BALTHAZAR VAN DER POL

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Balthazar van der Pol died on October 6, 1959, at Wassenaar, Holland. He was a pioneer in the field of radio, and a well-known personality in the field of international telecommunications, but he also pursued the mathematical problems encountered in radio work so far that his work has formed the basis of much of the modern theory of non-linear oscillations, and he has given his name to the most typical equation of that theory.

Balth van der Pol was born at Utrecht on January 27, 1889; the son of Balthazar van der Pol and G. C. Steffens. He studied at the University of Utrecht from 1911 to 1916 and graduated 1916 *cum laude* in Physics. In that year he came to England to study under Professor J. A. Fleming at University College, London, and proceeded to Cambridge in 1917 to work at the Cavendish Laboratory under J. J. Thomson, where he met Appleton, with whom he had much in common at that stage. On June 22, 1917, in London he married Pietronetta Posthuma, who survives him; he had two daughters and a son.

On returning to Holland he submitted a thesis on "The effect of an ionised gas on electro-magnetic wave propagation and its application to radio, as demonstrated by glow-discharge measurement", for which he was awarded the degree of doctor of science *cum laude* by the University of Utrecht on April 27, 1920. From 1919 to 1922 he was assistant to Professor H. A. Lorentz in the research laboratory of Teyler's Institute, Haarlem, and in 1922 he was appointed head physicist in the research laboratory of N.V. Philips' works at Eindhoven, where he later became director of scientific radio research. In 1927 he was made a Knight of the Order of Oranje Nassau for establishing the first radio-telephonic communication between the Netherlands and the Dutch East Indies. He remained at Philips until 1949, and concurrently with his appointment at Philips he held the professorship of theoretical electricity at the Technical University, Delft, from 1938 to 1949. From 1945 to 1946 he was in addition President of the Temporary University at Eindhoven which was founded to replace other Netherlands universities in occupied territories, and for his work as president he was made a Knight of the Order of the Netherlands Lion in 1946.

From 1949 to 1956 he was Director of the Comité Consultatif International des Radiocommunications (C.C.I.R.) in Geneva, and as the permanent executive officer of the C.C.I.R. he was the technical adviser to the International Telecommunications Union (I.T.U.) on the planning and development of radio communications during the post-war years.

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After his retirement he held a temporary professorship at the University of California, Berkeley, for a year and then the Victor Emanuel Professorship at Cornell.

He was an outstanding figure in the various societies established to promote the study of radio. He became a member and then fellow and life member of the Institute of Radio Engineers (U.S.A.) in 1920; he was Vice-President of it in 1934, and was awarded its Medal of Honour in 1935 for contributions to circuit theory. He was a Founder-member of the Nederlandsch Radiogenootschap, for a time its President, and later an honorary member. From 1934 to 1952 he was Vice-President of the Union Radio Scientifique Internationale (U.R.S.I.) and from 1957 to 1959 he represented it on the Executive Board of the International Council of Scientific Unions. He travelled widely in connection with his work, attending plenary sessions of the U.R.S.I. or telecommunications' conferences nearly every year in various foreign countries.

He was an honorary member of the Institute of Radio Engineers of Australia. Many foreign academies and universities also honoured him; the Danish Academy of Technical Sciences awarded him the Valdemar Poulsen Gold Medal in 1953 for outstanding contributions to radiotechnics and particularly for research in the field of electro-magnetic oscillations and wave-propagation and for international scientific cooperation and organization of technical questions relating to radio communication; the Technical University of Warsaw gave him an honorary degree in 1956, and the University of Geneva in 1959, and the French Academy of Sciences made him a Corresponding Member in 1957.

He was well liked and respected by the vast number of friends with whom he came in contact throughout the world. He never spared himself in his devotion to the pursuit of knowledge and human understanding on a wide international basis, and though primarily a physicist he had a special feeling for the mathematical aspects of his own subject and for the theory of numbers in itself.

