OBITUARY

IVAN MATVEEVICH VINOGRADOV

Ivan Matveevich Vinogradov was born on 14 September (New Style) 1891. His father Matveĭ Avraam'evich was priest at the graveyard church (*pogost*) of the village of Milolyub in the Velikie Luki district of Pskov province in western Russia. His mother was a teacher. He early showed an aptitude for drawing and, instead of an ecclesiastical school (as would have been normal for a son of the clergy), his parents sent him in 1903 to the modern school (*real'noe uchilishche*: that is, with a scientific as opposed to a classical orientation) in Velikie Luki, whither his father had moved with his family on his translation to the Church of the Holy Shroud (*Pokrovskaya Tserkov'*) there.

In 1910, on completing school, Vinogradov entered the mathematical section of the Physico-mathematical Faculty of the University at the Imperial capital, St. Petersburg. Amongst the staff were A. A. Markov, whose lectures on probability he is said to have known by heart, and Ya. V. Uspenskiĭ (= J. V. Uspensky, later of Stanford University, U.S.A.), both with interests in number theory and probability theory. There had been a long tradition in these subjects (Chebyshëv in both, Korkin, Zolotarëv, Voronoĭ in number theory). Vinogradov was attracted to number theory, and showed such ability that on completing the course in 1914 he was retained at the university for training as an academic. He successfully completed the extensive Master's examination and in 1915, on the initiative of V. A. Steklov, was awarded a bursary. His earliest work, under the direction of Uspensky, was on quadratic residues. He was asked to find a simple proof of the law of quadratic reciprocity but instead, possibly motivated by probabilistic considerations, looked at the distribution of quadratic residues to a given prime in short intervals. He obtained an estimate for the error term which, apart from the constant, remains the best unconditional estimate known. (It was found independently by Pólya in 1918, see §6 below.) He went on to generalize a method which had been used by Voronoĭ for 'Dirichlet's divisor problem' to obtain estimates for the number of integral points between an interval of the x-axis and the curve $y = f(x)$ when the second derivative $f''(x)$ is bounded: his error term was later shown by Jarník to be the best possible in the general case (see §5 below). He also used similar ideas to obtain bounds for the sum of $\exp(2\pi i f(x))$ over short intervals, a problem to which he was to recur.

During the war and in the immediate post-revolutionary period there was little communication with mathematicians in the West. News of Weyl's work on trigonometric sums and of that of Hardy and Littlewood on Waring's Problem, for example, did not reach Russian mathematicians. Similarly, Vinogradov's work was largely unknown outside Russia, though preprints were prepared of some of it and sent by Uspensky to Edmund Landau and a number of other foreign mathematicians.

The years 1918–20 were spent in the State University of Perm (later Molotov, in eastern european Russia), first as docent and then as professor. At the end of 1920,

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I. M. VINOGRADOV
1891–1983

Vinogradov returned to Petrograd to be simultaneously professor at the polytechnic and docent at the university. At the polytechnic he gave a course of higher mathematics on original lines, and at the university one on number theory, which was the basis of his well-known *Osnovy teorii chisel*. (The text has few original features but the exercises lead the reader step by step to a number of deep results, including those of the author mentioned above. See §8 below.) In 1925 he became professor at the university and leader of the section (*kafedra*) for number theory and probability. During this period his mathematical activity continued unabated and his work became known abroad. Landau in his influential three-volume *Vorlesungen über Zahlentheorie* (1927) gave prominence to 'die Winogradoffsche Methode'.

In 1929 Vinogradov was elected Academician. This marks the beginning of his activity as scientific organizer and administrator at national level. Together with S. I. Vavilov he worked out plans for a radical reorganization of the Physico-mathematical Institute of the Academy. In 1930–32 he was Director of the Demographic Institute and in 1932–34 of the mathematical section of the Physico-mathematical Institute (both in Leningrad).

In 1934 the Physico-mathematical Institute was split into the Lebedev Physical Institute and the Steklov Mathematical Institute. Vinogradov was the first director of the latter and remained director until his death. In the meantime the Academy moved its seat from Leningrad to Moscow, and the Steklov Institute was established in its familiar building in Vavilov Street. A Leningrad branch was set up later.

