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ALFRED YOUNG

1873–1940.

H. W. TURNBULL.

Alfred Young was born at Birchfield, Farnworth, near Widnes, Lancashire, on 16 April 1873. He died after a short illness on Sunday, 15 December 1940. He was the youngest son of Edward Young, a prosperous Liverpool merchant and a Justice of the Peace for the county. His father married twice and had a large family, eleven living to grow up. The two youngest sons of the two branches of the family rose to scientific distinction: Sydney Young, of the elder family, became a distinguished chemist of Owen's College, Manchester, University College, Bristol, and finally, for many years, of Trinity College, Dublin. He was elected Fellow of the Royal Society in his thirty-sixth year and died in 1937. Alfred, who was fifteen years his junior, was elected Fellow in 1934, at the age of sixty, in recognition of his mathematical contributions to the algebra of invariants and the theory of groups, a work to which he had devoted over ten years of academic life followed by thirty years of leisure during his duties as rector of a country parish. Recognition of his remarkable powers came late but swiftly; he was admitted to the Fellowship in the year when his name first came up for election.

In 1879 the family moved to Bournemouth, and in due course the younger brothers went to school and later to a tutor, under whom Alfred suffered for his brain power, being the only boy considered worth keeping in. Next, he went to Monkton Combe School, near Bath, and there again his unusual mathematical ability was recognised. Thence he gained a scholarship at Clare College, Cambridge, where he matriculated in 1892. At Clare he formed his life-long friendship with G. H. A. Wilson, another distinguished mathematician, who eventually became Master of the College. Young was a good oarsman and rowed in the Junior Trial Eights as a freshman and in the Scratch Fours of 1893. His college friends still remember him as a shy, clever lad with a great humility of spirit which so marked him in his youth and indeed throughout his life. Early in his third year at college his interest in research began, and his enthusiasm doubtless diverted him from the subjects laid down for the Tripos examination, for which he was prepared by the celebrated coach, Webb of St. John's College. In 1895 he graduated as tenth Wrangler, his friend Wilson being placed fifth. It was a brilliant year; Bromwich was Senior Wrangler, Grace and Whittaker were bracketed second, and thereafter followed Hopkinson, Godfrey, and Maclaurin twelfth Wrangler, to whom one of the three Smith's Prizes was subsequently awarded. Young, who according to Grace was the most original man of his year, would probably have occupied a higher place in the list had he directed his attention to the examination schedule; but in turning to his research, as Wilson tells us, undoubtedly he chose the better path. In the following year Young was placed in the Second Class of the Mathematical Tripos, Part II. From 1901 to 1905 he lectured at Selwyn College, Cambridge, but resigned that appointment shortly after his election to a Fellowship at his own college, where he was also Bursar until 1910. His work received recognition when he was approved for the degree of Sc.D. at Cambridge in 1908.

Young had always intended to take Holy Orders, but it was not until 1908 that he was ordained, when he accepted a curacy at Christ Church, Blacklands, Hastings. Two years later he was presented by his College to the living of Birdbrook, a village of Essex about twenty-five miles east of Cambridge and on the borders of Suffolk. There he lived and worked for thirty years, quietly and faithfully performing the duties of a parish priest, beloved by his congregation, who respected and wondered at his great scholastic gifts so modestly set forth. He was always a welcome visitor in their homes, and he conducted the services in the parish church with dignity and sincerity, and was an excellent preacher. He readily undertook the responsibilities of his calling further afield, was appointed in 1923 to be Rural Dean of Belchamp, and in 1929 Chaplain to the Bishop of Colchester. As recently as November 1940 he was installed as Honorary Canon of Chelmsford Cathedral. In 1926 he accepted an invitation from the University of Cambridge to give a course of lectures on Higher Algebra, and this he continued to do for several years during the Lent, and occasionally the May, Term. In 1931 he was awarded the Honorary LL.D. Degree of the University of St. Andrews.

In 1907 Alfred Young married Edith Clara, daughter of Mr. Edward Wilson, of Sheffield, by whom he is survived. There were no children of the marriage. Their home was a typical country rectory, set in an old world garden full of colour and of great charm, where a warm welcome awaited a visitor from Cambridge or elsewhere, young or old, who sought out in this secluded corner of Essex a master of abstract algebra, and found more than a mathematician, a friend. After a thirty-mile bicycle ride (these were the days before the motor bus had become ubiquitous) the shade and peace of the rectory garden on a summer day were greatly refreshing.