The scientific honours which came to van der Pol were, as has been stated, for his contribution to physics, and his best known mathematical work also had its roots in physical experiments. In the early days he did a good deal in collaboration with Sir Edward Appleton, who writes as follows :—

"Van der Pol's radio work was begun at Cambridge and was under two heads, experimental and theoretical. We can say that Heaviside had 'invented' the ionosphere to explain Marconi's remarkable transmission by radio over the Atlantic in 1901. In his theoretical work van der Pol gave a quantitative proof that, if one neglected the influence of a reflecting layer, experiment and theory were in flagrant disagreement in the case of long-distance propagation. To do this van der Pol familiarised himself with the theoretical work on the diffraction of radio waves round a conducting earth and used this to make a direct comparison between (*a*) signal strength predicted and (*b*) signal strength received, in a practical case of radio transmission. The paper in question is in the *Phil. Mag.* Sept. 1919 [3] and is worth consulting. (The ionosphere was not actually 'discovered', and located, until 1925, when Appleton and Barnett had finished their work.)

People had already begun to speculate as to the nature of the process by which radio waves were deviated by the 'invented' ionosphere, or Heaviside Layer, as it was then called. True reflection is due to a sharp discontinuity of properties—sharp within a wave-length that is. But the idea of ionic (or electronic) refraction was also mooted by Eccles (see van der Pol's *Phil. Mag.* paper) and this required that the refractive index of the ionised medium should be *less* than unity. In glass it is greater than unity. The relevant formula for an ionised medium is

$$K = \mu^2 = 1 - \frac{4\pi N e^2}{m p^2},$$

where K = dielectric constant, μ = refractive index. Van der Pol's experiment was to show that the dielectric constant of ionised gas (electric gas discharge tube) became less than unity as N increases; and ultimately can become negative with larger N. In this work van der Pol showed great experimental skill. He made his own apparatus, as was customary in those days of J. J. Thomson. His triode oscillator gave him waves of about 3 metres, which was about as short as anyone else had produced in those days. All this work on the dielectric constant of ionised air formed the basis of his Doctor's thesis.

It was round about this same period that van der Pol and I collaborated in the study of non-linear phenomena using triode circuits to check our theories. Together we worked at (*a*) oscillation hysteresis and (*b*) forced vibrations in a non-linear system. Van der Pol's work on relaxation oscillations was, of course, entirely his own."

Van der Pol's contributions to the mathematical theory of non-linear oscillations fall into four main groups all connected in some way with the differential equation of the triode oscillator in the form which now bears his name: viz.

(1) $$\ddot{x} - k(1-x^2)\dot{x} + x = 0, \quad k > 0,$$

or with a forcing term

(2) $$\ddot{x} - k(1-x^2)\dot{x} + x = bk\lambda \sin \lambda t.$$

In 1920 [5] he formulated the equations of the triode oscillator in the form

$$\frac{di}{dt}+C\frac{d^2v}{dt^2}+\frac{1}{R}\frac{dv}{dt}+\frac{1v}{L}=0, \quad i=\psi(kv). \tag{3}$$

In the experiments with which he was concerned the function $\psi(kv)$ appeared to be very nearly linear, but had a slight curvature so that he assumed

$$\psi(kv)=-\alpha v+\beta v^2+\gamma v^3, \quad \alpha>0, \quad \gamma>0. \tag{4}$$

The extraordinary feature of the experiments and of the equation (1) is that there is one and only one stable periodic motion of finite amplitude and all other solutions of the equation tend to it, except the trivial solution $x=0$, or $v=0$, as the case may be. This behaviour is quite unlike that of the solutions of any linear equation and remains true for (1) however small k is and however nearly sinusoidal v is in (3) provided that (4) holds. It is this phenomenon which has made the equation (1) so famous. Assuming that $v=a\sin\omega t$ and $\omega^2=1/(LC)$ approximately, van der Pol using a method given by Lorentz in lectures at Leiden showed that

$$a^2=\frac{4}{3\gamma}\left(\alpha-\frac{1}{R}\right), \tag{5}$$

and obtained the frequency correction

$$\omega^2=\frac{1}{LC}-\frac{a^2\beta^2}{3C^2}. \tag{6}$$

Since the approximations showed that the term βv^2 has little effect on the amplitude a, in most of the later work it was assumed that $\beta=0$. In this case by simple changes in the parameters and variables (3) can be reduced to the form (1) which will be used in what follows. Other forms in which $\psi(kv)$ is represented by a polynomial with 5 or more terms were considered by Appleton and van der Pol from the experimental point of view, and by Appleton and Greaves*, and by Greaves, from the strictly mathematical point of view*. In some of these there is more than one non-zero periodic solution.