Simultaneously with this administrative activity, Vinogradov was creating important mathematics, which will be described in the second part of this memoir. He was mathematically active almost to the end of his life, in particular in refining and finding applications of his 'method of trigonometric sums'.

With the passage of time, Vinogradov accumulated further administrative responsibilities. From 1950 he was chief editor of the mathematical section of the Academy's *Izvestiya* and from 1958 he presided over the National Committee of Soviet Mathematicians. Within the Steklov Institute one has the impression that he remained firmly in charge to the end. He may have spent more time in his dacha as he grew older, but important decisions (and some not so important) had to be referred to him. As do many eminent Soviet mathematicians, he took an interest in the teaching of mathematics at school, and up to his death was chairman of the USSR Commission on School Mathematics Reform.

Although only of medium height, he was physically extremely strong, and proud of it. His old teacher Uspensky would recall that on one occasion Vinogradov came into an office and said 'I am the strongest man in the world'. Another mathematician present burst into laughter until Vinogradov came up behind him, wrapped his arms around him, and started to squeeze him in a bear hug. Uspensky could hear the ribs cracking, and was able to free the victim only by taking a bronze inkstand and smashing it against Vinogradov's knuckles until the blood dripped on the floor. The other mathematician was A. S. Besicovitch, later of Trinity College, Cambridge and FRS — himself no weakling. Besi's version of the incident was rather different: 'Vinogradov entered and said "I am world's strongest mathematician". Then I said "No". Then Vinogradov squoze me and he squoze me and he squoze me until I agreed "You are world's strongest mathematician"'. There may have been a play of words in the phrase 'strongest mathematician'. Dame Mary Cartwright recalls that, when she visited the Soviet Union at the end of the second world war, a talking point was that Vinogradov could lift the taller Lavrentiev. He retained his robust health until the end. Even in his 90th year he would scorn the lift and make his way with rapid strides to his office in the Institute. He died after a brief illness in his 92nd year on 20 March, 1983.

In conversation Vinogradov would usually employ only Russian and gave the impression that he did not speak any other language. Professor Hayman recalls, however, that at the dinner for the I.M.U. meeting at Dubna (before the Moscow congress) his youngest daughter, then ten years old, sat next to Vinogradov, who talked to her 'very nicely in quite good English'. He left the Soviet Union only rarely, but on two of those occasions visited the U.K. — in 1946 for the Newton Jubilee celebrations of the Royal Society and in 1958 for the Edinburgh International Congress of Mathematicians.

Professor K. Chandrasekharan, a former Secretary-General and President of the International Mathematical Union, has made the following comments:

> 'Vinogradov headed the U.S.S.R. delegation to the third General Assembly of IMU at St. Andrews, which was held just before the International Congress of Mathematicians at Edinburgh in 1958. The U.S.S.R. along with many other countries in Eastern Europe had joined IMU the year before, partly as a result of the combined efforts of Hopf, Kuratowski and Paul Alexandroff. At St. Andrews the constitution of IMU was considerably changed to accommodate the views of the enlarged membership. Vinogradov headed the Soviet delegation again at the fifth General Assembly at Dubna in 1966.'
>
> 'He and I used to converse in English. We have met for long hours, sometimes discussing mathematics and mathematicians, at other times about other things. I had no difficulty in understanding his English, and his responses showed that he understood what I said. I know that he could read and understand German just as well, though he never particularly wanted to speak German.'
>
> 'He was a marvellous and meticulous host. You might recall that he personally marked all the invitation cards to the dinner he gave (at his own expense) at the Sovietskaya in 1971. No one who has been at his home as a guest can forget his bountiful hospitality.'

(The last reference is to a banquet for the participants in the international colloquium held in Moscow in honour of Vinogradov's 80th birthday.)

Among numerous honours from the Soviet government were Hero of the Soviet Union (twice), Order of Lenin (five times), Order of the Hammer and Sickle (twice), Order of the October Revolution, and both the Stalin (now State) and the Lenin Prize. The Soviet Academy accorded him its highest honour, the Lomonosov Gold Medal. He was elected Foreign Member of the Royal Society in 1942 and became an Honorary Member of the London Mathematical Society in 1939. He was also an honorary member of many other academies and societies in the USSR and abroad.