"He was charming to us", an undergraduate wrote after such a visit, "and I remember how delighted I was with the Rectory, and how he told us of the excellence of the beer brewed with the water of his pond". He was a practical mechanic, and had a device by which all the water in the house was pumped up with the help of a little motor engine which was run round to the pump each day. He had also successfully set up a small electric light plant to supply the house. His expert knowledge of its working seemed odd amid those rustic surroundings, till one recalled his interests in the more practical mathematics besides the theory of groups, and the paper he had published on the electromagnetic properties of coils. He was very methodical in planning his duties and his leisure; he was a willing correspondent and would take great pains to answer the mathematical queries of his friends, sharing with them liberally his own abundant thoughts on algebra, invariants and geometry. He could take up or lay aside and again, after several weeks, resume a formidable piece of algebraic computation, without apparently losing the threads of the arguments and with the utmost composure. Every year he and his wife would regularly take their holiday shortly after Easter at a South Coast resort, and every Tuesday they would go over to Cambridge and thus keep in touch with their University friends, a practice which doubtless started when Young returned to the lecture room at the call of the University. His quiet determination and his unhurried devotion to the things of the mind and of the spirit were very impressive. Of such might Whittier have been thinking when he wrote: "And let our ordered lives confess, The beauty of Thy peace". Young was intellectually alert to the end of his life, and during his last year he was constantly working at his ninth memoir, to which he attached great importance. It was nearly finished and lay on his desk awaiting the final touches. During the last few days of illness he asked his doctor whether he could hope to live to finish his work.

My friend W. L. Edge has supplied a picture of the lecture course when Young resumed his teaching at Cambridge: "I remember (who could forget?) very well my experiences of attending his first lectures. This was only a course of one lecture a week for one term; you can see for yourself how much he got through. . . . Doubtless it is all standard work to you, but it will be interesting to see how the old warrior entered the lists again and what he considered should be given to his first hearers. I went along on 19 January 1926, in my third year, just two terms before my Tripos, to Clare . . . there were eleven of us and I was the only undergraduate who ventured. Others in the class were, I think, Cooper, now at Belfast; Broadbent, now at Greenwich; L. H. Thomas, who got a Smith's Prize and a Trinity Fellowship and went to America; Dirac certainly, and F. P. White, the only M.A. And I remember the tall clerical figure entering the room, and his surprise at so large an audience, and shaking hands with White with obvious pleasure. And so to linear transformations and Aronhold's symbolic notation. . . . At the end of his last lecture in March, Young said that he was so pleased that people had turned up that he would lecture again in the following term. And he and I were both surprised, and I very embarrassed, when no other member of the class but myself showed up in April. It was my Tripos term, but I was not going to miss his lectures ! . . . One lecture fell during the General Strike, and no preparation of room, blackboard, chalk or anything had been made by the college. So Young sat down beside me and wrote out the notes with his own hand".

Young had a quiet humour. "I remember an occasion", writes Wilson, "when he said to me with a grave face: 'I have lost all sense of personal security'. It appeared that the maidservant at his Cambridge lodgings had used some of his manuscripts to light the fire rather than waste clean paper for that purpose".

Young wrote his first mathematical paper in 1899 and continued to write and to publish for over forty years. With the exception of his work on electromagnetism in 1918, every paper was devoted to one theme, the algebra of groups. It began with the algebraic theory of invariants, a subject which was first explicitly started a hundred years ago, in 1841, by Boole, and then developed by Cayley, Salmon and Sylvester, and later by Macmahon and Elliott. It provided the analytical aspect of geometrical projection and of those properties of a figure which remain unchanged for any such projection. This led, first of all, to the discovery of algebraic forms which were invariant for the corresponding linear transformations, and then to the search for the basic set of forms, out of which all other invariants of a given system of ground forms could be constructed. Such a form is the binary n-ic

