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HERMANN WEYL*

M. H. A. NEWMAN.

Hermann Weyl was born on 9 November, 1885, the son of Ludwig and Anna Weyl, in the small town of Elmshorn near Hamburg. When his schooldays in Altona ended in 1904 he entered Göttingen University and there remained (except for a year at Munich), first as student and then as Privatdozent, until his call to Zurich in 1913.

In spite of the great variety of mathematical stimulation of the Göttingen years, this was the only period of comparable length in which he devoted himself to a single branch of mathematics—analysis, and to a single theme, the problems that arose naturally out of his dissertation, on singular integral equations. Towards the end of this period two causes combined to turn his attention to wider fields. First, in the session 1911-1912 he lectured on the theory of Riemann surfaces, and was led by his sense of the inadequacy of existing treatments to plunge deep into the topological foundations. Secondly, in 1913 he accepted the offer of a chair at the Institute of Technology in Zurich, where his colleague for one year was Einstein, who was just then discovering the general theory of relativity. Weyl was soon launched on the series of papers on relativity and differential geometry which culminated in the book *Raum-Zeit-Materie*.

* This notice is a shortened form of one, written in collaboration with H. Davenport, P. Hall, G. E. H. Reuter and L. Rosenfeld, which appeared in the *Biographical Memoirs* of the Royal Society (1957). A full bibliography will be found at the end of that Memoir.

Later still this work led on, through his analysis and generalization of the Lie-Helmholtz space-problem, to his third great theme, the representation theory of the classical groups, and its application to quantum theory. In the decade 1917-1927, he was at the height of his powers. A stream of papers appeared, not only on his main themes, but on any mathematical topics that interested him—and that meant in almost all parts of mathematics.

The years at Zurich were happy ones, during which, he says, the worst that happened to disturb his peace was a series of offers of chairs by foreign universities. He declined an invitation to succeed Klein at Göttingen in 1923, but a second invitation, to succeed Hilbert in 1930, he accepted, after still more painful hesitations. His short stay as professor in Göttingen was clouded over by the threat of coming political events. In 1933 he decided that he could not stay in Germany after the dismissal of his colleagues by the Nazis, and he accepted an offer of permanent membership of the Institute for Advanced Study, then newly founded in Princeton. There he worked as a member till his retirement in 1951, and he remained an emeritus member till his death in 1955, spending half his time there and half in Zurich. Of the Institute he said that it is the finest workshop for a mathematician that it is possible to imagine.

He married in 1913 Helene Joseph, the daughter of a doctor in Ribnitz in Mecklenburg, and there were two sons of the marriage. All who were visitors at the Weyls' house in Mercer Street will remember her charm and gaiety. She shared to the full his taste for philosophy and for imaginative and poetical literature, and was the translator of many Spanish works, including the writings of Ortega y Gasset, into German. She died in 1948.

In 1950 he married Ellen Bär, born Lohnstein, of Zurich, and from that time had the happiness of spending half of each year in Zurich. He died suddenly, of a heart attack, on 9 December, 1955.

The last public event of his life was the 70th birthday gathering, at which he was presented with a volume of " Selecta " from his own works. A wider circle of his friends had a last happy glimpse of him at the Amsterdam Congress in 1954, where he delivered the address on the work of the Fields Medallists (Kodaira and Serre), a *tour de force* which showed him, in his 69th year, well abreast of those new theories which have changed the face of mathematics in the last 20 years.

Few mathematicians have left so clear an impression of themselves in their work. His life-long interest in philosophical problems, and his conviction that they cannot be separated from the problems of science and mathematics, has left its mark everywhere in his work. In the last year of his life he wrote a brief philosophical autobiography, which he called " Erkenntnis und Besinnung ", a title which he explained in these words: " In the intellectual life of man there can be clearly distinguished two domains: the domain of *action* (Handelns), of shaping and construction to which active artists, scientists, technicians and statesmen devote themselves; and a domain of *reflexion* (Besinnung) of which the fulfilment lies in insight, and which, since we struggle in it to find the *meaning* of our activity, is to be regarded as the proper domain of the philosopher." The essay itself traces his philosophical progress from Kant through Husserl's "Phenomenology" and Fichte's Idealism to his discovery in 1922 of the medieval mystic Eckehart, who gave him for a time "that access to the religious world which I had lacked ten years earlier. . . . But my metaphysical-religious speculations, aroused by Fichte and Eckehart, never came to a clear conclusion; that was in the nature of things." He turned, under the stimulus of writing his book on the philosophy of science (1927) to the astringent pages of Leibniz. "Auf den metaphysischen Hochflug folgte die Ernüchterung."