Acknowledgments. Uspensky's version of the 'squeezing' episode is taken from G. M. Petersen, *Obiter Dicta* (Christchurch, New Zealand, 1983). Petersen states that Besicovitch confirmed the details to him in conversation. We are indebted to Dr Harry Burkill for drawing our attention to this account and to Professor Walter Hayman, FRS for Besicovitch's version and other comments. We should like also to thank Dr Geoffrey Howson, Dr Frank Smithies, Professor K. Chandrasekharan and Dame Mary Cartwright, FRS, for their help. Use has been made of biographical material in B. N. Delone, *Peterburgskaya Schkola Teorii Chisel* (Izdatel'stvo Akad. Nauk, 1947) and in articles in *Uspekhi Mat. Nauk* 17.2 (March 1962) 201–204, *ibid.* 36.6 (November 1981), 1–20 and *Izvestiya Akad. Nauk* (*ser. mat.*) 47 (1983) 691–706, which appeared on the occasion of Vinogradov's 70th birthday, 90th birthday and death respectively.

VINOGRADOV'S RESEARCH

1. *Introduction*

Vinogradov's great forte was in the use of trigonometric sums in attacking stubborn problems in analytic number theory. The general importance of trigonometrical sums in the theory of numbers was first shown in a seminal paper by Weyl [33]. Weyl showed that a sequence (a_n) of real numbers is uniformly distributed modulo one if and only if for each natural number m the trigonometrical sum

$$S_N = \frac{1}{N} \sum_{n=1}^{N} e(ma_n)$$

where $e(z) = \exp(2\pi i z)$, tends to 0 as $N \to \infty$. Weyl also introduced a fundamental technique for showing that $S_N \to 0$ when a_n is a polynomial in n whose leading coefficient is irrational. In the 1920s, Hardy and Littlewood, and van der Corput, extended and developed Weyl's methods to attack other problems in analytic number theory. However, it was Vinogradov who, in a series of important papers in the 1930s, brought the method of trigonometrical sums to full fruition. He introduced and developed two fundamental methods, which could be briefly described as 'the bilinear form technique' and 'the mean value theorem'. They have enabled great progress to be made on a whole range of problems. For example, in what is probably his most celebrated piece of work [1937, 5], he was able to combine the bilinear form technique with the Hardy–Littlewood method so as to reduce the Goldbach ternary problem to that of checking a finite number of cases. Moreover, these methods are still the best line of attack we have for tackling many of the problems to which they were first applied, and the recent researches of Vaughan [29, 30], Heath-Brown [11, 12] and others have demonstrated that they are not yet exhausted either in depth or in scope.

The bulk of Vinogradov's output is concerned with the development of these methods. However, early in his career Vinogradov made important contributions to the theory of the distribution of power residues, non-residues, indices and primitive roots, the basic underlying inequality concerning Dirichlet characters being discovered independently by Pólya [25] and Vinogradov [1918, 1] during the first world war. Also, Vinogradov several times came back to his very first interest, the error term in the asymptotic formula discovered by Gauss for the mean value of the class number of purely imaginary quadratic fields. His final work on this subject is still definitive.

2. *The bilinear form technique*

This technique can be outlined as follows. The problem under consideration is made to depend on a non-trivial estimate for a bilinear form of the kind

$$F = \sum_m \sum_n x_m \, y_n \, f(m,n), \tag{1}$$

where in any particular application the x_m, y_n are special arithmetical functions whose l^2 means can be quite accurately estimated, where $f(m,n)$ is an oscillatory function such as $e(\alpha m^k n^l)$ with α a real number, and where the double sum is over a set A of lattice points in a region of the kind

$$A = \{(m,n): M < m \leqslant 2M,\ N < n \leqslant 2N,\ m^k n^l \leqslant P\}.$$