$$f = \sum_{r=0}^{n} \binom{n}{r} a_r x_1^{n-r} x_2^r$$

and the study resolved itself into the theory of annihilators, that is, of certain differential operators linearly composed of terms such as

$$a_{r-1} \frac{\partial}{\partial a_r}.$$

At an early stage it was supposed that, for binary forms higher than the quartic, the invariant theory was essentially different from that for lower forms. Indeed, in his second *Memoir on Quantics* (1855), Cayley had stated his conclusion that whereas the number of different irreducible invariants and covariants for a quartic was finite, this was no longer true of the quintic. But this surmise was upset in 1869 when Gordan startled the mathematical world by proving the finiteness of such systems for a binary form of any order. Gordan followed this up with the publication of his *Programm* at Erlangen in 1875 which widened the scope of the finiteness to all systems of binary forms. Finally, the theory was extended to all higher types of form by Hilbert in 1890. The influence of Gordan's work was apparent in the successful use of generating functions by Sylvester and MacMahon, who made important advances both in detailed systems and in the general theory. A friendly rivalry sprang up between the English mathematicians and the Continental algebraists, Gordan with his followers, the former following the non-symbolic method, as it is called, and the latter the symbolic. The work of the English school is ably expounded by Elliott in his *Algebra of Quantics*, which brings us to the beginning of the twentieth century.

The development by the symbolic method, which went on in Germany, grew out of certain hyperdeterminants invented by Cayley. In this branch of the theory the coefficient a_r in the form f was regarded as a numerical multiple of $\partial^n f / \partial x_1^{n-r} \partial x_2^r$ or, let us say, $\partial_1^{n-r} \partial_2^r f$, where $\partial_1 = \partial / \partial x_1$, $\partial_2 = \partial / \partial x_2$. All such forms together with all their invariants and covariants were expressed in terms of the ∂_1, ∂_2 belonging to the variables x_1, x_2, and of analogous symbols for any further cogredient variables. Clebsch and Aronhold perfected the technique of these symbols. Thereupon Clebsch and Gordan used them systematically for invariant theory with conspicuous success. They proved that every rational integral invariant of binary forms could be expressed as a polynomial aggregate of symbols $\partial^2/\partial x_1 \partial y_2 - \partial^2/\partial x_2 \partial y_1$ (the hyperdeterminants of Cayley) and of $y_1(\partial/\partial x_1)+y_2(\partial/\partial x_2)$, the polar operators (the First Fundamental Theorem); that all invariantive properties could be deduced by means of certain specified determinantal identities which left these characteristic hyperdeterminantal and polar structures unimpaired (the Second Fundamental Theorem); that every such invariant could be expressed rationally and integrally in terms of a *finite* number of invariants (Gordan's Theorem, 1869); and they established an important expansion (the Clebsch-Gordan series) which enables one to deal with forms involving many sets of variables by means of forms in fewer variables and their polars.

These symbolic advances, together with the generating functions and perpetuants of MacMahon, opened up a wide field of enquiry, and it was this into which Young entered, in company with his friend J. H. Grace, soon after and possibly even before they took their degrees. Their interest was first aroused in 1895 by reading Meyer's newly published *Bericht über den gegenwartigen Stand der Invariantentheorie*, which opened up a vast world of algebra and gave them their first ideas of modern mathematics. They came to grips with the symbolic method, which fascinated them, and at once began to make important contributions to a subject hitherto very little known in England. With the publication of their treatise, the *Algebra of Invariants*, in 1903, a new era dawned for the teaching and progress of Higher Algebra. This excellent book, with its fine display of algebraic technique and geometrical insight, had a considerable influence on the younger geometers and algebraists at Cambridge, particularly in the decade preceding the last war. Both authors were masters of their subject, Grace as a geometer and Young as an algebraist. The pages of the book have a deceptively simple appearance, owing to the extraordinary compactness of the symbolic notation, where, for example, an invariant $a_0 a_4 - 4a_1 a_3 + 3a_2^2$ of a binary quartic appears in the guise $\frac{1}{2}(ab)^4$. The book ends with four lively appendices, bringing the latest results to the notice of the reader—and one almost expects to see a stop press column on the last page!