In mathematical logic, too, things seemed a little less sure at the end of his life than at the beginning, but he held steadily to his view that postulation cannot replace construction without the loss of significance and value. This belief he held so seriously that he deliberately kept away throughout his life from those mathematical theories which make essential and systematic use of the Axiom of Choice.

The literary graces with which he liked to adorn his work gave it an unmistakable flavour but the mathematical form of his presentations was even more characteristic. His strong preference for arguments that stem from the central core of the problem, rather than verifications—even easy verifications—by computation, and his liking for pregnant verbal statements where others might use symbols, more easily seized by the mathematician's eye, sometimes made close demands on the reader's attention, but the reward was doubly great when the passage was understood. But his absorption in conceptual analyses and general theories never extinguished his zest for formal mathematical detail. He considered examples to be the life-blood of mathematics, and his books and papers are full of them. In the obituary of Hilbert he mentions with admiration the many examples by which Hilbert illustrated the fundamental theorems of his algebraic papers—"examples not constructed *ad hoc*, but genuine ones worth studying for their own sake!" The *dictum* of Hilbert on the subject, in *Mathematical Problems*: "The solution of problems steels the forces of the investigator: by them he discovers new methods and widens his horizon. One who searches for methods without a definite problem in view is likely to search in vain."—this he took as a precept in his own mathematical life.

He was indeed not only a great mathematician but a great mathematical writer. His style was leisurely by modern standards, but it had a wonderful richness of ideas. His discoveries will surely not only long survive as mathematics, but will be read in his own incomparable accounts of them.

Throughout his life Weyl wrote papers from time to time on topics in *analysis*, but the long series of papers (1908-1915) in which (following Hilbert's precept) he applied the new theory of integral equations to eigenvalue problems of differential equations establishes most clearly his stature as an analyst.

The papers written in this period fall into two groups. In the first (1909-1910) the theory of singular eigenvalue problems is developed for ordinary differential equations. In the classical Sturm-Liouville problem the equation

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)+\left(\lambda-q(x)\right)u=0 \qquad (1)$$

is considered on a finite interval with one boundary condition at each end. There is then a discrete sequence of eigenvalues and eigenfunctions, and any smooth function satisfying the boundary conditions can be expanded in a series of eigenfunctions. Weyl took up the "singular" case, where either the interval is semi-infinite, or $p(x)$ and $q(x)$ are allowed to become singular at one end. He proved the following alternative for the boundary condition at the singular end. Either the solutions are of integrable square for every λ, and thence a boundary condition must be imposed and the expansion takes the classical series form; or if for at least one λ not all solutions are of integrable square, no boundary condition can be imposed. In this case the expansion usually involves Stieltjes integrals with respect to the eigenvalue parameter λ, and the eigenfunctions occurring in the expansion may be "improper", *i.e.*, not of integrable square. Weyl hoped to extend this theory to partial differential equations but published nothing further on it. A satisfactory extension has indeed only been found quite recently, and the subject is very much alive at present. A second early group of papers in analysis (1911-1915) are on the asymptotic distribution of natural frequencies of oscillating continua, *e.g.* membranes, elastic bodies and electromagnetic waves in a cavity with reflecting walls.

Weyl's outstanding quality as an analyst is already shown by these early papers. In all of them the argument moves by clearly visible steps, each involving difficult work ranging from the estimate of the Green's function near the boundary to delicate questions that arise when the associated homogeneous integral equation has non-trivial solutions. A distinctive character is given to the work by the combination of a professional analyst's technical powers with the deliberate selection of problems of concrete physical interest. In this earliest phase of his work he already considered it a duty to use advances in analysis to solve the problems of natural science.

Apart from his work on almost periodic functions which is intimately connected with the "Peter-Weyl Theorem" (see later) the remaining papers which Weyl wrote from time to time on topics in analysis had more the character of occasional pieces, but all bore the mark of his immense skill as an analyst.

In the *theory of numbers* his papers were few in number, but of great influence. The memoir on the uniform distribution of numbers mod. 1, (1916) was of fundamental importance for almost all later work in the analytic theory of numbers. A preliminary note on the subject had appeared in 1914.