If one takes $f(m,n)$ to be 0 outside the region A, then (1) can be written in the form $F = X\mathfrak{F}Y^T$, where $X = (x_m)_{M+1}^{2M}$, $Y = (y_n)_{N+1}^{2N}$, $\mathfrak{F} = (f(m,n))$, and a suitable bound for F can be obtained in many cases of interest from the observation that

$$|F|^2 \leqslant |X|^2 |Y|^2 \lambda,$$

where λ is the largest eigenvalue of the Hermitian matrix $\mathfrak{F}\mathfrak{F}^*$. It is well known that the eigenvalues of a Hermitian matrix (a_{rs}) are bounded by the maximum of $\sum_s |a_{rs}|$. Thus

$$|F|^2 \leqslant |X|^2 |Y|^2 \max_r \sum_s \left|\sum_m f(m,r)\bar{f}(m,s)\right|. \tag{2}$$

Hitherto the most powerful techniques had been dependent on a more detailed knowledge of the x_m and y_n. The greater flexibility permitted by (2) in the choice of x_m and y_n enabled Vinogradov to make inroads into a variety of problems. However the inequality (2) is only likely to be of value if the $M \times N$ matrix $\mathfrak{F}$ is good in the sense that the rectangle $[M+1, 2M] \times [N+1, 2N]$ is not too thin. Thus there is something of an art in finding a means of relating the original problem to a bilinear form or bilinear forms in which $\mathfrak{F}$ is not too thin.

The earliest application of the method [1934, 6] is to show in combination with the Hardy–Littlewood method that the function $G(k)$ in Waring's problem, namely the least s such that every large natural number is a sum of at most s k-th powers of natural numbers, satisfies $G(k) < 32(k \log k)^2$ when $k \geqslant 3$. This can be compared with the earlier bound (which we state in a slightly simplified form)

$$G(k) \leqslant (k-2)\, 2^{k-2} + k^2 \log 2 + k + 5$$

of Hardy and Littlewood [**9**]. The bound was further rapidly reduced [1934, 9, 12; 1935, 10, 12] to $G(k) \leqslant 6k \log k + 3k \log 6 + 4k$ for $k \geqslant 14$ and to $G(k) \leqslant 2k^2 \log 2 + (2 - \log 16)\, k$ for smaller values of k [1934, 10; 1935, 8]. Vinogradov several times returned to this problem [1936, 12; 1938, 11; 1959, 2], and the last of these papers, which incidentally uses an ingenious combination of the bilinear form technique and the mean value theorem, gives the bound

$$\varlimsup_{k \to \infty} \frac{G(k)}{k \log k} \leqslant 2$$

which has never been improved.

In [1937, 2, 5] Vinogradov discovered a way of applying the method to the estimation of sums of the form

$$\sum_{p \leqslant P} e(\alpha\pi) \tag{3}$$

where α is a real number and p is used to indicate that the sum is over prime numbers. In particular this gave a means of establishing an earlier conditional theorem of Hardy and Littlewood [**8**] that every large odd natural number is the sum of three primes. In the earlier work on Waring's problem the argument is constructed around an appropriate good bilinear form (1). However, in this work there is a further difficulty. Vinogradov first uses the sieve of Eratosthenes to relate (3) to a bilinear form F, but unfortunately F is not good in the sense described above and it is only with considerable ingenuity that Vinogradov manages to relate F to other bilinear forms that are good. There followed a further series of papers [1938, 5; 1939, 1, 3, 4; 1940, 2; 1942, 3; 1948, 1] in which the estimates for (3) are sharpened and applied to questions concerning the distribution of αp modulo one.

A variant of this method was also developed in another series of papers [1938, 8, 9; 1950, 3; 1952, 1; 1953, 3; 1966, 1] so as to treat the sum

$$\sum_{p \leqslant P} \chi(p+k),$$

where χ is a Dirichlet character to a prime modulus, and to give information regarding the distribution of $\operatorname{ind}(p+k)$ for a given prime modulus. The paper [1950, 3] also introduces a very efficient way of exhausting a rectangular hyperbola with rectangles that are not too thin for (2) to be applied. This has been used recently by Montgomery and Vaughan [**22**] in work in which, on the assumption of the generalized Riemann hypothesis, a best possible version of the Pólya–Vinogradov inequality [**25**; 1918, 1] for character sums is obtained.