From the outset Young's rapid and skilful handling of symbolic algebra bore all the signs of genius. Grace likens him to Ramanujan, not only for what each achieved but for what each ignored. Young at once found his own solutions for the complete systems of the binary octavic and septimic forms: and, of his ingenious device for treating the octavic as the symbolic square of a quartic, Grace says "I could not have thought of that in fifty years". Young began his research in algebra by solving the problem of binary quartic types—the invariant theory of any number of quartics; and it was through the practice of the symbolic methods upon such problems that he was led to his first great discovery, which he called *Quantitative Substitutional Analysis.* He was dealing with functions of a finite number of variables; and these variables always occurred in sets, let us say a, b, c, ..., each of which could be manipulated as a single vector, or as a column of a determinant. In the course of the work innumerable varieties of alternative expressions were produced, many of which only differed among themselves by sign, or else by derangements of these vectors, in much the same way as a three-rowed determinant, $|a_1 b_2 c_3| = \Delta$, assumes two values $\pm\Delta$ and six alphabetical forms, when the letters are permuted without disturbing the suffix sequence. His functions were, of course, usually much more complicated than single determinants; nevertheless the determinant provided the clue to a general theory which comprehends all the details of the algebra. Now functions such as these are evidently closely connected with the theory of finite groups, and it became clear to Young that sets of functions, which at first sight were quite distinct, but which on examination proved to belong to the same group, could be dealt with by a single prescription if only the properties of that particular group were thoroughly known. Young therefore set himself to extricate the group properties of these functions, and accordingly he expressed the functions as well as the relations between such functions by means of a new kind of symbolic operator depending at once on a substitution group.

This operator, which consisted of two main ingredients N_i and P_i, can best be explained by a simple example: thus, if $f(a, b, c)$ is a function of three variables, then

$$f(a, b, c)+f(b, a, c) = Pf(a, b, c),$$

$$f(a, b, c)-f(b, a, c) = Nf(a, b, c),$$

where $P = 1+(ab)$, $N = 1-(ab)$ and (ab) denotes the operation of interchanging a with b in the function $f(a, b, c)$. For n letters such a P has $n!$ positive terms, and forms the positive symmetric group, while N forms the negative symmetric group with the same terms, half of which have a negative sign. For n such elements a_1, a_2, ..., a_n, Young writes

$$P = \{a_1 a_2 \dots a_n\}, \quad N = \{a_1 a_2 \dots a_n\}'.$$

In the above two-letter illustration it will at once be seen that, when f is the determinant Δ, then $P\Delta = 0$, $N\Delta = 2\Delta$. Moreover, if *any* expression $\phi = \Sigma\lambda a_1 b_2 c_3$ is taken which involves each letter and each suffix once in each term, besides a numerical coefficient λ, then the effect of the three-letter operators P and N upon ϕ are $\mu\Delta_+$ and $\mu\Delta$, where $\mu = \Sigma\lambda$, and Δ_+, Δ are the *permanent* and the *determinant* respectively. These examples serve to explain how Young transferred the whole emphasis from the operand f to the substitutional operator, and very soon he had elaborated a considerable theory of these operators.

The main result was contained in *the method of the tableau* (1900). A tableau is an arrangement of the variable in rows and columns, equal or diminishing both downwards and from left to right. A function, depending on, let us say, five permutable variables, possesses (among others) the tableau

$$\begin{matrix} \times\times\times \\ \times\times \end{matrix} \qquad \begin{matrix} a\,b\,c \\ d\,e \end{matrix}.$$

From this model Young constructs the operators

$$P_1 P_2 N_1 N_2 N_3 = \{abc\}\{de\}\{ad\}'\{be\}'\{c\}',$$

where each P refers to a particular row, and each N to a particular column. The sum of all 5 ! such expressions, due to the permutations of all five letters, Young calls $T_{3,2}$, the suffixes denoting the lengths of the rows of the tableau. Clearly there are as many such shapes (with the longest row and column always at the top and on the left) as there are partitions of n, the number of letters. For four letters there are five shapes

$$\times\times\times\times, \quad \begin{matrix} \times\times\times \\ \times \end{matrix}, \quad \begin{matrix} \times\times \\ \times\times \end{matrix}, \quad \begin{matrix} \times\times \\ \times \\ \times \end{matrix}, \quad \begin{matrix} \times \\ \times \\ \times \\ \times \end{matrix}, \qquad (1)$$

