PETER GUTHRIE TAIT
NEW INSIGHTS INTO ASPECTS OF HIS LIFE AND WORK; AND ASSOCIATED TOPICS IN THE HISTORY OF MATHEMATICS

Elizabeth Faith Lewis

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ELIZABETH FAITH LEWIS

This thesis is submitted in partial fulfilment for the degree of Ph.D.
at the University of St Andrews.

2014

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Contents

List of Figures

List of Tables

Abstract

In this thesis I present new insights into aspects of Peter Guthrie Tait's life and work, derived principally from largely-unexplored primary source material: Tait's scrapbook, the Tait–Maxwell school-book and Tait's pocket notebook. By way of associated historical insights, I also come to discuss the innovative and far-reaching mathematics of the elusive Frenchman, C.-V. Mourey.

P. G. Tait (1831–1901) F.R.S.E., Professor of Mathematics at the Queen's College, Belfast (1854–1860) and of Natural Philosophy at the University of Edinburgh (1860–1901), was one of the leading physicists and mathematicians in Europe in the nineteenth century. His expertise encompassed the breadth of physical science and mathematics. However, since the nineteenth century he has been unfortunately overlooked—overshadowed, perhaps, by the brilliance of his personal friends, James Clerk Maxwell (1831–1879), Sir William Rowan Hamilton (1805–1865) and William Thomson (1824–1907), later Lord Kelvin.

Here I present the results of extensive research into the Tait family history. I explore the spiritual aspect of Tait's life in connection with *The Unseen Universe* (1875) which Tait co-authored with Balfour Stewart (1828–1887). I also reveal Tait's surprising involvement in statistics and give an account of his introduction to complex numbers, as a schoolboy at the Edinburgh Academy. A highlight of the thesis is a re-evaluation of C.-V. Mourey's 1828 work, *La Vraie Théorie des quantités négatives et des quantités prétendues imaginaires*, which I consider from the perspective of algebraic reform. The thesis also contains: (i) a transcription of an unpublished paper by Hamilton on the fundamental theorem of algebra which was inspired by Mourey and (ii) new biographical information on Mourey.

Acknowledgements

I am pleased to record my special thanks to those who have helped make my post-graduate studies both pleasant and productive.

At the University of St Andrews, my supervisors, *Professor Edmund Robertson* and *Dr. John O'Connor*: for generously sharing their expertise; for always knowing the path ahead and the best way to proceed; for their patience as I developed and their encouragement along the way. *Professor Peter Jupp*: for his valuable comments on Tait's contribution to statistics, and my interpretation of it, and his continued interest in my research. *Dr. Colin Campbell*: for his company on our conference trip to Hungary, his tales of the Edinburgh Academy and his support throughout. *Dr. Isobel Falconer*: for her enthusiasm about Tait as a research topic. And *Stefanie Eminger*, my friend and fellow researcher: for sharing the experience with me.

At the University of Dundee, *Ross MacDonald*: for explaining aspects of Scots succession law in relation to Tait's assets.

In Edinburgh, the *trustees of the James Clerk Maxwell Foundation*: for arranging access to the archives preserved in India Street; for kindly giving their permission to publish material from Tait's scrapbook and the Tait–Maxwell school-book; and for providing me with contact details for living members of the Tait family, including *Susan Rutherford* (*Tait*), the paternal great-granddaughter of P.G.T., who generously shared information on her family history. Also *Andrew McMillan*, Honorary Archivist at the Edinburgh Academy: for facilitating a very pleasant visit to the Academy archives in November 2013; and for giving his kind permission to publish some material from the archive, especially the mathematics paper Tait and Maxwell attempted in 1847 as part of the Academical Club Prize competition.

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Further afield, in Dublin, the *staff at the Manuscripts Library of Trinity College, Dublin*: for facilitating a productive research trip in February 2014; and for their kind permission to publish some of the correspondence between Hamilton, De Morgan and Salmon, and an unpublished paper by Hamilton on the fundamental theorem of algebra. In Paris, the *staff at the Archives de Paris*: for their expert assistance in looking for biographical information on Mourey during my visit to the archives, also in February 2014. And in Hungary, *Professor Péter Kötesi* from the University of Miskolc: for organising memorable conferences in Sàrospatak, Hungary in May 2012 and in Cluj-Napoca, Romania in May 2014.

Finally at home, my family: for their love, support and prayers.

Foreword

Peter Guthrie Tait (1831–1901) was one of the most eminent physicists and mathematicians of the nineteenth century. Few could claim a greater reputation than Professor Tait: for his original researches, in physical science (experimental and theoretical) and in mathematics; for his contribution to the Royal Society of Edinburgh and for his powers of exposition in the lecture theatre. Beyond the nineteenth century, however, Tait has been unfortunately overlooked. Consequently, the life and work of P. G. Tait has vast scope as a research topic.

I first encountered Tait during my undergraduate days at St Andrews. My senior honours dissertation was on knot theory and Tait featured, as a key figure, in the account that I gave of its historical developments.

The benefits of choosing Tait as a Ph.D. topic were numerous. First, it seemed that there was something to be learned about every aspect of his life, personal and professional—a multitude of avenues to original research. He was a fascinating character—known to be blinkered, stubborn, fiercely patriot and prejudiced but at the same time, a loyal friend and family man. He was also a first-class teacher and a solid contributor to the corpus of scientific knowledge. His expertise and interests were truly diverse, covering the full breadth of physical science and mathematics. He was also well connected, being intimately associated with James Clerk Maxwell (1831–1879), Sir William Rowan Hamilton (1805–1865) and William Thomson (1824–1907), later Lord Kelvin. Second, there was the practical aspect: Tait was local and largely-unexplored primary source material was available nearby.

However rich Tait as a research topic proved to be, it was always my intention to explore any avenue of research which promised to lead to something of historical interest or significance. Tait was to be used as a springboard to any topic in the history of mathematics worthy of further research.

