Leçons sur l'approximation des fonctions d'une variable réelle, (Paris, Gauthier-Villars, 1919), 152 p.

Leçons de mécanique analytique (Paris, Gauthier-Villars), I (1924), 216 p.

Les nouvelles méthodes de la théorie du potentiel et le problème généralisé de Dirichlet (Actualités scientifiques et industrielles, 578; Paris, Hermann 1937), 46 p.

Le potentiel logarithmique, balayage et représentation conforme (Louvain, Uystpruyst and Paris, Gauthier-Villars, 1949), 452 p.

PAPERS

A number of short or less important papers are omitted. The list is arranged under the year of review in the *Jahrbuch über die Fortschritte* (after 1940, *Mathematical Reviews*) *e.g.* the papers appearing in 1899 carry the prefixes **99a**, **99b**, **99c**, ... *Annales de la Société Scientifique de Bruxelles* is shortened to *Brux.*

91a. " Sur une démonstration des formules de Fourier generalisées ", *Brux.*, 15A, 39–41.

92a. " Etude des intégrales à limite infinie...", *Brux.*, 16B, 150–180.

92b. " Recherches sur la convergence des intégrales définies ", *Journal de Math.*, (4) 8, 421–467.

93a. " Mémoire sur l'intégration des équations différentielles ", *Mémoires couronnés de l'Acad. de Belgique*, 47, 1–82.

95a. " Sur la géometric non-euclidienne ", *Mathesis* (2) 5, supp. 5, 6–15.

96a. " Démonstration simplifiée du théorème de Dirichlet sur la progression arithmétique ", *Mémoires couronnés de l'Acad. de Belgique*, 53, 1–32.

96b. " Recherches analytiques sur la théorie des nombres premiers ", *Brux.* 20B, 183–256, 281–362, 363–397 and 21B, 351–368.

96c. " Recherches arithmétiques sur la composition des formes binaires quadratiques ", *Mémoires couronnés de l'Acad. de. Belgique*, 53, 1–59.

98a. " Sur les valeurs moyennes de certaines fonctions arithmétiques, " *Brux.*, 22A, 84–90.

99a. " Sur le fonction $\zeta(s)$ de Riemann et le nombre des nombres premiers inférieur à une limite donnée ", *Mémoires couronnés de l'Acad. de Belgique* 59, 1–74.

99b. " Réduction des intégrales multiples généralisées ", *Journal de Math.*, (5) 5, 191–204.

00a. " Sur la surface de révolution minimum ", *Brux.*, 24A, 48–52.

02a. " Sur les relations qui existent entre les racines d'une équation algébrique et celles de sa derivée ", *Brux.*, 26B, 1–12.

07a. " Le mouvement instantané le plus général d'un solide ", *Brux.*, 31A, 73–77.

08a. " L'objet de la démonstration mathématique et la réalité ", *Bull. sci. de l'Acad. de Belgique* (1908), 1131–1156.

08b. " Sur l'approximation des fonctions...", *Bull. Sci. de l'Acad. de Belgique* (1908), 193–254.

08c. " Sur la convergence des formules d'interpolation entre ordonnées équidistantes ", *Bull. sci. de l'Acad Belgique* (1908), 319–410.

10a. " Démonstration nouvelle d'un théorème fondamental de la théorie des covariants des formes binaires ", *Brux.*, 34B, 223–229.

10b. " Sur les polynômes d'approximation et la représentation approchée d'un angle ", *Bull, sci. de l'Acad. de Belgique*, (1910), 808–844.

10c. " Réduction des intégrales doubles de Lebesgue. Application à la définition des fonctions analytiques ", *Bull. sci. de l'Acad. de Belgique*, (1910), 768–798.

10d. " Sur les enveloppes de courbes planes qui ont un contact d'ordre supérieur avec leurs enveloppées ", *Atti della Reale Accademia dei Lincei*, 28, 43–50.

11a. " Sur la méthode de l'approximation minimum ", *Brux.*, 35B, 1–16.

11b. " Un nouveau cas de convergence des séries de Fourier ", *Rendiconti Palermo*, 31, 296–299.

11c. " Sur les polynômes d'approximation à une variable complexe ", *Bull. sci. de l'Acad. de Belgique*, (1911), 199–211.

12a. " Sur l'unicité du développement trigonométrique ", *C.R. Acad. Sci.* 155, 951–953, and *Bull. sci. de l'Acad. de Belgique*, (1912), 702–718 and (1913), 9–14.

15a. " Sur l'intégrale de Lebesgue ", *Trans. American Math. Soc.*, 16, 435–501.

15b. " Sur la définition de la différentielle totale et sur les intégrales curvilignes...", *Brux.*, 38A, 67–72.

15c. " Démonstration simplifiée du théorème fondamental de M. Montel sur les familles normales de fonctions ", *Annals of Math.* (2) 17, 5–11.

15d. " Sur les propriétés générales des fonctions elliptiques ", *Brux.*, 38A, 157–161.