and therefore five operators T_4, $T_{3,1}$, $T_{2,2}$, $T_{2,1,1}$, $T_{1,1,1,1}$. Young found that, for all values of n, a certain positive linear combination of these T's was identically equal to unity, say

$$\sum_p A_{(p)} T_{(p)} = 1,$$

where each (p) denotes a different partition of n, and the coefficient $A_{(p)}$ is a non-zero perfect square rational number. In fact

$$A_{(p)} = \left(\prod_{r,s} (a_r - a_s - r + s)! \Big/ \prod_r (a_r + h - r)! \right)^2,$$

where a_r is the number of letters in the rth row of the corresponding tableau and h is the number of rows. This may well be called *Young's Theorem* (1900). From it he deduced the Clebsch-Gordan series and a host of other results. It acted as a powerful crystallising influence by turning an amorphous function f, depending on several sets of variables, into the highly organized but limited varieties of forms Tf. Moreover, many such forms Tf vanish identically (as in the example above), and so do many products $T_{(p)}T_{(q)}$. They certainly vanish if the partition (q) precedes (p) in the descending order as illustrated in (1). And, again, $T_{(p)} = A_{(p)}T^2_{(p)}$. It is remarkable that the whole of this theory was elaborated out of one simple basic fact—that it is impossible for a non-zero function to be simultaneously symmetric and skew symmetric in two of its variables. This fact is the unit out of which Young constructed his whole edifice. It was natural for the enquiry to be suggested, how to find the general solution for one or more substitutional equations such as were constantly occurring. The whole of Young's subsequent work provided a substantial answer to this.

These results were given in the early papers of 1900 and 1902 on Quantitative Substitutional Analysis, which at once attracted the notice of Burnside and Frobenius, the two greatest contemporary experts on the theory of groups. Indeed it had been at the request of Burnside, who refereed the papers, that they were thrown into a form which emphasized the underlying group theory. But Young published his work in ignorance of its close connection with that of Frobenius, who had begun to write long and deep memoirs (1896–1903) on the finite group, and had already used matrices and linear transformations to represent any such group. In particular, Frobenius had set himself the problem of finding, if possible, numerical coefficients such as should satisfy the equation $\Sigma\lambda A = (\Sigma\lambda A)^2$; that is to say, a certain linear combination of all the elements of the group was to be equal to its own square, with the significant proviso that elements belonging to any one and the same class should all have the same coefficient. The fundamental group property $AB = C$ affords an *a priori* reason for such a possibility. The successful answer to this question led Frobenius to study sets of coefficients involving *group characters*, whereby he opened up a wide field of research which is still far from exhausted.

When Young's work appeared Frobenius at once saw its close relation with his own, and explained the connection and adapted the tableau method to his own work in his "Die charakterischen Einheiten der symmetrischen Gruppe" (*Berliner Sitzungsberichte*, 1903, 349). Young first learnt of these memoirs through Burnside in 1906, but apparently it was not until after the last war—a period of inevitable mathematical inactivity for so many—that Young had fully mastered them. He had certainly started his research again in 1922 when he wrote his *Ternary Perpetuants*, and he had already been re-reading Frobenius for several months when I first met him in the summer of 1925. The reading was slow and painstaking for, as he remarked, the German was involved and he was no linguist. This opportunity to visit Young occurred when I sought his advice on the problem of extending Pascal's Theorem of the hexagram inscribed in a conic to the decagon inscribed in a quadric surface. Algebraically the hexagon satisfies the condition

$$(123)(156)(264)(345)-(456)(423)(531)(612)=0,$$

where each expression (123) denotes the determinant of the coordinates for three of the points. The function on the left is skew symmetric in each pair of the six symbols; and a corresponding sum of terms containing five four-rowed determinants of ten points in space was known to exist. E. Study had conjectured that it would consist of four terms, but I had found it to have five, even in the special case when three of the ten points were in line. In reply to a query whether the five-term expression could be rendered quite general, Young, who was at once interested in the problem, suggested the trial of a certain simple operator, adding that it would produce either a zero result or else the desired form. Happily I was able to report to Young that it turned out to be non-zero; but, alas, the series had sixty times as many terms as Study had conjectured. This, however, did not daunt Young, who thought in terms of factorial n as easily as most of us with n. He at once went into the general theory of the symmetric group of ten letters, and in an amazingly short time produced a highly elaborate but complete account of the linear invariants of ten quadrics.