The definition of uniform distribution (Gleichverteilung) modulo 1 of a sequence $\alpha_1, \alpha_2, \ldots$ of real numbers is a very simple and natural one. For any γ between 0 and 1, let $A(N, \gamma)$ denote the number of those of $\alpha_1, \ldots, \alpha_N$ which have fractional parts between 0 and γ. Then the condition is that

$$\frac{A(N, \gamma)}{N} \to \gamma \tag{2}$$

as $N \to \infty$, for each γ. The basic theorem of Weyl's memoir is that a sequence is uniformly distributed if and only if, for each integer h other than 0,

$$\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h \alpha_n} \to 0 \tag{3}$$

as $N \to \infty$. This theorem is so elegant that it seems strange, on looking back, that it was not discovered earlier, especially as the proof is not particularly difficult.

The theorem that, if θ is any irrational number the sequence $\theta, 2\theta, 3\theta, \ldots$ is uniformly distributed modulo 1, which had already been proved in a variety of ways by others, is an immediate deduction from Weyl's theorem. For from the elementary estimate

$$\left| \sum_{n=1}^{N} e^{2i\pi h n\theta} \right| \leqslant |\sin \pi h\theta|^{-1}, \tag{4}$$

it follows that for each integer h the sum on the left is bounded independently of N, and the condition (3) is satisfied.

A more important deduction was that the sequence $f(1), f(2), \ldots$ is uniformly distributed modulo 1, if $f(x)$ is a polynomial:

$$f(x) = \theta x^k + \theta_1 x^{k-1} + \ldots + \theta_{k-1} x,$$

with at least one irrational coefficient. To establish this, Weyl developed a method of estimating exponential sums of the form

$$S = \sum_{n=1}^{N} e^{2\pi i f(n)}.$$

The resulting estimate, known as *Weyl's inequality*, played (in various forms) a vital part in later work of Hardy and Littlewood, van der Corput, Vinogradov and many others.

To express the inequality in a simple and explicit form, it is necessary to make some supposition about the irrational character of one of the coefficients in $f(x)$, and this was done by others, notably Hardy and Littlewood, Landau, and Vinogradov. If we suppose, for example, that

$$\left|\theta - \frac{a}{q}\right| < \frac{1}{q^2},$$

where a and q are relatively prime integers and

$$N^{1/k} < q < N^{1-1/k},$$

then $S = O(N^{1-1/k})$. This result played a vital part in Hardy and Littlewood's work on Waring's Problem, in which Weyl himself took a keen interest. It was also the essential tool for the estimation of the Riemann zeta-function, a task to which Weyl also contributed. It was not until 1936 that a more effective method of estimating exponential sums of the above kind was developed by Vinogradov, and even so his method gives an improvement only for large values of k.

Enough has been said to make clear the importance of the memoir of 1916, though only part of its contents have been indicated. It remains a paper which can be read and re-read with profit today.

Weyl returned to the theory of numbers in 1939-1940 with his book *Algebraic theory of numbers*, based on lectures he gave at Princeton in 1938-1939. This is a text-book, but a text-book written in a highly individual style and with a particular theme. Weyl contrasts the relative merits of the two principal methods of developing algebraic number theory: that based on *ideal theory*, due to Dedekind, and that based on *divisor theory*, due to Kronecker and Hensel. The two are equivalent in their effects, but Weyl gave a preference to the second, for reasons which he explains. It is interesting to observe that in spite of his many other interests Weyl still retained his mastery of the detail of a subject of such a formidably technical nature as algebraic number theory.