In two papers [1941, 2; 1948, 2] Vinogradov indicates how the method for estimating (3) can be adapted to treat sums in which the summation variable is restricted by multiplicative constraints other than that of being prime. For instance, sums in which the number of prime factors of the summation variable is of fixed parity are considered. However, a more general treatment of such sums has occurred only fairly recently with the advent of a more systematic method (Vaughan [**29, 30**], Heath-Brown [**11, 12**]) of relating them to good bilinear forms of the kind (1).

Vinogradov [1951, 2; 1953, 2; 1954, 2] introduced a further variant of the method which avoids the use of trigonometric sums in discussing the distribution modulo one of sequences such as αp, where α is a real number, by instead treating directly the sum

$$\sum_{p \leqslant P} (K(\alpha p) - l)$$

where K is the characteristic function of a given subinterval, of length l, of the torus $\mathbb{R}\backslash\mathbb{Z}$.

3. *The mean value theorem*

Mordell [**23**] introduced an apparently highly specialized method for bounding complete trigonometrical sums

$$S(p; f) = S(p; \boldsymbol{a}) = \sum_{x=1}^{p} e(f(x)/p),$$

where $f(x) = \sum_{j=1}^{k} a_j x^j$ and p is a prime number. He considered the mean value

$$M = p^{-k} \sum_{\boldsymbol{a}} |S(p; \boldsymbol{a})|^{2b}$$

where the sum is over elements of $\mathbb{Z}_p^k$. The number M can be interpreted as the number of solutions of the system

$$\begin{aligned} x_1 + \ldots + x_b &= y_1 + \ldots + y_b \\ \vdots \quad & \quad \vdots \\ x_1^k + \ldots + x_b^k &= y_1^k + \ldots + y_b^k \end{aligned} \qquad (1)$$

with x_j and y_j in $\mathbb{Z}_p$. When $b = k$, it can be shown that the y_j are a permutation of the x_j and hence that $M \leqslant k!\, p^k$. By considering the effect of the linear transformation $x \mapsto ux + v$ on the polynomial f, Mordell showed that for a given polynomial f with $p \nmid a_k$ there are roughly p^2 other polynomials f modulo p for which $|S(p; f)|$ has the same value. This leads to a non-trivial estimate for $S(p; f)$. In 1934 Vinogradov [1934, 3] considered the case when the prime p is replaced by a composite number, and then shortly after there appeared a series of papers [1935, 4, 6, 11; 1936, 5–13] in which the process is adapted to estimate much more general sums.

Let $f(x) = \alpha_1 x + \ldots + \alpha_k x^k$ where the α_j are real numbers, and write

$$S(\boldsymbol{\alpha}) = S(f) = \sum_{x \leqslant N} e(f(x)).$$

In this context it is more appropriate to consider the continuous mean value

$$J(N) = \int_{\mathscr{U}_k} |S(\boldsymbol{\beta})|^{2b} \, d\boldsymbol{\beta}$$

where $\mathscr{U}_k$ denotes the unit hypercube $[0, 1]^k$. Now the mean value $J(N)$ is the number of solutions of the system (1) in natural numbers x_j, y_j not exceeding N. There is considerable difficulty both in satisfactorily relating a given sum to the mean and in estimating the number of solutions of (1) for appropriate values of b. The process devised by Vinogradov is technically much more complex and leads to less precise conclusions than in Mordell's much more special situation. However that anything could be done at all was a major achievement. There has been much work done on refining the process, and the most satisfactory method of estimating $J(N)$ so far discovered is that of Karatsuba [**17**].

Vinogradov applied this method to a number of different problems. In their work on Waring's problem, Hardy and Littlewood [**9**] had obtained an asymptotic formula for the number of representations of a number as the sum of s k-th powers provided that s is sufficiently large in terms of k. In [1935, 7] Vinogradov obtained a radical reduction in the required size of s, and further reductions occur in [1936, 7, 12; 1942, 2].

A natural area for the application of the method is to the distribution of the fractional parts of polynomials and this is done in the sequence [1936, 3, 4, 6, 9, 11, 13; 1961, 1, 2].