The first chapter in this thesis provides the reader with a concise introduction to P.G.T.—a biographical sketch covering various aspects of his personal and professional life. It is supplemented by a family tree report for the Tait family in Appendix A. The second chapter discusses a remarkable book, co-authored by Tait and the

Scottish physicist and meteorologist, Balfour Stewart (1828–1887). Therein, Tait and Stewart proposed hypotheses in an attempt to unite the latest scientific theories with the established doctrines of Christianity. Reviews of *The Unseen Universe*, which was originally penned anonymously, have been sourced from Tait's scrapbook. The third chapter reveals Tait's surprising involvement in statistics. Knowledge of Tait's contribution to statistics has come from an entry in his pocket notebook. An unpublished poem by Tait on the Franco–Prussian War, which was also discovered in the notebook, is transcribed in Appendix B. The fourth chapter in this thesis is an account of Tait's first introduction to complex numbers, based on an entry from the Tait–Maxwell school-book. This particular school-book entry is transcribed in Appendix D. Supplementary material associated with Tait's schooldays at the Academy appears in Appendix C, including a transcription of a mathematics paper attempted by Tait and Maxwell as part of the Academical Club Prize competition in 1847. In Appendix E, I provide additional insights into historical developments regarding the geometrical representation of complex numbers, focusing on the contributions of Adrien-Quentin Buée (1748–1826) and Joseph Gergonne (1771–1859). By associated historical insights we are lead on to the final chapter, on C.-V. Mourey's 1828 work, *La Vraie Théorie des quantités négatives et des quantités prétendues imaginaires*. After exhaustive research, I am able to provide new biographical information on Mourey who has remained an unknown to historians of mathematics for the past 186 years. In Appendix F, I provide a transcription of an unpublished paper by Hamilton on the fundamental theorem of algebra which was inspired by Mourey. I discovered the paper in one of Hamilton's notebooks in the Manuscripts Library of Trinity College, Dublin. A guide to Mourey's notation and terminology is given in Appendix G.

Principal unpublished sources

This thesis is based primarily on three unpublished sources: Tait's scrapbook, the Tait–Maxwell school-book and Tait's pocket notebook. Other principal sources are discussed in a bibliographical essay at the end of the thesis (page 198). The scrapbook and the school-book are preserved by the James Clerk Maxwell (J.C.M.) Foundation at Maxwell's birthplace, 14 India Street, Edinburgh.[1] Tait's pocket notebook has been in my possession since January 2011. Prior to that it was kept by the Edinburgh Mathematical Society. After submission I intend to hand it over to the J.C.M. Foundation.

Tait's scrapbook is a collection of miscellaneous documents once belonging to Tait which includes: newspaper cuttings of reviews, letters to editors, political cartoons, articles written by Tait on golf science, obituary tributes, etc.; a limited amount of correspondence (some from Maxwell, Thomson and Forbes); original poems, some by Tait, others by Maxwell; copies of examinations set by Tait and syllabuses he taught; records of Tait's school and university examination results; copies of addresses given by Tait, Thomas Andrews etc.; edited drafts for submission for publication; and so on. Some material has been inserted by Tait and some must have been put in after his death—obituaries, for instance—presumably by the family.[2] The scrapbook is signed on the inside cover: 'Edith Tait, 1878'. Edith was Tait's eldest child. She would have been eighteen in 1878. The scrapbook was presented to the Foundation by Murray Tait (P.G.T.'s paternal great-grandson) in 2003.

The Tait–Maxwell school-book dates from 1846–1847. It belonged to Tait and was originally intended as a fair-copy book: some entries are written carefully in ink, and are signed and dated; however, there is also an abundance of rough pencil

[1] For more information on Maxwell's birthplace see [1]. For a biographical sketch of P.G.T., supplemented with extracts from the scrapbook, see [2].

[2] In some places Tait's annotations appear on the actual pages of the scrapbook, rather than on the cuttings pasted in, which is evidence that Tait started the scrapbook. See Figure 2.1 (page 27) and Figure 2.2 (page 28) for instance.

work, with workings-out and schoolboy sketches subsequently fitted into available space. Entries in the school-book include: (factually dubious) notes on the history of enumeration; a table recording the positions of the satellites of Jupiter, as observed by Tait at the age of thirteen; a number of problems and solutions on the mensuration of heights and distances, which I have traced to a contemporary textbook [3]; a summary of a paper published in the *Transactions of the Royal Society of Edinburgh* on the geometrical representation of complex numbers (see Chapter 4); and copies of the MSS. which Tait and Maxwell exchanged during their final year at the Edinburgh Academy; but the bulk of the school-book is taken up with notes which Tait abstracted from the *Encyclopaedia Britannica* (7th edition, 1842) on 'Algebra' and 'Fluxions'. Dr John W. Arthur, a trustee of the James Clerk Maxwell Foundation, drew my attention to the possibility that Maxwell's entries in the school-book might be written in Tait's hand: I had originally thought that Tait and Maxwell passed the school-book between them and that they had each written in their own entries; but careful analysis of the handwriting suggests that in reality it was Tait who had produced a fair copy of Maxwell's manuscripts in the school-book, imitating Maxwell's handwriting.

Tait's pocket notebook dates from 1870. All the entries are written in pencil. Very few pages have been utilized. One of the pages is signed: 'P. G. Tait, College, Edinburgh'. The notebook contains: Tait's quaternion version of Green's theorem; an unpublished poem by Tait on the Franco–Prussian War; an entry written in French and a reminder to make amendments to an address which Tait was to give to Section A of the British Association for the Advancement of Science (B.A.A.S.) in 1871.