A slowly varying amplitude had been used by Appleton and van der Pol [26], but the method of slowly varying amplitudes, sometimes called the method of van der Pol, was a further development by him for dealing with the case of forced oscillations in which the period of the forced oscillation is very near that of what corresponds to the free oscillation in the linear case; that is to say, when the equation for forced oscillations has been reduced to the form (2), it is the case in which k is small and

* See [18], where a complete account of the theory of non-linear oscillations up to 1934 is given with numerous references.

λ near 1. There are several interesting physical phenomena connected with this case when the parameters corresponding roughly to b and λ vary, in particular synchronisation, or the suppression of the free oscillation by the forced near resonance. Some of these with a brief mathematical treatment were discussed by Appleton*, but the method used by van der Pol in [8] and [16] formed the basis of a very great deal of later work, especially in the U.S.S.R. It consists in putting

$$x = a_1 \cos t + a_2 \sin t,$$

and assuming that a_1, a_2 vary slowly; in this way he obtained equations of the form

$$\dot{a}_1 = A_1(a_1, a_2), \quad \dot{a}_2 = A_2(a_1, a_2), \tag{7}$$

where A_1, A_2 are polynomials. Although van der Pol was able to deal with many of the phenomena observed, he pointed out some for which only a complete solution of his equations of the form (7) could establish a complete connection between the observed results and the mathematics. This point was taken up by Andronov and Witt* working under Mandelstam and Papalexi, who applied the topological methods of Poincaré and Bendixon to the particular equations involved, and thus initiated one of the main developments of the subject from the point of view of pure mathematics. The Russians, some† using amplitude and phase instead of separate amplitudes, developed the method analytically for more general equations with small non-linear terms, and gave a rigorous treatment of many of the results about periodic and almost periodic solutions, using the results of Fatou and Denjoy.

In his paper [9] on "Relaxation Oscillations" van der Pol was the first to discuss oscillations which are *not* nearly linear. He set out physical arguments, which in fact correspond closely to the real variable arguments, for the equation (1) to have, for *all* $k > 0$, a single stable periodic solution, and he obtained graphical solutions for $k = 1$ and $k = 10$, by means of the (x, y) or phase plane, where $y = \dot{x}$. The equation (1) is then written

$$y\frac{dy}{dx} = k(1-x^2)y - x,$$

and the periodic solution appears as a limit cycle of Poincaré, but van der Pol did not use the name limit cycle or refer to Poincaré. He also obtained the following relations for the periodic solution of (1) which hold for all k:

$$\int x^4\, dt = \int x^2\, dt = \int \dot{x}^2\, dt = \int x^2 \dot{x}^2\, dt,$$

* E. V. Appleton, *Proc. Cambridge Phil. Soc.*, 21 (1922), 231–.

* See A. Andronov and A. Witt, *Archiv für Electrotechnik*, 24 (1930), 99–110.

† See N. Kryloff and N. Bogoluiboff, "Introduction to non-linear Mechanics" (*Annals of Maths. Studies*, 1943, 87–99). and reference 16 given there.

where the integration is over the period. The last is obtained by multiplying by $\dot{x}$ and integrating, and the others by using x and $\int x$ instead of $\dot{x}$.

Finally van der Pol was also very much interested in the form of the periodic solution of (1) for k large, and obtained an approximation for the period $2k(\frac{3}{2}-\log 2)$, which is large for k large, and he discussed many physical and biological systems, including the human heart, having oscillations with similar features. The oscillation consists of rather square-headed wave, a slow descent from about $x=2$ to near $x=1$ followed by a rapid descent to about $x=-2$, and a corresponding half-wave from $x=-2$ to $x=2$ and the higher harmonics are by no means negligible. Many people consider that most of the corresponding physical phenomena are better represented by solutions of two differential equations of the first order connected by some other relations at the points $x=\pm 2$, $x=\pm 1$, where $\dot{x}$ is discontinuous. Van der Pol made suggestions about the equation (2) for k large and varying λ or b which led to most interesting developments in pure mathematics. Starting from the idea of a relaxation oscillation, van der Pol with van der Mark [27] constructed a system corresponding roughly to equation (2) with k large, and produced in it subharmonics of the forced oscillation $bk\lambda \sin \lambda t$. Moreover, by increasing and decreasing λ they found that two subharmonics with different periods could occur for the same values of the parameters. Although, as he explained to me later, the system was very unsymmetrical (which explained the subharmonics of even order which were observed), he was right in suggesting* that similar phenomena occurred for (2), and this was the starting point of much of the work of Littlewood† and myself including many topological results.