During my visit to Birdbrook he also told me that he was gradually mastering Frobenius, though the work was very abstract and he always preferred to embark upon a theory by way of a practical problem. Already he had learnt much from Frobenius and had greatly improved his own substitutional analysis by inventing *standard forms* and a new matrical representation of his T operators. These important results were published two years later (1927) in the third memoir of the series, in which he paid generous tribute to Frobenius. The original stages of Young's theory had suffered from a defect which is also inherent in the symbolic invariant theory; in it results would certainly be complete, but they might often be redundant. The discovery of standard forms removed the redundancy without impairing the completeness, and there were certain numerical checks, issuing from the theory of Frobenius, which would safeguard the accuracy of the results. It is indicative of the care and deliberation with which Young worked that he should have turned aside from this general theory to the problem of ten quadrics, which he treated as a challenge to his newer technique, before publishing the main result.

By standard forms he meant arrangements of the tableau such as the ten following, for the case of four letters,

$$abcd, \quad \begin{array}{l} abc \\ d \end{array}, \quad \begin{array}{l} abd \\ c \end{array}, \quad \begin{array}{l} acd \\ b \end{array}, \quad \begin{array}{l} ab \\ cd \end{array}, \quad \begin{array}{l} ac \\ bd \end{array}, \quad \begin{array}{l} ab \\ c \\ d \end{array}, \quad \begin{array}{l} ac \\ b \\ d \end{array}, \quad \begin{array}{l} ad \\ b \\ c \end{array}, \quad \begin{array}{l} a \\ b \\ c \\ d \end{array}$$

Here the alphabetical order is strictly preserved in each row and down each column. The 4! arrangements of each of the five shapes are now seen to be reduced to 1, 3, 2, 3, 1 arrangements respectively. Young found that all others can be expressed in terms of these standard arrangements, so that the series $\Sigma AT = 1$ can be greatly simplified. The resulting forms are also linearly independent and give exact information about the number of functions of a particular type. The most general function of the type $T_{(p)}f$ then has exactly q^2 arbitrary constants, where q is the number of standard forms of the partition (p). As an illustration of this remarkable result, which may be called Young's *Standard Theorem*, we may take the above case of four letters. In the list of ten standard forms the sum of the squares of the subsets $1^2+3^2+2^2+3^2+1^2$ must be 4!; and in general $\Sigma q^2 = n!$. Also, in terms of the original coefficients $A_{(p)}$, $q = n! \sqrt{(A_{(p)})}$.

The original $T_{(p)}$ is now replaced by a modified form

$$T'_{(p)} = \Sigma \lambda_{rs} P_r \sigma_{rs} N_s$$

where each of r and s is summed from 1 to q, while λ_{rs} is numerical, and the P, σ, N are definite substitutional expressions arising from the tableau. This leads to a matrix $[\lambda_{rs}]$ of q rows and columns which completely specifies the function $T'_{(p)}f(x, y, \ldots, z)$, derived from any function f of n variables $x, y, \ldots, z$, which may undergo derangement. In fact the sum or product of two such modified T operators obeys the matrix law.

For the general function $f(x, \ldots)$ the numbers λ_{rs} in each matrix are quite arbitrary: so for four variables there are five matrices of orders 1×1, 3×3, 2×2, 3×3 and 1×1 respectively, thus possessing altogether 24 elements. Furthermore, any modification of the function due to symmetry, or to skew symmetry, or to any other such property of the variables, is at once visible in the matrices—blank spaces appear. For example, if $f(a, b, c, d)$ is symmetric in a, b, it is then capable of at most twelve values, by interchange of the variables in all possible ways. In this case Young found that half the rows of the matrices would be blank. Exactly which rows then survive provided a very interesting problem, and it was analysed directly from the standard forms; for the case in point Young proved that this amounted to rejecting each standard form wherein the symmetrical letters a, b occur in the same column. A glance shows that this leaves only 1, 2, 1, 1, 0 standard forms; and these tell us the numbers of non-zero rows, which have, of course, the same respective numbers of elements in them as before, namely, 1, 3, 2, 3, 1. Thence the full number of constants is found, by multiplying together respective pairs, to be $1+6+2+3+0 = 12$, correctly.