Figure 0.1: Peter Guthrie Tait, an etching by William Brassey Hole, 1884. From the "Quasi Cursores", the gallery of portraits of the Principal and Professors of Edinburgh at the time of the Tercentenary in 1884. Reproduced with the kind permission of the National Portrait Gallery, London.

CHAPTER 1

AN INTRODUCTION TO P. G. TAIT

This chapter provides a biographical sketch of P. G. Tait, covering various aspects of his personal and professional life. It is supplemented by a family tree report for the Tait family in Appendix A.

1.1 Concise biography

Peter Guthrie Tait (Figure 0.1, page 1) was, in his time, 'one of the most renowned scientists and mathematicians in Europe'.[3] He was born in Dalkeith, Midlothian on 28 April 1831 and educated at the Edinburgh Academy—where he established a lifelong friendship with fellow student, James Clerk Maxwell (1831–1879)—and at the Universities of Edinburgh and Cambridge.[4,5] At Edinburgh he studied mathemat-

[3] [4, p120]

[4] For a book on the history of the Edinburgh Academy see [5]. The Academy's *Register* [4] is another useful resource, providing concise biographies of former pupils: Tait appears on page 120.

[5] James Clerk Maxwell (1831–1879): Professor of Natural Philosophy at Marischal College, Aberdeen (1856–1860) and King's College, London (1860–1865); then Cavendish Professor of Physics at the University of Cambridge (1871–1879); author of the celebrated *A Treatise on Electricity and Magnetism* (1873). From the Edinburgh Academy, Tait and Maxwell both went on to the University of Edinburgh: Tait remained at Edinburgh for only one session, while Maxwell remained for three sessions before following Tait to Cambridge. Maxwell originally entered Peterhouse but transferred to Trinity, advised by Forbes that it would be easier to obtain a fellowship at Trinity. [6] After Cambridge, the pair maintained their friendship principally through correspondence, rather than face-to-face meetings. Through this correspondence they were of influence to one another, in the sharing of expertise and in the sounding of ideas. Forfar and Pritchard write: 'each had the highest regard for the abilities of the other which were freely put at the other's disposal'. [7] For an obituary tribute to Maxwell written by Tait see [8]. For a contemporary biography of Maxwell see [9].

ics under Philip Kelland (1808–1879) and natural philosophy under James David Forbes (1809–1868).[6,7,8] He received his B.A. from Cambridge in 1852. He came out as Senior Wrangler (the second Scot on record) and First Smith's Prizeman.[9] A fellowship at Peterhouse kept him in Cambridge until 1854.

From Cambridge Tait went to Belfast as Professor of Mathematics at the Queen's College, where he established a number of happy and profitable associations.[10] Through Thomas Andrews (1813–1885), Vice-President and Professor of Chemistry, he was introduced to experimental work and to Sir William Rowan Hamilton (1805–

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[6]Philip Kelland (1808–1879): Professor of Mathematics at the University of Edinburgh (1838–1879), the first Scot to hold the position. See [10] for an obituary tribute to Kelland, written by Tait and George Chrystal, Professor of Mathematics at the University of Edinburgh (1879–1911) and previously Regius Professor of Mathematics at St Andrews (1877–1879).

[7]James David Forbes (1809–1868): Professor of Natural Philosophy at the University of Edinburgh (1833–1860); Principal of the United College, St Andrews (1860–1868); invented the seismometer in 1842; remembered for his work on glaciers. For a biography of Forbes in which Tait deals with his scientific work see [11].

[8]For a book on the history of the University of Edinburgh see [12]: pages 247–250 are on Tait.

[9] [13] *Senior Wrangler*: the student who placed first amongst graduates taking first-class degrees in mathematics in a given year at Cambridge. According to Craik, the first Scot to be Senior Wrangler was Alexander Ellice in 1833. [14, p45f(no.70)] *Smith's Prizeman*: recipient of the Smith's Prize, an annual £25 prize awarded to two students who had excelled in examinations in mathematics and natural philosophy at Cambridge. The influence of the Cambridge mathematical Tripos on the contribution to physics made by students and professors at Cambridge during the nineteenth century is discussed in [15]: note, there is more emphasis on Thomson and Maxwell than Tait.

[10]In 1908 the Queen's College in Belfast—along with the Queen's Colleges in Cork and Galway and the Royal University—became the Queen's University of Belfast and the National University of Ireland. For a book on the history of the Queen's College, Belfast see [16]: Tait is mentioned briefly on pages 164–165 of volume I.

1865), the discoverer of quaternions.[11,12] After Hamilton, Tait is considered to have been the leading expounder of the quaternionic theory and the foremost advocate for its use in physics.[13] In 1860 Tait returned to Edinburgh to take up the Chair

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[11] Tait and Andrews worked together investigating the density of ozone and the action of electric discharge on oxygen and other gases. In fact, Tait's first published papers were written in conjunction with Andrews and came out of their work on the density of ozone. These papers were presented to the Royal Society of London and printed in Andrews' *Scientific Papers* [17], where there is a biographical memoir on Andrews written by Tait and Alexander Crum Brown (1838–1922), who was Tait's brother-in-law and Professor of Chemistry at the University of Edinburgh (1869–1908).