Van der Pol did much to popularize his subject; he was an engaging lecturer, and often took the opportunity of bringing together phenomena over a wide field of science which could be elucidated by a single mathematical relation such as the equation for relaxation oscillations. His summary [18] of work on non-linear oscillations up to 1934 was a quite masterly account of the theory up to that time. It gives many references, including some to Russian work, but it only seems to have been observed later that if $K<0$, $K'>0$ Rayleigh's equation‡

$$\ddot{\theta}+K\dot{\theta}+K'\dot{\theta}^3+n^2\theta = 0$$

can be reduced to (1) by differentiating and putting $nt=\tau$, $x=(3K'/|K|)^{\frac{1}{2}}\theta$. Rayleigh's treatment however does not go much further than a rough approximation for the amplitude.

* See [18], 1080, 1081.

† See J. E. Littlewood, *Acta Math.*, 97 (1957), 267–308, where further references will be found.

‡ Lord Rayleigh, *Phil. Mag.*, 15 (1883), 229–235.

The book by van der Pol and Bremmer on the Operational Calculus is an interesting treatment of one of the methods which are useful for linear differential equations, but the method is probably not the most popular one in use nowadays, at any rate in this country.

In his later years van der Pol became very keenly interested in the theory of numbers, and Rankin writes:—

"Of van der Pol's papers on the theory of numbers [2] is perhaps the best known. In it he combined his knowledge of radio technology and number theory to advantage. In order to investigate the behaviour of the Riemann zeta-function $\zeta(s)$ on the line $\mathrm{Re}\, s = \frac{1}{2}$ he derives the formula

$$\frac{\zeta(\frac{1}{2}+it)}{\frac{1}{2}+it} = \int_{-\infty}^{\infty} \{e^{-x}[e^x]-e^{\frac{1}{2}x}\}\, e^{-ixt}\, dx.$$

The 'saw-tooth' function within the braces was cut on the circumference of a paper disk and a beam of light was projected past the teeth on to a photocell. The electric current so produced eventually yielded a record, rather like an anemometer trace, of the modulus of $\zeta(\frac{1}{2}+it)$, from which the first 73 zeros could be read off with decreasing accuracy for increasing values of t.

The branch of number theory, however, which lay closest to his heart was the theory and applications of theta-functions. His published work on this subject is contained in four papers [**1**, **6**, **7**, **9**]; mention should also be made of his highly individual 'Lectures on a modern unified approach to elliptic functions and elliptic integrals' (mimeographed notes) given at Cornell University in 1958.

In [**6**] he gave an extremely neat, elegant and quite elementary proof of the relation

$$\theta_3^4 = \theta_0^4 + \theta_2^4$$

by dividing the lattice of integral points (m, n) into two sublattices according to the parity of $m+n$. The other three papers [**1**, **5**, **9**] are concerned with relations between powers of theta-functions and other modular forms such as Eisenstein series, the modular invariant and the modular discriminant; these relations include differential equations as well as algebraic equations. Perhaps the most important idea in these papers is the introduction of the function

$$M_k = \theta_0^{4k} + \theta_2^{4k} + (-\theta_3^4)^k,$$

which satisfies the recurrence relation

$$M_{k+3} - \tfrac{1}{2}M_2 M_{k+1} - \tfrac{1}{3}M_3 M_k = 0,$$

and in terms of which other modular forms can be simply expressed.

Mathematicians who attended the International Congress at

Amsterdam in 1954 will also remember van der Pol from the 'Gaussian prime tablecloths' displayed there. In two papers [7, 8] he studied the patterns made by the Gaussian primes and by primes in the quadratic field $k(\rho)$ generated by a primitive cube root ρ of unity. These patterns were obtained by marking the position of a prime number in the complex plane by a black square and black hexagon respectively. For $k(\rho)$ his diagram contains all primes with norm less than 10,000."