Young took a modest but wholehearted pleasure in these results, and his enthusiasm was infectious. "I am delighted", he wrote some months later (1926), "to find someone else really interested in the matter. The worst of modern mathematics is that it is now so extensive that one finds there is only about one person in the universe really interested in what you are". The tide turned in his favour with the appearance of the third memoir. Within a few years his method of the tableau and the standard theorem appeared in Weyl's *Theory of Groups and Quantum Mechanics* (2nd ed., 1930), as a means of elucidating the properties of quantum numbers, while the Clebsch-Gordan series, which had been largely responsible for substitutional analysis, was now found to be of fundamental importance for the whole of spectroscopy. Also during the last decade an interest in the theory of group characters has developed among several of the younger algebraists throughout the country.

During the last fourteen years of his life Young wrote a steady series of papers elaborating his theory and applying it to the problems of invariants and their generating functions. In a letter to a friend, he wrote (1930): "For the last two years I have been working at a paper on the application of substitutional analysis to invariants, but though I have obtained a good many interesting results that I think might be worth publishing, yet I feel that it is too scrappy as yet to write out. . . . My ambition is at the moment to present the complete system for a single cubic in any number of variables. Quite do-able if things turn out as I hope; but I am a confirmed optimist, and so suffer many defeats".

Something should perhaps be said in detail of this later work. The fourth memoir was mainly illustrative and technical, but it brought out the close relation between the matrices $[\lambda_{rs}]$ and the group matrices of Frobenius and Schur (1908). The fifth dealt with the group of rotations and reflexions of the hyperoctahedron in n dimensions. In the sixth a proof of Frobenius' generating function for characters of the symmetric group was obtained by the method of the tableaux, together with semi-normal (or triangular) group matrices. The seventh and eighth memoirs, together with the communication to the Royal Society (1935), were devoted to invariant theory; in the eighth was included an illustration from the invariants of a quaternary cubic, which has an irreducible system of six forms of degrees 8, 16, 24, 32, 40, 100. They had been discovered seventy years earlier by Clebsch and Salmon independently, but of this Young was unaware. The impression left by these examples, and by all the replies to his friends which he readily supplied on particular problems of geometry or invariants, was that in his hands the method of the tableau was irresistible. Known and unknown results alike were treated summarily and afresh; he merely reaffirmed (or corrected where they were wrong) the old and recorded the new.

The motif which ran through these last few memoirs related to certain well-known forms called gradients (homogeneous and isobaric polynomials in the coefficients of the ground forms). Gradients were fundamental in the nineteenth century progress of invariant theory and in all the work of Elliott. Their behaviour at bottom depended on additive properties of sets of the positive integers occurring among their indices and suffixes. But so, also, were the properties of the forms which the method of the tableau produced. When the semi-normal matrices were used, Young found a marked parallelism in these two methods, which he called respectively the method of leading gradients and that of irreductible forms. Indeed, for the binary case, parallelism became coincidence; and on this evidence, and his own instinct for algebraic truth, he surmised the following theorem:

In general the complete set of leading gradients is defined in the same way as the complete set of irreductible forms.

It is probable that his most recent and unfinished work deals with this, and in any case it is greatly to be hoped that the pages are complete enough to make their publication possible.

Young's work is never easy reading, for it lacks that quality which helps the reader to grasp the essential point at the right time. The very closest and constant attention is required to pick out some of the most fundamental results from a mass of detail. One could almost suppose that he camouflaged his principal theorems. His work resembles a noonday picture of a magnificent sunlit mountain scene rather than the same in high relief with all the light and shade of early morning or sunset. The craftsmanship is accurate and logical, and the ideas underlying many of the proofs are very beautiful. His powers of combining deep insight into abstract algebraic theory with an uncanny technical manipulative skill in all the practical applications give him a place unrivalled among his contemporaries. His humility, and perhaps his isolation and lack of teaching experience among undergraduates, prevented him from realising the importance of clarifying the crucial passage from abstract theory to detailed practice. It is hard to believe that processes which to the mind of Young were intuitively clear cannot yet be made part of our common mathematical heritage; for if they can, then there is a great future for algebra.