Perhaps the most important application of the method is in estimating the order of magnitude of the Riemann zeta-function $\zeta(s)$ in the critical strip. The bounds obtained for $\zeta(s)$ can be used, via function-theoretic techniques, to give information about the zero free region (and consequently about the error term in the prime number theorem). In this way, Chudakov [**4, 5**] was able to show that $\zeta(s)$ is free of zeros in the region

$$\sigma > 1 - C(\log \tau)^{-\gamma}, \quad s = \sigma + it, \quad \tau = |t| + 3,$$

where γ is any fixed number with $\frac{3}{4} < \gamma < 1$. This may be compared with the earliest inequality

$$\sigma > 1 - C(\log \log \tau)/(\log \tau)$$

of Littlewood [**21**] obtained via Weyl's technique. Later Vinogradov himself turned to this problem [1942, 2; 1958, 2, 3]. The last of these papers leads to the zero free region

$$\sigma > 1 - C(\log \tau)^{-2/3} (\log \log \tau)^{-1/3}, \quad s = \sigma + it, \quad \tau = |t| + 3, \tag{2}$$

which had been obtained independently by Korobov [**18, 19**] at about the same time by a somewhat different variant of the technique. Elegant accounts of these methods can be found in Walfisz [**31**] and Richert [**26**]. The zero free region (2) for the Riemann zeta function is still the best that is known.

4. *Hybrid methods*

In another long series of papers [1937, 4, 6; 1938, 3, 4, 6; 1939, 1; 1941, 1; 1946, 1; 1947, 1; 1956, 2; 1958, 1; 1959, 1] Vinogradov combines the bilinear form technique with the mean value theorem so as to obtain bounds for trigonometrical sums over primes p

$$\sum_{p \leqslant P} e(f(p)) \tag{1}$$

where f is a polynomial, or $f(x)$ is of the form Cx^{α} with α real. There is a good deal of flexibility in how the arguments can be arranged, and the recent papers of Ghosh [**7**] and Harman [**10**] indicate that the methods have not been exhausted yet. There are two obvious applications of estimates for sums of the kind (1). They can be combined with the Hardy–Littlewood method to show that for $s \geqslant s_0(k)$ every large natural number is the sum of at most s k-th powers of prime numbers (the Goldbach–Waring problem) and they can be used to study the distribution of the sequence $f(p)$ modulo one. This Vinogradov does in [1937, 3; 1938, 3; 1942, 1] and [1937, 3, 7; 1938, 3; 1940, 3] respectively. The work of extending the methods systematically to the Goldbach–Waring problem was taken on by Hua, and the results are summarized in Hua's monograph [**15**].

The most striking application of hybrid arguments is that alluded to above in connection with the function $G(k)$ in Waring's problem.

5. *Lattice point problems*

Vinogradov's earliest published research [1917, 1, 2] is on the asymptotic formula discovered by Gauss [**6**, Art. 302],

$$\sum_{d=1}^{n} h(-d) = \frac{4\pi}{21\zeta(3)} n^{3/2} - \frac{2}{\pi^2} n + R(n)$$

where $h(-d)$ is the class number of the purely imaginary quadratic field $\mathbb{Q}(\surd(-d))$. He shows that the error term $R(n)$ satisfies $R(n) = O(n^{5/6}(\log n)^{2/3})$. The problem is very similar to that of estimating the number of lattice points in a large sphere, and Vinogradov bases his argument on the theorem, proved by elementary methods, that if the function $f(x)$ satisfies

$$\frac{1}{A} \leqslant f''(x) \leqslant \frac{C}{A}$$

on the interval $[a, b]$, where C is a constant, then

$$\sum_{a<n\leqslant b} \{f(n)\} = \tfrac{1}{2}(b-a) + O\left(\left(\frac{b-a}{A}+1\right)(A \log A)^{2/3}\right).$$

This can be interpreted as a statement about the number of lattice points under the curve $y = f(x)$, and Jarnik [**16**] showed that the theorem is essentially best possible.