In 1913 there appeared *Die Idee der Riemannschen Fläche*. This book, in which Weyl revealed his full powers for the first time, marked the beginning of the widening of his mathematical interests. By its declared subject it belonged to analysis, and indeed it contained a masterly exposition of the classical theory of algebraic and analytic functions on Riemann surfaces, culminating in a proof of the uniformization theorem. But it was the plan, revolutionary at that time, of placing " geometrical " function theory on a basis of rigorous definition and proof, hitherto enjoyed only by the Weierstrass theory, that gave the book its unique character, and forced Weyl to plunge deep into the *topology* of manifolds. In his Lectures on Algebraic Functions of 1891-1892 Felix Klein had shown that the notion of a Riemann surface need not be tied to the multiply-sheeted coverings of a sphere to which Riemann had confined himself, but could be extended to include any surface provided with *local uniformizing variables* (conformal maps of the members of an open covering on to a circular domain). When Klein delivered his lectures there were no means available of giving exact form to these ideas: the lack of topological notions made it impossible even to define a Riemann surface precisely. In 1910 and 1911 L. E. J. Brouwer published his papers on the topology of simplicial manifolds. Weyl saw at once that here was the basis for an exact treatment of Klein's ideas, with Hilbert's proof of the Dirichlet Principle as the instrument for establishing the existence of differentials on the surface. To these ingredients, which, as might be expected, he modified and simplified to suit his purpose, he added others of his own. In order to prove, as he wished, that the "analytisches Gebilde" can itself be regarded as a surface, he needed a thoroughgoing axiomatic definition of a surface, which should make it clear that the "points" can be mathematical objects of any kind (in this case pairs of power series). The notion of a *neighbourhood-space*, as a set in which certain subsets are associated with each point as its neighbourhoods, had been introduced by Hilbert in 1902 (*Math. Ann.*, 56, reprinted as Anhang IV to *Grundlagen der Geometrie*), but his definition remained unused and almost unnoticed. Weyl revived and clarified Hilbert's definition, and showed for the first time how it could be applied. The conditions which make the restriction to manifolds were not separated, as they would be today, from the general topological axioms, but the notion of a *topology* as a designated family of subsets was clearly brought into view. He could now define a *surface* to be a (connected) triangulable 2-manifold, and a *Riemann surface* to be a surface on which, for each point p_0, certain complex-valued continuous functions are designated *regular at* p_0, again subject to suitable axioms. The *analytisches Gebilde* (a notion also here made exact for the first time) is a set of "function elements", f, *i.e.* pairs $(t^\mu P, t^\nu Q)$ where P and Q are power series in t ($P \neq$ constant) and μ, ν are integers. The process of direct continuation having been defined in a natural way (pp. 8 and 9) a topology is set up by taking as a neighbourhood of f_0 the set of all its direct continuations to points in $|t| < \epsilon$, for some $\epsilon > 0$. This topological space is clearly a 2-manifold with "local uniformizers". It is proved rather laboriously that it can be triangulated, and is, therefore, a Riemann surface. But it is more than that, since two meromorphic functions on the surface, determined by the elements $(t^\mu P, t^\nu Q)$ are given as part of the definition. The first main task is to prove, conversely, that on any given Riemann surface there can be constructed a pair of meromorphic functions which make it into an "analytisches Gebilde". It is for this purpose that a particularly elegant proof of the Dirichlet Principle is included (§12).

Another of the new ideas which Weyl brought to his task had to wait more than 20 years to be independently rediscovered in more general form by topologists. This was the isolation of the topological part of the proof of the duality between the differentials and the 1-cycles on the surface. The " curve-functions " introduced in §11 are 1-dimensional *co-chains* on the Riemann surface: the equation $F(\gamma) = f(p_2) - f(p_1)$ on p. 68 states precisely that F is the co-boundary of f, and shows that the symbol $F \sim 0$ has the meaning that is given to it in homology theory. The duality theorem, that the 1-dimensional connectivities derived from cycles and co-cycles are equal, is established in this section.

Still another substantial contribution made in this book to the topology of the subject is the treatment of the covering surface. This notion had been used by Poincaré, but only Weyl's exact definitions and proofs made clear what precisely are the parts played by the topological and the function-theoretic properties.

Weyl's interest in *general relativity*, and through it in *differential geometry*, began through his giving a course of lectures on the subject in Zurich after the departure of Einstein to Berlin—lectures which were the nucleus from which the book *Raum-Zeit-Materie* grew, through a series of revisions and expansions, to the great treatise of 1923 (5th edition). This book is too well known to need lengthy description. It gave Weyl his first opportunity to combine discussion of the philosophical questions in which he was so deeply interested with technical mathematics. On the mathematical side, it is distinguished, as might be expected, for the precision of the results. Nowhere else, for example, is there to be found so thorough and exact a discussion of the central orbit, finishing with rigorous inequalities for the maximal and minimal distances—a useful piece of information for discussion of the motion over long periods of time.

Weyl's own principal contribution to the subject was his " unified field theory " of gravitation and electricity—the beginning of the quest on which Einstein spent so many fruitless years. The two papers of Weyl on the subject (1918) have been more influential in differential geometry than in relativity theory. Weyl took up Levi-Civita's idea (1917) of the " parallel displacement " of a vector, but made the decisive innovation of freeing it entirely from dependence on a Riemann metric. An infinitesimal *affine structure* on a differentiable n-manifold is determined, in a given co-ordinate frame, by the choice of n^3 functions Γ^i_{jk} and the parallel vector $(\xi^i + d\xi^i)$ at $(x^i + dx^i)$ to (ξ^i) at (x^i) is then defined by

$$d\xi^i = -\Sigma\, \Gamma^i_{rs}\, \xi^r\, dx^s.$$

Upon this basis definitions of geodesics and of curvature can be constructed in the usual way. This was the starting point of the rapid development of projective differential geometry (" geometry of paths ") which took place, particularly in America under the leadership of O. Veblen, after the first world war. It also greatly clarified the geometrical theory of Lie groups, in the works of E. Cartan and others.