[12] William Rowan Hamilton (1805–1865): Andrews Professor of Astronomy at Trinity College, Dublin, Astronomer Royal of Ireland and Director of Dunsink Observatory; remembered as the discoverer of quaternions and for his work on optics and dynamics; knighted in 1835. Tait's correspondence with Hamilton began in August 1858; an introduction having been secured by Thomas Andrews. Tait had bought a copy of Hamilton's *Lectures on Quaternions* in 1853, while at Cambridge. [18, p13] Tait was keen to understand the 'physical applications of the method', which was his primary interest in the quaternionic theory. [18, p119] The correspondence also proved beneficial to Hamilton: encouraged by Tait's enthusiasm, Hamilton resumed his interest in quaternions and began work on his *Elements of Quaternions* (published posthumously in 1866). Hamilton acknowledged his debt to Tait in a letter to Tait dated 21 January 1859. See [18, p131]. Unfortunately, a misunderstanding over Tait's plans to publish his own treatise on quaternions threatened the cordial relationship. Hamilton had misunderstood the planned scope of Tait's *Elementary Treatise on Quaternions*: he had given Tait permission to publish a set of examples on the application of the quaternionic method but he believed Tait was to publish, instead, a full exposition of the theory which was to appear before Hamilton's *Elements*. Hamilton became suspicious of Tait's motives and in a letter to Augustus De Morgan (dated 14 November 1860), Hamilton expressed his fear of having been deceived by Tait. See [19, pp361–362]. To allay Hamilton's fears, Tait agreed to delay the publication of his book until Hamilton's *Elements* had appeared. Tait's *Elementary Treatise on Quaternions* was finally published in 1867. Tait and Hamilton met for the first time in 1859, at the meeting of the B.A.A.S. in Aberdeen. For an obituary tribute to Hamilton written by Tait see [20]. For a contemporary biography of Hamilton see [19]. For selected correspondence between Hamilton and Tait, in relation to the application of quaternions, see [21].

[13] In his biography of Hamilton, Hankins describes Tait as 'Hamilton's chief disciple'. [19, p316] Tait's biographer, Knott writes: 'Tait was one of the very few who really appreciated the immense

of Natural Philosophy at the university there. He was elected a Fellow of the Royal Society of Edinburgh (R.S.E.) in 1861.

Amongst Tait's chief contributions to mathematics was his work on quaternions and knot enumeration.[14] In his experimental researches, he investigated thermal and electric conductivity and thermo-electricity, devising the first thermoelectric diagram in 1873.[15] His experimental work in connection with the *Challenger* Expedition in the late 1870s led him to his invention of a pressure-measuring instrument called the Tait Gauge and to further important scientific work on: the compressibility of water, glass, mercury, etc.; the physical properties of fresh water and

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value of Hamilton's work.' [18, p14]

[14] It was Thomson's theory of vortex atoms which brought about Tait's mathematical work on knots: persuaded by Thomson, Tait chased a full classification of the forms of knotted vortex rings, under the assumption that a unique form of vortex ring existed for each of the elements. Working with the two-dimensional projection of the knots, Tait took an intuitive approach to investigating the number of different knots which exist with the same number of crossings and how these knots are represented by the scheme of the knot, i.e. a sequence of letters recording the order in which the labelled crossings are encountered when traversing the projection. Tait's work was taken further by the Revd. Thomas P. Kirkman (1806–1895) as far as eleven crossings. Tait, Thomson and Maxwell all worked on the topology of knots and many new concepts were developed in the correspondence exchanged between them. [22] Some of Tait's papers on knots were reprinted in volume I of his *Scientific Papers* (Cambridge : at the University Press, 1898). For Chris Pritchard's paper on Tait's knot theory see [23]: in addition to explaining the roles of Thomson and Kirkman and the previous work done by Listing, Pritchard also describes the knot-specific terminology invented by Tait and highlights the modern-day applications of knot theory. The 'Third Man' of the title is a reference to Tait: Pritchard has Tait as the 'third man of natural philosophy' (after Maxwell and Thomson).

[15] During the 1870s Tait undertook experiments in his Edinburgh laboratory to determine the effect of temperature on the thermo-electric properties of metals. His thermo-electric researches had been inspired by the earlier work of J. D. Forbes and William Thomson. Tait's thermo-electric diagram was a graphical representation of the relationship between thermo-electric-motive force and temperature: Tait found that for most metals electro-motive force followed a parabolic law, while thermo-electric power followed a linear law. For Tait's papers on his thermo-electric diagram see [24] first, then [25].

sea water; the effects of pressure on the maximum density point of water; and so on.[16] In 1875, in collaboration with the physicist and chemist, Sir James Dewar (1842–1923), he conducted experiments on Crookes' radiometer which led him to the true dynamical explanation of the phenomena which Thomson described as 'one of the most interesting and suggestive of all the scientific wonders of the nineteenth century'.[17,18] Between 1885 and 1892 Tait published a series of five papers on the kinetic theory of gases, giving in the fourth, according to Thomson, the first proof of the Waterston–Maxwell equipartition theorem of the average equal partition of energy in a mixture of two different gases.[19] Tait, who was especially fond of golf, combined mathematics and experimental work when he undertook research into golf

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[16] The *Challenger* Expedition was a four-year voyage of scientific discovery of the oceans, conducted between 1872 and 1876. Tait was asked by the scientific leader of the expedition, Sir Charles Wyville Thomson (1830–1882) to determine the corrections that would need to be applied to the temperature readings taken by the self-recording deep-sea thermometers used during the expedition, in order to compensate for the high-pressure conditions. Tait had known Wyville Thomson at Queen's, Belfast: Thomson was Professor of Mineralogy and Geology at Queen's when Tait was Professor of Mathematics; and in 1870 Thomson came to the University of Edinburgh as Professor of Natural History. [26] For Tait's published papers relating to the *Challenger* Expedition consult his *Scientific Papers*, especially [27].

[17] [28, p367] Thomson's explanation of the Crookes' radiometer phenomena is this: 'The phenomena to be explained is that in highly rarefied air a disc of pith or cork or other substance of small thermal conductivity, blackened on one side, and illuminated by light on all sides, even the cool light of a wholly clouded sky, experiences a steady measurable pressure on the blackened side.' [Ibid.] Tait and Dewar's results were communicated to the R.S.E. on 5 July 1875; however, the only published record is an article, 'Charcoal Vacua' published in *Nature* on 15 July 1875.