Vinogradov returned [1927, 4; 1934, 10; 1935, 9; 1949, 1; 1955, 1; 1960, 1; 1963, 1, 2] to this problem throughout his career, and his last paper on this subject establishes the estimate $R(n) = O(n^{2/3}(\log n)^6)$, which has never been improved.

6. *The Pólya–Vinogradov inequality*

Much of Vinogradov's early work on quadratic residues and non-residues [1918, 1, 2], residues and non-residues of higher powers [1926, 1, 5; 1927, 1, 5], and indices and primitive roots [1925, 1, 2; 1926, 3] is based on the inequality

$$\left| \sum_{n=1}^{M} \chi(n) \right| < q^{1/2} \log q \tag{1}$$

where χ is a non-principal Dirichlet character modulo q. This first appears in a paper [1918, 1] on the average value of the class number of real quadratic fields, where a proof of (1) is given using 'finite Fourier series'. In [1918, 4] Vinogradov establishes (1) in the case when χ is the Legendre symbol by using instead classical Fourier series. Quite independently Pólya [**25**] had discovered the same inequality and established it by the latter approach, and then Schur [**28**] had immediately discovered the 'finite Fourier series' proof. The inequality (1) is now known as the Pólya–Vinogradov inequality. It gives a non-trivial estimate for the character sum whenever $M < q^{1/2}\log q$ and is quite close to being best possible (see [**22, 24**]).

In [1919, 1] Vinogradov introduced the following argument. Let G denote the least positive quadratic non-residue modulo p, and suppose that $H < G^2$. Let χ denote the Legendre symbol modulo p. When $G \leqslant n \leqslant H$, $\chi(n) = -1$ if and only if n is divisible by exactly one prime q with $G \leqslant q \leqslant H$ and $\chi(q) = -1$. Thus

$$\sum_{n \leqslant H} \chi(n) = H - 2 \sum_{\substack{G \leqslant q \leqslant H \\ \chi(q) = -1}} [H/q].$$

Suppose further that H is large and the sum on the left is small compared with H. Then crudely estimating the sum on the right by prime number theory gives an inequality of the form

$$H(1-\varepsilon) \leqslant 2H \log \frac{\log H}{\log G},$$

whence

$$G \leqslant H^{\lambda} \quad \text{with} \quad \lambda = \exp\left(-\frac{1-\varepsilon}{2}\right).$$

The inequality (1) tells us that H just a bit larger than $p^{1/2}\log p$ is permissible. A slightly more careful analysis indicates that the choice $H = p^{1/2}\log^2 p$ is close to optimal, and one obtains

$$G = O(p^{1/2\sqrt{e}}(\log p)^{2/\sqrt{e}}).$$

Vinogradov made the plausible conjecture that for each $\varepsilon > 0$, $G = O(p^{\varepsilon})$. Burgess [**2**] has developed a very deep and difficult method that gives non-trivial estimates for (1) when M is as small as $p^{1/4+\varepsilon}$ and the technique described above then gives $G = O(p^{(1/4\sqrt{e})+\varepsilon})$, which is still the best that is known.

In a series of papers in the early thirties, Vinogradov used variants of the ideas underlying the proof of (1) to study the distribution of quadratic residues and non-residues [1933, 3; 1934, 4], of residues of higher powers [1933, 4; 1934, 2], and of indices and primitive roots [1930, 1; 1933, 3; 1934, 1, 5, 11]. However, much of this work has been superseded by that of Mordell [**23**], Weil [**32**] and Burgess [**2, 3**].

7. *Miscellaneous Research*

One early paper [1924, 1] contains a proof of Hilbert's theorem (1909) that for each natural number k there is a number $s(k)$ such that every natural number is the sum of at most $s(k)$ k-th powers of natural number (Waring's problem). Whilst the proof makes use of Weyl's technique of estimating exponential sums as developed by Hardy and Littlewood in their work on Waring's problem, and of Fourier series, it is otherwise quite elementary, and in particular makes no use of the Hardy–Littlewood circle method. Landau [**20**, VI Teil, Kap. 5] showed that the method could be used to give quite a decent upper bound for the function $g(k)$ in Waring's problem.