In drawing up this notice I have been very much indebted to Mrs. Gunnery, Mr. G. H. A. Wilson, Prof. E. T. Whittaker, Mr. J. H. Grace, Dr. A. C. Aitken and Mr. W. L. Edge for supplying family, academic or mathematical details.]

Published Work

The series on Quantitative Substitutional Analysis

1. "Quantitative Substitutional Analysis, I", *Proc. London Math. Soc.* (1), 33 (1901), 97–146.
2. "Quantitative Substitutional Analysis, II", *Proc. London Math. Soc.* (1), 34 (1902), 361–397.
3. "Quantitative Substitutional Analysis, III", *Proc. London Math. Soc.* (2), 28 (1928), 255–292.
4. "Quantitative Substitutional Analysis, IV", *Proc. London Math. Soc.* (2), 31 (1930), 253–272.
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6. "Quantitative Substitutional Analysis, VI", *Proc. London Math. Soc.* (2), 34 (1931), 196–230.
7. "Quantitative Substitutional Analysis, VII", *Proc. London Math. Soc.* (2), 36 (1934), 304–368.
8. "Quantitative Substitutional Analysis, VIII", *Proc. London Math. Soc.* (2), 37 (1934), 441–495.
9. "Quantitative Substitutional Analysis", *Journal London Math. Soc.*, 3 (1928), 14–19.
10. "Some generating functions", *Proc. London Math. Soc.* (2), 35 (1933), 425–444.
11. "The application of Quantitative Substitutional Analysis to invariants", *Phil. Trans. Royal Soc.*, 234 (1935), 79–114.

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12. "The irreducible concomitants of any number of binary quartics", *Proc. London Math. Soc.* (1), 30 (1899), 290–307.
13. "The invariants of lowest degree for any number of quartics", *Proc. London Math. Soc.* (1), 32 (1901), 384–404.
14. "On covariant types of binary n-ics", *Proc. London Math. Soc.* (2), 1 (1904), 202–208.
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19. " On certain classes of syzygies ", *Proc. London Math. Soc.* (2), 3 (1905), 62–82.
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23. " Binary forms with a vanishing covariant of weight four or five ", *Journal London Math. Soc.*, 8 (1933), 182–187.
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25. *The algebra of invariants* (Cambridge University Press, 1903). In collaboration with J. H. Grace.
26. " Perpetuant syzygies ", *Proc. London Math. Soc.* (2), 2 (1905), 221–265. In collaboration with P. W. Wood.
27. " The linear invariants of ten quaternary quadrics ", *Trans. Cambridge Phil. Soc.*, 23 (1926), 265–301. In collaboration with H. W. Turnbull.

LIBRARY

The Council wish to remind members that a substantial portion of the Society's Library was destroyed or damaged at University College, London, in the autumn of 1940. They do not ask for immediate gifts of books, since there is no suitable place for their deposit, but they hope that members will do what they can later to replace the books which have been lost.

ADDENDUM

" ON A CHAIN OF THEOREMS DUE TO H. COX "*

H. W. RICHMOND.

The theorem of § 3, p. 106, is, of course, the well-known configuration† of 8 points and 8 planes discovered by Möbius in 1828. Cox may have been unaware that the theorem was already known, but my omission of Möbius' name was careless and regrettable. To recognise, as Cox did, that this result suffices to prove the whole chain of theorems appears to me as great an achievement as the ingenious reasoning devised by Clifford for the special case of lines and circles in a plane.

A correspondent points out that Cox's argument has been rediscovered by Baker‡, who uses it to prove Clifford's chain, noticing, but not drawing attention to, its much wider scope.

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* *Journal London Math. Soc.*, 16 (1941), 105–107.

† See, for example, Baker, *Principles of geometry*, 1 (1922), 61.

‡ Baker, *loc. cit.*, 4 (1925), 29.