[18] Sir James Dewar (1842–1923): Lecturer in Chemistry (1869) and later Professor of Chemistry (1875) at the Royal (Dick) Veterinary College; Jacksonian Professor of Natural Experimental Philosophy at Cambridge (1875–1923) and Fullerian Professor of Chemistry at the Royal Institution, London (1877–1923). [29]

[19] [28, p366] For Tait's papers on the kinetic theory of gases see volume II of his *Scientific Papers* (Cambridge : at the University Press, 1900).

ball aerodynamics.[20,21] During his career he published some 365 papers and twenty-two books.[22] He remains best remembered, however, as the co-author with William Thomson (1824–1907) of the epoch-making *Treatise on Natural Philosophy*—a con-

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[20]Tait published on golf science in *Nature*, *Golf* and the *Proc.* and *Trans. Roy. Soc. Edinburgh*. Some of his papers were republished in volume II of his *Scientific Papers*. Denley and Pritchard have published a paper [23] on Tait's golf science in which they explain aspects of Tait's research on the subject and go through some of his early papers. They also recount a charming anecdote in which Tait's son, Freddie, who was a championship golfer and used to take part in Tait's experiments, proved Tait's results wrong by exceeding the maximum distance a golf ball could be made to travel by an experienced golfer as stated by Tait.

[21]Chris Pritchard discusses various aspects of Tait's scientific work in [30]: the topology of knots, golf ball aerodynamics and quaternions.

[22]For Tait's published papers, consult the *Proc.* and *Trans. R.S.E.* and his *Scientific Papers*, published in two volumes by Cambridge University Press (1898 and 1900). For a detailed bibliography see [31]. Tait's principal publications include: *Dynamics of a Particle* (1856) with Steele, *Sketch of Elementary Dynamics* (1863) with Thomson, *Treatise on Natural Philosophy* (1867) with Thomson, *Elementary Dynamics* (1867) with Thomson, *Elementary Treatise on Quaternions* (1867), *Sketch of Thermodynamics* (1868), *Elements of Natural Philosophy* (1873) with Thomson, *Introduction to Quaternions* (1873) with Kelland, *Recent Advances in Physical Science* (1876), *Heat* (1884), *Light* (1884), *Properties of Matter* (1885), *Dynamics* (1895) and *Newton's Law of Motion* (1899). For the *Encyclopaedia Britannica*, he contributed the articles: 'Light', 'Mechanics', 'Quaternions', 'Radiation and Convection' and 'Thermodynamics'. He wrote biographical essays on: Hamilton, Maxwell, Balfour Stewart, Kelland, Forbes, Rankine, Kirchhoff, Stokes and Listing. He also published papers on atmospheric, meteorological and astronomical phenomena; graph theory and recreational mathematics; and education, including the Cambridge mathematical Tripos.

temporary treatment of the subject in terms of the new physics of energy.[23,24,25]

In recognition of his achievements, Tait received the Keith Prize and the Gunning Victoria Jubilee Prize of the R.S.E., and the Royal Medal of the Royal Society

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[23] William Thomson (1824–1907), later Lord Kelvin: Professor of Natural Philosophy at the University of Glasgow (1846–1899); remembered for his work in the areas of thermodynamics, electricity and submarine telegraphy and for his success in patenting a number of inventions (a mirror galvanometer and a mariner's compass, to name but two); he was knighted in 1866 in recognition of the part he had played in the installation of transatlantic telegraph cables; he took the title Baron Kelvin of Largs in 1892. Tait's association with Thomson began in 1860, when Tait took up the Chair of Natural Philosophy at Edinburgh. Tait was already acquainted with William's brother, the engineer, James Thomson (1822–1892) who he had known at Queen's, Belfast. Although William Thomson was based in Glasgow, he visited Edinburgh regularly for the meetings of the R.S.E. It was soon after meeting that Tait and Thomson decided that they ought to produce a book together on natural philosophy. [28, p364] *Treatise on Natural Philosophy*, also known as "Thomson and Tait" or *T and T′*, was pronounced "'one of the greatest books which have appeared since the *Principia*—a book not only profound, but full of original methods of treatment"'. [32, pp203–204] Knott said of the work: 'The publication of Thomson and Tait's *Natural Philosophy* was an event of the first importance in the history of physical science. No more momentous work had been given to the world since the days of the brilliant French mathematicians, Laplace, Lagrange, and Fourier.' [18, p176] The ambitious project was a huge undertaking which demanded eighteen years of collaboration and in that time only the first of a series of planned volumes ever saw publication. In fact, it is said that the work would never have been published had it not been for Tait's 'dogged persistence': Thomson disliked book writing and over the years he became much wearied by the project and his enthusiasm waned. [33, p35] For a time Tait used *T and T′* as a teaching guide at Edinburgh. [18, p21] It was referred to by his students as the "'Student's First Glimpse of Hades"'. [34, p45] For a contemporary biography of Thomson see [35].

[24] See [36] for Crosbie Smith's contextual history of the science of energy in the nineteenth century. Special emphasis is put on the network of the "'North British" physicists and engineers': William Thomson; Macquorn Rankine (1820–1872), Professor of Civil Engineering and Mechanics at Glasgow; James Clerk Maxwell; Tait and Fleeming Jenkin (1833–1885), Professor of Engineering at the University of Edinburgh. [36, p1] Chapter 10 in this book is on Tait and Thomson's *Treatise on Natural Philosophy.*

[25] *T and T′* is one of a select number of books to feature in Grattan-Guinness' *Landmark Writings in Western Mathematics 1640–1940*: see [37].

of London; in addition to a number of honorary degrees and fellowships.[26] He died on 4 July 1901, at the age of seventy, shortly after his retirement from the Edinburgh chair. As a lasting memorial to Tait, a second chair in the natural philosophy department was instituted in the University of Edinburgh in 1922.[27]

1.2 Selected biographical highlights

In this section I cover, in brief, a variety of aspects of Tait's life and character.