In his next work on additive number theory [1928, 1, 2; 1929, 1] Vinogradov introduced an important innovation. In their work on Waring's problem, Hardy and Littlewood considered the generating function

$$f(z) = \sum_{m=1}^{\infty} z^{m^k}$$

and its s-th power

$$f(z)^s = \sum_{n=0}^{\infty} R_s(n)\, z^n.$$

The coefficient $R_s(n)$ is the number of representations of n as the sum of s k-th powers. They applied Cauchy's integral formula

$$R_s(n) = \frac{1}{2\pi i}\int_{\mathscr{C}} f(z)^s z^{-n-1}\, dz$$

where $\mathscr{C}$ is a circle centre 0 of radius ρ, $0 < \rho < 1$ and found an alternative way of evaluating the integral asymptotically when $s \geqslant s_0(k)$ and n is sufficiently large. Vinogradov instead considers the finite sum

$$S(\alpha) = \sum_{m=1}^{N} e(\alpha m^k)$$

where $N = [n^{1/k}]$. Now

$$S(\alpha)^s = \sum_{m=1}^{sn} R_s(m, n)\, e(\alpha m)$$

where $R_s(m, n)$ is the number of representations of m as the sum of s k-th powers, none of which exceed n. In particular $R_s(m, n) = R_s(m)$ when $m \leqslant n$. Thus a special case of Cauchy's integral formula, namely the trivial orthogonality relation

$$\int_0^1 e(\alpha h)\, d\alpha = \begin{cases} 1 & \text{when } h = 0 \\ 0 & \text{when } h \neq 0 \end{cases}$$

gives

$$\int_0^1 S(\alpha)^s e(-\alpha n)\, d\alpha = R_s(n).$$

The analysis then proceeds in an analogous but simpler manner to that of Hardy and Littlewood. The simplification that results from the replacement of infinite series by finite sums is such that thereafter all applications of the Hardy–Littlewood method adopt this approach.

The series of papers [1926, 2, 4; 1927, 2, 3; 1935, 5] is concerned with the distribution of the fractional parts of polynomials with the leading coefficient irrational. The techniques are largely based on Weyl's method. Of particular interest is [1927, 2]. Let $\| \theta \|$ denote the distance of θ from a nearest integer. It is relatively easy to show that, for example, if α is irrational, then

$$\| \alpha n^k \| < n^{\varepsilon - 2^{1-k}}$$

for infinitely many natural numbers n. The question then arises as to whether the solutions of such an inequality can be localized in the following sense. Is it true that, for some $\gamma_k > 0$, for each large N there is an $n \leqslant N$ such that $\| \alpha n^k \| < N^{-\gamma_k}$? When $k = 1$ we know both from the classical theory of continued fractions and from an argument of Dirichlet that the answer is yes and we may take $\gamma_1 = 1$. In [1927, 2] Vinogradov resolved this question affirmatively for every k. Questions then arise concerning general polynomials and the best possible value for γ_k, and progress has been made in these directions by Heilbronn [**13**], Schmidt [**27**], Baker [**1**] and others. This is still an important ongoing area of research.

8. *Articles and monographs*

Vinogradov was assiduous in periodically producing accounts of his methods in the form of monographs [1937, 1; 1947, 2; 1971, 2; 1976, 1; 1980, 1] two of which have been translated into English [1954, 3; 1978, 1]. There have also been collections of selected works [1952, 2; 1953, 4; 1984, 1], the last two of which are in English.

Vinogradov wrote a textbook on elementary number theory which has gone through a large number of editions [1936, 1; 1938, 1; 1940, 1; 1944, 1; 1949, 2; 1953, 1; 1965, 2; 1971, 1] and been translated into English [1954, 1; 1955, 2] and German [1956, 1]. One of the contributors to this biographical memoir was highly stimulated as a student by the exercises, many of which are concerned with the estimation of trigonometrical sums.

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Department of Pure Mathematics
and Mathematical Statistics
16 Mill Lane
Cambridge CB2 1SB

J. W. S. CASSELS

Department of Mathematics
Huxley Building
Imperial College
180 Queen's Gate
London SW7 2BZ

R. C. VAUGHAN