From relativity Weyl turned in 1923, by a natural transition, to the problem of finding the " inner reason " for the structure of general metric space, *i.e.* deducing the Riemann assumption of a metric based on a quadratic form from axioms about the group of " movements " in the space. For the classical constant-curvature spaces Helmholtz had characterized the group of movements as the smallest which allows free mobility, *i.e.* contains just one element which carries a point, a directed line through the point, and so on, into another arbitrary system of the same kind. He sketched a proof, which was made exact by Lie, that such a group coincides with the group of linear transformations leaving a quadratic form invariant. Weyl's problem was to formulate and prove a corresponding theorem for infinitesimal geometry. This he did by analyzing the meaning of the assumption that the metric, *i.e.* the group of movements, uniquely determines the affine connection, and he showed that this assumption and the conservation of volume suffice to characterize the group of infinitesimal rotations at a point as the set of linear transformations that leave a non-degenerate quadratic differential form invariant.

From the Lie-Helmholtz space-problem Weyl's attention soon moved to the general problem of the *representation of continuous groups.*

The three great papers on this subject which appeared in 1925-1926 in the *Mathematische Zeitschrift* were considered by Weyl himself to be his greatest single contribution to mathematics. They are concerned with the representations of a semisimple Lie group $\mathbf{a}$ by linear transformations of a finite dimensional vector space V over the complex field. Up to this time Lie groups had been considered almost exclusively from a local, and mainly from an infinitesimal, point of view. Weyl's work contains the first important contributions to the global study of Lie groups and, as such, has been the stimulus to numerous later investigations.

The essence of Weyl's method was to pass from the given semisimple group $\mathbf{a}$ to the universal covering group $\mathbf{a}_u$ of the adjoint group $\tilde{\mathbf{a}}_u$ associated with the compact real form of the complex Lie algebra $\mathbf{a}^\circ$ of $\mathbf{a}$, and to infer the properties of representations of $\mathbf{a}$ from those of $\mathbf{a}_u$ by algebraic arguments. The problem was thus reduced to a discussion of the representations of the simply connected and compact Lie group $\mathbf{a}_u$. For a compact group there is available the powerful operation of integrating over the group manifold—the natural analogue of summation over the elements of a finite group. Weyl established very simply the existence of an invariant volume element on any compact Lie group, with respect to which the volume of the whole group is finite. The notion of an invariant integration over a group had first been applied by Hurwitz to the rotation group and by Schur to the representations of the real orthogonal groups. Schur proved the complete reducibility of the representations and the orthogonality relations for the characters of these groups. Weyl extended Schur's results systematically to all the compact

Lie groups. More important still, by bringing these global calculations into relation with the fundamental results of E. Cartan on the representations of semisimple Lie algebras, he was able to unify the global and infinitesimal points of view.

To complete the analogy between finite groups and compact Lie groups it was necessary to prove the completeness of the system of orthogonal functions formed by the coefficients of the inequivalent irreducible unitary representations. This was done in 1927, in a paper written with F. Peter, by applying E. Schmidt's methods in the theory of integral equations to a discussion of the eigenfunctions of Hermitian kernels of the form

$$K(s, t) = \int x(sr^{-1})\,\bar{x}(tr^{-1})\,dr,$$

where x is a continuous function on this group.

The Peter-Weyl paper preceded by only a few years the construction by Haar in 1933 of a left-invariant measure on any (separable) locally compact topological group. For compact groups, the Haar measure is also right-invariant, and the methods of Peter and Weyl could be carried over with scarcely any change to this more general case. Their work marks a decisive forward step in group-analysis, and points the way to the theory of almost periodic functions on a group, due to von Neumann, and more distantly to the modern theory of representations of locally compact topological groups by unitary transformations of a Hilbert space.