16a. " Sur les zéros de $\zeta(s)$ de Riemann ", *C.R. Acad. Sci.*, 163, 418–421 and 471–473.

17a. " Sur les expressions qui s'écartent le moins de zéro dans un intervalle ", *Bull. de la Soc. Math.*, 45, 53–56.

18a. " Sur la meilleure approximation des fonctions d'une variable réelle par des expressions d'ordre donné ", *C.R. Acad. Sci.* 166, 799–802.

20a. " Les fonctions à variation bornée et les questions qui s'y rattachent ", *Darboux Bulletin* (2), 44, 267–296.

23a. " Sur les fonctions quasi-analytiques de variables réelles ", *C.R. Acad. Sci.*, 176, 635–638.

23b. " Le temps et la relativité restreinte ", *Bull. sci. de l'Acad. de Belgique*, (5) 9, 569–611.

24a. " Sur une propriété des fonctions entières considérées dans un demi-plan ", *Rend. sem. mat. R. Univ. Roma* (2) 1, 61–67.

24b. " Quatre leçons sur les fonctions quasi-analytiques de variable réelle ", *Bull. de la Soc. Math.*, 52 (1924) 175–203.

25a. " On the approximation of functions of real variables and on quasi-analytic functions ", *Rice Institute Pamphlet* 12, 101–172.

26a. " Sur les théorèmes d'existence de la théorie du plan osculateur ", *Brux.*, 46A, 524–542.

28a. " Sur les enveloppes de courbes planes ", *Brux.*, 48A, 5–9.

28b. " Sur les fonctions presque périodiques de H. Bohr ", *Brux.*, 47A, 141–158 and 48A, 56–57.

29a. " Sur l'équation différentielle linéaire...", *Journal de Math.* (9) 8, 125–144.

29b. " Sur l'unicité de la détermination de l'intégrale d'une equation linéaire d'ordre n par n points dans le domaine complexe ", *Brux.*, 49A, 11–22.

30a. " Application de l'intégrale de Lebesgue au problème de la représentation d'une aire simplement connexe sur un cercle ", *Brux.*, 50A, 23–34.

30b. " Critérium d'existence de la derivée normale d'un potential de surface en un point de la surface active ", *Brux.*, 50A, 132–139.

30c. " Sur la représentation conforme des aires planes multiplement connexes ", *C.R. Acad. Sci.* 190, 782–783 and 191, 1414–1418, *Annales de l'Ecole Normale* (3), 47, 267–309.

31a. " Sur la représentation... connexes ", *Bull. sci. de l'Acad. de Belgique* (5) 17, 10–27 and *C.R. Acad. Sci.* 192, 128–131.

31b. " Sur quelques extensions de la méthode du balayage de Poincaré et sur le problème de Dirichlet ", *C.R. Acad. Sci.* 192, 651–653.

31c. " Sur l'expression asymptotique de la formule de Lord Kelvin donnant le rapport des résistances d'un fil en courant alternatif ou continu ", *C.R. Congrès nat. sci. Bruxelles*, 1930, 78–83.

32a. " Extension de la méthode du balayage de Poincaré et problème de Dirichlet ", *Annales Institut Henri Poincaré*, 2, 169–232.

32b. " Utilisation de la méthode du balayage dans la théorie de la représentation conforme," *Bull. sci. de l'Acad. Belgique* (5) 18, 385–400.

32c. " Propriétés des fonctions harmoniques...", *C.R. Acad. Sci.* 195, 11–14 and 92–94.

32d. " Mouvement quasi-pendulaire...", *Brux.*, 52A, 16–22 and 83–98.

32e. " Sur la résolution de l'équation de Gauss $\sin(z-q) = \lambda \sin^4 z$ ", *Brux.*, 52A, 306–313.

33a. " Expression nouvelle d'une fonction harmonique positive dans une aire et nulle en tout point du bord sauf un ", *Brux.*, 53A, 113–122.

33b. " Sur l'extension de la méthode du balayage à une aire connexe non étalée ", *Bull. sci. de l'Acad. de Belgique* (5) 19, 1217–1229.

33c. " Propriétés des fonctions harmoniques dans un domaine ouvert limité par des surfaces à courbure bornée ", *Annali Pisa* (2) 2, 167–197.

34a. " Détermination des fonctions harmoniques dans une aire A et qui s'annulent en tout point de bord sauf un ", *Brux.*, 54A, 55–67.

38a. " Points irréguliers. Détermination des masses par les potentiels ", *Bull. sci. de l'Acad. de Belgique* (5) 24, 368–384 and 672–689.

38b. " Potentiel et problème généralisé de Dirichlet ", *Math. Gazette*, 22, 17–36.

50a. " Sur les extrémales...", *Pont Acad. Sci. Acta* 12 (1948), 141–159.

56a. " Le théorème de Picard au point de vue topologique ", *Brux.*, 69, 37–49 and 70, 81–86.

57a. " Fonctions périodiques douées de valeurs exceptionelles ", *Brux.*, 71, 73–88.