1.2.1 Reputation as a lecturer

In 1860, aged twenty-seven, Tait returned to Edinburgh from Belfast—with a wife, Margaret and a daughter, Edith—to take up the Chair of Natural Philosophy in the University of Edinburgh, succeeding Forbes who had gone to St Andrews as Principal of the United College. Tait had successfully outrivaled a number of strong candidates for the post, including: James Clerk Maxwell (Marischal College, Aberdeen), E. J. Routh (Peterhouse, Cambridge) and Frederick Fuller (King's College, Aberdeen).[28] Tait stood out from the group for combining abilities in original scientific investigation with remarkable powers of exposition in the lecture theatre.

Tait's biographer, C. G. Knott (1856–1922) described Tait as being 'unsurpassed' as a lecturer.[29] Professor Robert Flint (1838–1910), Professor of Moral Philosophy at St Andrews (1864–1876) and Professor of Divinity at Edinburgh (1876–1903), described Tait as 'one universally recognised to have had not only a genius of the

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[26] Tait received honorary degrees from the Universities of Ireland (Sc.D., 1875), Glasgow (LL.D., 1901) and Edinburgh (LL.D., 1901), and honorary fellowships of the Edinburgh Mathematical Society (1883) and Peterhouse College, Cambridge (1885). He was an honorary member of the academies of Denmark, Holland, Sweden and Ireland.

[27] [12, p252]

[28] [18, p16]

[29] [18, p19]

first order for research, but rare gifts as a teacher'.[30]

Several of Tait's former students at Edinburgh have produced written accounts bearing witness to his gifts as an expositor and to his presence in the lecture theatre. J. M. Barrie (1860–1937) recalled:[31]

> Never, I think, can there have been a more superb demonstrator. I have his burly figure before me. The small twinkling eyes had a fascinating gleam in them; he could concentrate them until they held the object looked at; when they flashed round the room he seemed to have drawn a rapier. I have seen a man fall back in alarm under Tait's eyes, though there were a dozen benches between them. These eyes could be merry as a boy's, though, as when he turned a tube of water on students who would insist on crowding too near an experiment, for Tait's was the humour of high spirits. I could conceive him at marbles still, and feeling annoyed at defeat. He could not fancy anything much funnier than a man missing his chair.[32]

Preserved in Tait's scrapbook is an article published in the University of Edinburgh's newspaper, *The Student*, in which two former students of Tait's at Edinburgh recorded their experience when they returned to his natural philosophy class in 1888. The following is an extract from their account:

> We were not long seated before the door leading to the Physical Laboratory opened, and the tall figure of the Professor slipped round it. The noises, more pronounced than formerly we thought, soon ceased under the steady gaze which followed the peculiar and well-remembered bow. This seems to be made when the motion towards the audience is combined with that in a direction parallel to it; but the full resolution into its components still remains for some bright young student.
>
> The old gown was yet extant, and the dark jacket, fastened by a single button at the neck, still seemed as far as ever from the *toga virilis*.[33]

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[30] [38, p60]

[31] Sir James Matthew Barrie (1860–1937): Scottish playwright and novelist; creator of Peter Pan; Rector of the University of St Andrews (1919–1922) and Chancellor of the University of Edinburgh (1930–1937). [39]

[32] [34, pp46–47]

[33] *Toga virilis*: 'the toga of manhood, assumed by boys at puberty; hence in figurative context'; white in colour. [40]

> The lecturer has lost nothing of his ancient power of graphic illustration and charming style. The conclusions are obtained from various trains of reasoning, and apt reference to everyday commonplaces clinches the argument, while the subtle humour of the man is every now and then revealed, and serves to keep his audience in good fettle. Details are skilfully subordinated, and the principles stand out in bold relief. The brilliant experiments are all necessary, and none are shown for effect. Performed at the right moment, they never fail, and make all seem clear as at noon.[34] One feels that the students are not the only ones who thoroughly enjoy the Natural Philosophy lectures, and indeed it is said that sometimes when dealing with the more complex parts of Dynamics the lecturer alone seems perfectly contented.
>
> Listening to Professor Tait, one feels that it is quite impossible ever to forget the facts talked about and their relations and consequences. But sad experience, gained in the examination hall, has made it quite plain that it is the lucidity of the lecturer who makes everything transparent to us for the time, not we who can see through Carnot's Reversible Cycle and Simple Harmonic Motion on first hearing about them.[35]

Tait's presence in the lecture theatre was no doubt partly on account of his height and his uncommonly strong physique. The physicist and mathematician, Alexander Macfarlane (1851–1913), who was a former student of Tait's at Edinburgh and who became a disciple of the quaternionic system, described Tait's physical appearance in the following terms:

> Tait was not only an intellectual, but likewise a physical, giant. I am nearly six feet high, but standing beside Tait, I used to feel diminutive. He was well-built, and muscular. He wore a long beard, the hair on the top of his head had disappeared at an early date, and left exposed a massive forehead. To protect his head while lecturing it was his custom to wear a skull cap. On the street he wore a sack-coat and a soft felt hat, and with cane in hand, was always walking rapidly.[36]

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[34] According to Barrie, Tait's experiments were not always successful. On hand, should the equipment fail, was Tait's mechanical assistant, Thomas Lindsay, son of James Lindsay who was mechanical assistant to both Forbes and Tait. [34, pp50–51]

[35] [41]

[36] [42, p42]

Professor Tait is captured in lecturing mode in Figure 0.1 (page 1). It appears, from the apparatus set up on the bench and the formulae written on the chalk-board, that he is explaining the fundamental principles of electrostatics. From Knott we learn that the apparatus Tait is operating is the Holtz machine.[37]

In 1868 Tait further contributed to the natural philosophy department, when he undertook the development of laboratory facilities and secured a grant for this purpose with the support of the Principal of the university, Sir David Brewster (1781–1868). With the benefit of the new laboratory, Tait could offer students the opportunity to undertake their own original investigations, after some initial training in research methods provided by Tait and his assistants. Many of these students were actually put to work helping Tait in his experimental researches. Tait's laboratory at Edinburgh was modelled on William Thomson's Glasgow laboratories.[38] Explaining

[37] [18, pp49–50] The Holtz machine converts mechanical energy into static electricity by a process of induction. The static electricity is then stored in the Leyden jar capacitors, also shown in the sketch (Figure 0.1, page 1). The formula $Q = CV$ gives the voltage-charge relationship for the capacitor: Q is the charge on the capacitor, C is a constant called the capacitance and V is the voltage across the capacitor. The formula $W = \frac{Q^2}{2C}$ gives the energy stored in the capacitor W, in terms of charge and capacitance.