In his book *The classical groups*, Weyl gave a connected account of the representations and invariants of these groups. His concept of an invariant is much wider than that which had dominated the theory of algebraic invariants in its heyday in the second half of the 19th century. If $x, y, \ldots$ are variable vectors chosen from representation spaces of a linear group $\mathfrak{g}$, then an invariant of $\mathfrak{g}$ is a polynomial $f(x, y, \ldots)$ in the coordinates of $x, y, \ldots$ such that $f(x, y, \ldots) = f(\sigma x, \sigma y, \ldots)$ for all σ in $\mathfrak{g}$. Relative invariants are defined in a similar way. In the classical period of invariant theory, one normally considered only the representation spaces provided by algebraic forms of a given degree in the coordinates of the space V on which $\mathfrak{g}$ acts. Weyl's point of view is not only more general, but is clearly the natural one to take today when, largely through his own efforts, the representation theory of the classical groups has been so adequately developed. The result is that Weyl has succeeded in bringing this attractive theory once more into the main stream of algebraic thought.

Weyl's work in *mathematical logic* began with his monograph *Das Kontinuum* (1918), and he never departed greatly from the position he there took up (though he published a more extended exposition in 1921). A characteristic opening paragraph declared his purpose. " In this little book I am not concerned to disguise the ' solid rock ' on which the house of analysis is built with a wooden platform of formalism, in order to talk the reader into believing at the end that this platform is the true foundation. What will be propounded is rather the view that the house is largely built on sand. I believe I can replace this shaky part of the foundation by strong and reliable supports, but they will not carry everything that is nowadays generally believed to be secure. The rest I abandon: I can see no other possibility."

The "sandy part" was the part of mathematics which (he said) involves a "vicious circle", namely the kind of definition called "impredicative" by Russell, who also saw here a danger to the stability of mathematics. As a measure of protection against the appearance of "semantic" paradoxes Russell had enunciated the principle: "No totality can contain members defined in terms of itself." This principle appears to be violated by the usual definition of the *least upper bound*, $\sup E$, of a set, E of real numbers, when these are defined as Dedekind "segments" (lower classes of Dedekind cuts); namely, $\sup E = \bigcup E$, the union of all the members of E. It is easily shown that $\bigcup E$, a set of rational numbers, is a segment. But its definition involves a reference to the "totality" of all real numbers, of which it is itself a member, since E is in general defined as the set of all real numbers having a certain property.

It was to give formal expression to his principle that Russell introduced the "ramified" theory of types, of which Weyl's *Kategorien* and *Stufen*, in *Das Kontinuum*, are a version. The proliferation of real numbers of different types to which this theory leads makes analysis quite intractable, and led Russell to the desperate expedient of the Axiom of Reducibility, which simply postulates that for every sentence with a single free variable, x, of level α, there exists a "first-order" sentence defining the same class, that is a sentence of the lowest type possible for sentences with the free variable x. Weyl rejected this way out of the difficulty. "Russell, in order to extricate himself from the affair, causes reason to commit hara-kiri, by postulating [the existence of an equivalent first order sentence] in spite of its lack of support by any evidence". (*Philosophy of Mathematics and the Natural Sciences*, p. 50.)

His own cure was the drastic one of allowing only "first order" definitions, and throwing away the parts of mathematics that failed to survive the purge. This means that bounded sets of real numbers need not have least upper bounds, and that we may not, in general, form the set $\mathbf{P}(E)$, of all subsets of a given set. He carried out in detail, in the book, the development of analysis as far as it would go on this basis. His set theory was of a genetic kind, only such sets being admitted as could be built up from "ground categories" by the use of allowed principles of construction. It thus resembled the Zermelo system (to which he refers) later developed by von Neumann; but in place of the more powerful Zermelo operations, such as formation of $\mathbf{P}(E)$, he uses only the combination of Boolean operations and quantification of type-0 variables, by means of a recursive iteration scheme. The theory of real numbers to which this leads is of the sort that has been made familiar by the Intuitionists. The extent of the sacrifice involved is much more accurately known at the present day. The movement of logic is now towards a re-interpretation of classical proofs in a constructive sense, rather than a policy of voluntarily jettisoning certain of the most powerful instruments of proof, which is unlikely to recommend itself to mathematicians in general. Nevertheless a return to the " naive " acceptance of the axioms of classical set-theory as self-evident truths, on which we can confidently build up mathematics, is now out of the question; and in this change of opinion Weyl's writings certainly played an important part. His advocacy of the intuitionist views and his clear and attractive expositions of them, in his papers and in the book mentioned above, first made them accessible to many mathematicians, and turned the revolutionary doctrine of his time into the orthodoxy of today.

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