On the occasion of the award of the Valdemar Poulsen Gold Medal of the Danish Academy he delivered a lecture on "Radio Technology and the Theory of Numbers" [4] in which he describes eight problems from radio technology whose solution demands a knowledge of number theory.

From his early years van der Pol was a keen musician; he played the piano, violin and violoncello, and when he was a student he composed a little. He gave an extremely interesting lecture on mathematics and music at the St. Andrews Colloquium in 1955. He possessed absolute pitch to an astonishing degree of accuracy which he measured scientifically over a long period. He used to sing middle C each morning when entering his office and have it checked on some electrical machine. The standard deviation of his error was amazingly small, but I do not know the precise figure.

Bibliography.

Papers on Oscillations

The following list includes some papers mainly concerned with the physical phenomena of electric oscillations which were the starting point of his mathematical work on oscillations, and papers in which the mathematical interest predominates. There are also papers discussing a very wide variety of cases in physiology, botany and so on, in which oscillations similar to certain electric oscillations occur. It is not a complete list of his physical papers, and some of those in Dutch are duplicates of later English papers.

1. "The production and measurement of short continuous electro-magnetic waves", *Phil. Mag.* (6), 38 (1919), 90–97.
2. "A method of measuring without electrodes the conductivity at various points along a glow discharge and in flames", *Phil. Mag.* (6), 38 (1919), 352–364.
3. "On the propagation of electromagnetic waves round the earth", *Phil. Mag.* (6), 38 (1919), 365–382, and (6), 40 (1920), 163.
4. "Over de amplitude van vrije en gedwongen triode-trillingen", *Tijdschr. Ned. Rad. Gen.*, 1 (1920), 3–.
5. "A theory of the amplitude of free and forced triode vibrations", *Radio Review*, 1 (1920), 701–710, 754–762.
6. "Trillingshysteresis bij een triode-generator met twee graden van vrijheid", *Tijdschr. Ned. Rad. Gen.*, 2 (1931), 125–.
7. "On oscillation hysteresis in a triode generator with two degrees of freedom", *Phil. Mag.* (6), 43 (1922), 700–719.
8. "Gedwongen trillingen in een systeem met nietlineairen weerstand (Ontvangst met teruggekoppelde triode)", *Tijdschr. Ned. Rad. Gen.*, 2 (1924), 57–.
9. "On 'relaxation oscillations' I", *Phil. Mag.* (7), 2 (1926), 978–992.
10. "The correlation of some recent advances in wireless", *Exp. Wire*, 3 (1926), 338–343.
11. "Relaxatietrillingen I", *Physica*, 6 (1926), 154–157.
12. "Over 'Relaxatietrillingen' I", *Tijdschr. Ned. Rad. Gen.*, 3 (1926), 25–.

13. " Over ' Relaxatietrillingen ' II ", *Tijdschr. Ned. Rad. Gen.*, 3 (1927), 94–.

14. " Über Relaxationsschwingungen I ", *Jahr. der draht. Tel. und Tel.* (*Zeit. fur Hochfreq. Tech.*), 28 (1926), 178–.

15. " Über Relaxationsschwingungen II ", *Jahr. der draht. Tel. und Tel.* (*Zeit. fur Hochfreq. Tech.*), 29 (1927), 114–118.

16. " Forced oscillations in a circuit with nonlinear resistance (reception with reactive triode) ", *Phil. Mag.* (7), 3 (1927), 65–80.

17. " The effect of regeneration on the received signal strength ", *Proc. I.R.E.*, 17 (1929), 339–.

18. " The nonlinear theory of electric oscillations ", *Proc. I.R.E.*, 22 (1934), 1051–1086.

19. " On oscillations ", *Norsk Rikskringkastings Forelesninger* (1936), 243–275.

20. " Oscillations de relaxation et demultiplication de frequence ", *Actualités Sci. et Indust. Congrès du Palais de la Decouverte*, I (1937), 69–79.

21. " Beyond radio ", *Proc. World Radio Convention* (Sydney, 1938), 3–15.

22. " Heaviside's operational calculus ", *Inst. of Elec. Eng.*, Heaviside Centenary Vol. (1950), 70–75.

23. " Note sur les propriétés des solutions d'une equation differentielle, que l'on peut déduire directement de l'équation differentielle elle-meme ", *Des Actes du Colloque International des vibrations non linéaires Ile de Porquerolles* (1951).