[38] See Graeme Gooday's Ph.D. thesis [43]: chapters 2,3,7 are especially relevant to this thesis. Gooday argues that Thomson's Glasgow laboratory (his chapter 2) became the authoritative example in laboratory teaching and precision measurement practices for subsequent laboratories, including Tait's in Edinburgh (his chapter 3) and Balfour Stewart's at Owens College in Manchester (his chapter 7). Gooday details the efforts of Thomson, Tait and Stewart to secure—in spite of problems with funding and accommodation—an experimental environment in which students could have: a practical illustration of the theoretical laws introduced in lectures, instruction in research methods and the opportunity to engage in their own research, working on customized experiments (which were often at the centre of their professor's own current researches) with the chance to make a real contribution to science. Gooday describes the approach of Thomson, Tait and Stewart in their respective laboratories as non-hierarchical; and thus they operated in a manner true to the 'Scottish "democratic" context'. [43, p2:19] In these laboratories there was little distinction between teaching and research, which Gooday characterizes as a 'uniquely Scottish form of natural philosophy pedagogy'. [43, p2:25] Thomson's was a largely organic approach, while Tait and Stewart followed a more organized plan of study in their laboratories.

the importance of the physical laboratories at Edinburgh and Glasgow, and the significance of Tait and Thomson's influence, Macfarlane writes:

> Prior to the founding of the Cavendish Laboratory at Cambridge, the facilities at Edinburgh and Glasgow for gaining an experimental knowledge of physics were the best in Great Britain and this was due to the circumstance that in these twin cities of the North, the chairs of physics were occupied by twin giants in physical science.[39]

1.2.2 The Tait Memorial Movement

Following Tait's death, his friends and former students expressed a desire to contribute something to a memorial to him. Unfortunately, the raising of funds had to be postponed, as the University of Edinburgh was already running an Extension Fund and they were not prepared to run the Tait Memorial Fund alongside. Consequently, fund-raising for Tait began in 1911, when the university's Extension Scheme was completed. Fortunately, this coincided with the publication of Knott's biography of Tait, which helped renew enthusiasm for the cause.

A Tait Memorial Committee was established in November 1911, at a meeting held in the physical laboratory at the university, with the Principal, Sir William Turner (1832–1916) presiding. Serving as Honorary Secretary to the committee was Prof. J. G. MacGregor (1852–1913), Tait's successor.[40] When MacGregor died suddenly in May 1913, Knott took on the role of Honorary Secretary.

At a subsequent meeting of the committee, it was decided that a fitting memorial to Tait would be the institution of a second chair in the natural philosophy department: 'The proposed chair would be connected with the department of the Professor's work in which he achieved especially conspicuous success, namely, the

[39] [42, p41]

[40] Prof. James Gordon MacGregor (1852–1913): born in Nova Scotia, Canada; Professor of Natural Philosophy at the University of Edinburgh (1901–1913); a student at the university during the 1870s; he undertook research in Tait's laboratory on the electric conductivity of saline solutions, c.1872. [12, p250]

application of mathematics to the solution of physical problems, including those which bear upon engineering and other departments of applied science'.[41] MacGregor had recalled that Tait himself had 'strongly urged the founding of a second chair in his department'.[42] For this purpose, the committee recommended raising a sum of £20–25,000. Appeals for funds were made to former students of Tait's (some ten thousand in number), men of science, businessmen and men of industry.[43]

Amongst the documents relating to the Tait Memorial Movement, preserved in the National Library of Scotland, there is an official letter from Sir James Alfred Ewing (1855–1935), Turner's successor as Principal, dated 9 November 1918.[44] He was writing to formally lend his support to the work of the committee. The following is an extract from his letter:

> The University of Edinburgh ought certainly to have a memorial of Tait. He was for many years a great figure in it as a teacher, a thinker, and founder of a school of research. During his Professorship here he exerted an influence that has profoundly affected the development of Physical Science. No memorial to him could be more fitting than the endowment of a professorship in the branch of science with which his name has and will always have a world-wide association, namely, the Mathematical

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[41] [44]

[42] [Ibid.] Tait was writing in *Macmillan's Magazine* in 1872.

[43] Those who contributed to the Tait Memorial Fund, according to [45], include: Prof. P. R. Scott Lang (1850–1926), who was assistant to Tait in the natural philosophy department at Edinburgh in the early 1870s, later Regius Professor of Mathematics at St Andrews (1879–1921) (£25); the physicist and mathematician, Alexander Macfarlane (1851–1913), who was a former student of Tait's at Edinburgh and a disciple of the quaternionic system (£10); the oceanographer, Sir John Murray (1841–1914), who is remembered for his work in connection with the *Challenger* Expedition, having first worked on constructing the apparatus for use in the expedition in Tait's laboratory in the 1870s, his home in Wardie, Leith was where Tait died in July 1901 (£100); J. S. Porter, Tait's brother-in-law (£100), the Revd. H. S. Reid and Mrs. Reid, Tait's son-in-law and daughter (£25); and Mrs. Tait, P.G.T.'s wife (£100).

