OBITUARY NOTICES

HIERONYMUS GEORG ZEUTHEN.

By the death of Prof. Hieronymus Georg Zeuthen, on January 6th, 1920, in the eighty-first year of his age, the Society has lost one of the oldest of its honorary members—a member of forty-five years standing—for it was in January 1875 that Drs. Klein, Kronecker, and Zeuthen were elected foreign members of the Society. The writer of this notice had not the privilege of personal acquaintance with Prof. Zeuthen, and wishes gratefully to acknowledge his obligation to the kindness of Prof. C. Juel, of Copenhagen, who has allowed him to quote from the memoir of Prof. Zeuthen which was read before the Royal Danish Academy of Science, and has also supplied a list of Prof. Zeuthen's publications.

Zeuthen was born in Jutland in 1839, and entered the University of Copenhagen as a student in 1857. His earliest productions were papers contributed to the Danish *Tidsskrift for Matematik*, which was founded in 1859; these were written during his student days or the years immediately following. His first work of importance was his dissertation for his Doctorate. He had gone, in 1863, to Paris to study under Chasles, the mathematician who undoubtedly exerted a greater influence upon him than any other. Chasles is the founder of Enumerative Geometry and of the Theory of Characteristics, and it was in these subjects that Zeuthen's powers first revealed themselves. His earliest work in this field was his Doctor's Thesis of 1865, translated and published in the *Nouvelles Annales de Mathématiques* in 1866, with the title "A New Method of Determining the Characteristics of Systems of Conics"—a work whose merit was immediately recognised. Zeuthen next studied surfaces of the second order and determined the characteristics in the elementary systems of such surfaces. It may be mentioned that, on learning that Chasles was writing on the same subject, Zeuthen withheld his results from publication, sending them in a closed envelope to the Danish Academy of Science, with the instructions that it should not be opened until after the publication of Chasles' treatise. Continuing investigations of a similar kind Zeuthen produced in 1873 his comprehensive "General Properties of Systems of Plane Curves with Application to

determine Characteristics in the Elementary Systems of the Fourth Order." The subject is one which has never attracted so much attention in this country as it has abroad. As a type of the results which Zeuthen obtained we may extract an example from Chap. XV of Pascal's *Repertorio di Mathematiche Superiori*, Vol. II, 1900. Since nine conditions determine a plane cubic curve, it was to be expected that a finite number of such curves will pass through r given points and touch $9-r$ given lines. Zeuthen determined the number of such curves, corresponding to values 9, 8, 7, ..., 0 of r, viz. 1, 4, 16, 64, 256, 976, 3424, 9766, 21004, 33616.

A variety of such results will be found quoted in this chapter; reference should also be made to Zeuthen's article "Abzählende Methoden," in the *Encyklopädie der Mathematischen Wissenschaften*, Bd. III, Heft 2 (1906), pp. 257–312. For quadric surfaces, determined by nine conditions of passing through certain points, touching certain planes, and touching certain lines, thirty separate cases have to be considered and the numbers of solutions in various cases range from 1 up to 128.

Other important works by Zeuthen of this period bear testimony to the brilliance of his powers. They include several which deal with the genus (or deficiency) of algebraic curves and allied matters. There is his beautiful geometrical proof, *Comptes Rendus*, Vol. 70 (1870), p. 743, that the genera of two curves whose points are in (1, 1)-correspondence must be equal (a theorem already proved by Riemann from consideration of Riemann surfaces, and algebraically by Clebsch and Gordan). Zeuthen's proof was obtained independently of a very similar proof published a few months earlier by Bertini [*Giorn. di Mat., Battaglini*, Vol. 7 (1869), p. 105]. The method of proof is as follows. If a moving point M of a given curve C and a moving point M_1 of a second given curve C_1 are in (1, 1)-correspondence, the intersection of the lines AM and A_1M_1 joining M and M_1 to two fixed points A and A_1 traces an algebraic curve; and by considering the class of this curve as calculated from the number of tangents to it from A and A_1, respectively, the theorem that the genus of C is equal to that of C_1 follows at once. It would be difficult to devise any proof more simple and fundamental than this. Zeuthen proceeded to use his method to extend the theorem to cases where the points of two curves are in multiple correspondence, and so established what is known as "Zeuthen's extended theorem upon genus". Here Zeuthen's geometrical method led to a result which had not previously been recognised, although it was remarked later that the theorem could be obtained from the classical theory. It is therefore fitting that the theorem should bear Zeuthen's name. Continuing to work in the same field Zeuthen applied the principles of correspondence to solid geometry, and in 1871 discovered a number

which is invariant in any point for point transformation of one algebraic surface into another. The value of this discovery was not, and in fact could not be, recognised at the time: but more than twenty years later, when such properties of surfaces were investigated by more modern methods by the Italian school of mathematicians, the invariant was rediscovered in 1895 by Segre, and now is known as the Zeuthen-Segre invariant of the surface. See Prof. H. F. Baker's Presidential Address to this Society (1912), *Proc. London Math. Soc.*, Vol. 12, p. 33, or *Encyklopädie*, Vol. III, Cap. 6, *b.*, p. 701.

Space will not permit more than a mention of Zeuthen's work upon cubic surfaces, or of his contribution to Vol. 10, Ser. 1, of the *Proceedings* of this Society in 1879. An arresting paper is that entitled "Sur les différentes formes de courbes planes du quatrième ordre" (*Math. Annalen*, Vol. 7, pp. 410–432), in which Zeuthen first examines the distinction (pointed out by v. Staudt) between the odd and even branches of a curve, and proves in a very simple manner the theorems concerning the intersections of two branches. He then shows that of the twenty-eight double tangents of a curve without nodes there are always four which are real, and either touch the *same* branch twice or are isolated, *i.e.* are real lines having two imaginary contacts with the curve; the eight points of contact of these four double tangents lie on a conic. All other real double tangents touch two different branches, each two branches external to one another necessarily giving rise to four double tangents. It is hardly to be doubted that this paper largely inspired the striking discoveries of Klein and Harnack, published in two famous papers in Vol. 10 of the *Math. Annalen*; and Klein's results as to the form of cubic surfaces are closely connected with it.

From about 1880 onwards Zeuthen's interests turned more and more towards the history of mathematics, chiefly, but by no means wholly, in classical times. He had published a short paper in 1876 on "Brahmegupta's trapeziums", but from 1880 he found a richer field for study in tracing the development of Greek mathematics. Thus it is probable that the name of Zeuthen is better known at the present day as a historian of mathematics than as an original discoverer in the subject. We will not here attempt to give a detailed account of his many writings upon Archimedes, Euclid, Apollonius, Diophantus, &c., or of those dealing with the later times of Descartes, Cardan, Fermat, Newton, Barrow. His most important historical work, *Die Lehre von den Kegelschnitten in Altertum*, was published in 1886.

To the end of his life Zeuthen continued to publish papers on mathematical, and for the most part geometrical or historical, subjects. In the

year 1919 (a year before his death) he published two papers—one on the origin of Algebra, and the other on the explanation of a paradox in Enumerative Geometry.

Almost the whole of Zeuthen's life was passed in Copenhagen, where he was for many years Professor at the University. The number of Zeuthen's publications amounts to nearly two hundred, and include besides the numerous articles in various periodicals, elementary textbooks, textbooks for students at the University or Polytechnic, papers read at various International Congresses in mathematics or philosophy, and (in addition to the history of Conic Sections already referred to) a *History of Mathematics* (1883), and a *History of Mathematics in the Sixteenth and Seventeenth Centuries* (1903). Until the last year of his life he was Secretary to the Danish Academy of Science.

H. W. R.