24. " On a generalisation of the non-linear differential equation

$$\frac{d^2 u}{dt^2} - \epsilon(1-u^2)\frac{du}{dt} + u = 0 \text{ ''},$$

Koninkl. Nederl. Akademie van Wetenschappen (A), 60 (1957), 477–480.

With E. V. Appleton :

25. " On the form of tree triode vibrations ", *Phil. Mag.* (6), 42 (1921), 201–220.

26. " On a type of oscillation-hysteresis in a simple triode generator ", *Phil. Mag.* (6) 43 (1922), 177–193.

With J. van der Mark :

27. " Frequency demultiplication ", *Nature*, 120 (1927), 363.

28. " The heartbeat considered as a relaxation oscillation and an electrical model of the heart ", *Phil. Mag.* (7), 6 (1928), 763–775.

29. " Le battement du coeur considéré comme oscillation de relaxation et un modèle electrique du coeur ", *L'Onde Electrique*, 7 (1928), 365–; *Ann. des P.T.T.*, 17 (1928), 1041.

30. " De hartslag als relaxatietrilling en een electrische model van het hart ", *Gen. bevord nat.-, genees- en heelk. A'dam*, 12 (1928), 614–.

31. " The heartbeat considered as a relaxation oscillation and an electrical model of the heart ", *Extr. d. arch. neerl. de physiol. de l'homme et des animaux*, 14 (1929), 418–.

32. " The production of sinusoidal oscillations with a time period determined by a relaxation time ", *Physica*, 1 (1934), 437–448.

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33. " Gittergleichrichtung ", *Zeitschrift fur Hochfrequenztechnik*, 29 (1927), 139–147.

With H. Bremmer :

34. " Modern operational calculus based on the two-sided Laplace integral ", *Proc. Koninklijke Nederlandsche Akademie van Wetenschappe*, 51 (1948), I and II, 1005–1012 and 1125–1136.

35. *Operational calculus* (Cambridge University Press, 1950).

Papers on the Theory of Numbers and allied topics

1. " On a non-linear partial differential equation satisfied by the logarithm of the Jacobian theta-functions, with arithmetical applications ", *Proc. Koninklijke Nederlandse Akademie van Wetenschappen* A, 54 (1951), I and II, 261–271 and 272–284.

2. " An electro-mechanical investigation of the Riemann Zeta function in the critica strip ", *Bulletin Amer. Math. Soc.*, 53 (1947), 976–981.

3. " Note on the gamma function ", *Canadian Jour. of Math.*, 6 (1953), 18–22.

4. " Radio Technology and the Theory of Numbers ", *Jour. of the Franklin Inst.*, 255 (1953), 475–495.

5. " The representation of numbers as sums of eight, sixteen and twenty-four squares ", *Koninkl. Nederl. Akademie van Wetenschappen Proc.* A, 57 and *Indag. Math.*, 16 (1954), 349–361.

6. " Démonstration élémentaire de la relation $\theta_3^4 = \theta_0^4 + \theta_2^4$ entre les différentes fonctions de Jacobi ", *L'Enseignement mathematique*, 1 (1955), 259–262.

7. " Merkwaardige eigenschappen van geheele getallen ", *Verslagen van der Maatschappij Diligentia*, 's Gravenhage, 23 (1946).

With P. Speziali :

8. " The primes in $k(\rho)$ ", *Koninklijke Nederlandse Akademie van Wetenschappen Proc.* A, 54 (1951), 9–13.

With J. Tonchard :

9. " Equations différentielles linéaires vérifiées par certaines fonctions modulaires elliptiques ", *Koninkl. Nederl. Akademie van Wetenschappen Proc.* A, 59, and *Indag. Math.*, 18 (1956), 166–169.

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1. " On series of the reciprocals of the Jacobian theta functions ", *Proc. Koninkl. Nederl. Akademie van Wetenschappen*, Series A, 63, and *Indag. Math.*, 22 (1960), 107–114.

With A. H. M. Levelt :

2. " On the propagation of a discontinuous electro-magnetic wave ", *Proc. Koninkl. Nederl. Akademie van Wetenschappen*, Series A, 63, and *Indag. Math.*, 22 (1960), 254–265.