[44] Sir James Alfred Ewing (1855–1935): Principal of the University of Edinburgh (1916–1929); previously a student of engineering at the university; he worked with MacGregor in Tait's laboratory during the 1870s. [12, pp250,273]

side of Physics. [...]

As myself an old pupil of Tait, grateful of his inspiration and cherishing his memory, I earnestly hope that the efforts of your Committee will secure for him this most appropriate memorial.[45]

The title of the proposed second chair was discussed at a meeting of the executive committee on 28 May 1920.[46] While the decision would ultimately be made by the University Court, the committee recommended: 'the Tait Chair of Natural Philosophy on the mathematical and theoretical side, which might be shortly termed Mathematical Physics'.[47]

Raising the requisite sum proved difficult. By May 1920 the total subscriptions amounted to just under £3000 and the committee realized that there was no prospect of them raising all of the money themselves—they would need to join forces with

[45] Ewing to Convener of the Tait Memorial Committee (9 Nov. 1918) in [46].

[46] The Tait Memorial Committee: Members of the executive sub-committee, according to [44]. (1) W. B. Blaikie (1847–1928): a civil engineer, astronomer and printer; a former pupil at the Edinburgh Academy (1858–1864). (2) B. Hall Blyth (1849–1917): a civil engineer and a student at the University of Edinburgh in the 1860s. (3) George A. Gibson (1858–1930): extra-academical lecturer in medicine at the University of Edinburgh; Prof. of Mathematics at the University of Glasgow (1909–1927) and President of the Edinburgh Mathematical Society (1888–1889). (4) Alexander Crum Brown (1838–1922): Prof. of Chemistry at the University of Edinburgh (1869–1908) and Tait's brother-in-law. (5) C. G. Knott (1856–1922): Tait's biographer; a former student of Tait's at Edinburgh; assistant to Tait in the natural philosophy department (1876–1883); appointed lecturer in mathematics (1891) and reader in applied mathematics at the University of Edinburgh. (6) J. G. MacGregor (1852–1913): Honorary Secretary of the Tait Memorial Committee; see footnote no.40. (7) Sir David Paulin (1847–1930): an actuary who founded the Scottish Life Assurance Co. (8) William Peddie (1861–1946): in charge of the natural philosophy laboratories at Edinburgh; assistant (1883) and lecturer (1892–1907) in the natural philosophy department; Professor of Physics at University College, Dundee (1907–1942) which was then a constituent college of St Andrews, and President of the Edinburgh Mathematical Society (1896–1897). [All eight were Fellows of the R.S.E.] And (9) Sir George M. Paul (1839–1926): Honorary Treasurer for the Tait Memorial Committee; an Edinburgh lawyer who studied law at the University of Edinburgh during the 1860s.

[47] [47]

other persons or bodies with the same aim. Funds were distributed to the R.S.E. (£1550) and the University of Edinburgh (£640). An amount was invested in war stock, with dividends presumably being dispersed periodically in support of the second chair. The Tait Chair of Natural Philosophy was eventually founded in 1922 and in 1923 the physicist, Charles Galton Darwin (1887–1962), grandson of Charles Darwin, the naturalist, became its first incumbent.[48]

1.2.3 Contribution to the Royal Society of Edinburgh

Tait was proposed for fellowship of the R.S.E. by Philip Kelland in December 1860 and was elected on 7 January 1861. He served the Society as Councillor (1861–1864), Secretary to the Ordinary Meetings (1864–1879) and General Secretary (1879–1901).[49]

Campbell and Smellie, in their history of the R.S.E., describe Tait as a 'devoted servant of the Society'.[50] They also credit Tait, together with Thomson, as having 'played a great role in maintaining the prestige of the Society'.[51] Knott, who was himself a Fellow of the Society, described Tait's role in the R.S.E. thus: 'To many of the frequenters of the meetings in the seventies and eighties, Tait was in fact the Royal Society [of Edinburgh]; and there is no doubt that he guided its affairs with consummate skill.'[52]

Following Tait's death, the Council of the Society formally expressed their feelings of loss in the minutes of the meeting of the Council held on 19 July 1901. The

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48 [12, p252] The title of the chair changed in 1966 to the Tait Chair of Mathematical Physics. According to [48], the current Tait Professor of Mathematical Physics is Professor R. D. Kenway, O.B.E., F.R.S.E.

49 [29]

50 [33, p45]

51 [33, p46]

52 [18, p30] Knott served the R.S.E. as Councillor (1894–1897, 1898–1901, 1902–1905), Secretary to the Ordinary Meetings (1905–1912) and General Secretary (1912–1922). He was proposed for fellowship by Tait, and others, in 1880. He received the Society's Keith Prize in 1893. [29]

original handwritten document, of which the following is an extract, is preserved in Tait's scrapbook. It is signed by Kelvin (William Thomson), who was serving as President of the Society at the time.

> During his forty years of Fellowship he [Tait] contributed a very large number of Papers, all of them original and interesting, some of them of the very highest importance, with a place assured for ever in the history of the development of science.
>
> His loss will be felt in the Society not only as a contributor, but perhaps even more as a wise counsellor and guide. The Council always felt that in his hands the affairs of the Society were safe—nothing would be forgotten, everything that ought to be done would be brought before them at the right time and in the right way. [...]
>
> [...] What the Council now feel is that a great man has been removed, a man great in intellect and in the power of using it, in clearness of vision and purity of purpose, and therefore great in his influence, always for good, on his fellow men; they feel that they and many in the Society and beyond it have lost a strong and true friend.[53]

The Society awarded Tait its Keith Prize on two occasions (1867–1869, 1871–1873) and once, their Gunning Victoria Jubilee Prize (1887–1890). The Keith Prize was awarded '"for the most important discoveries in Science made in any part of the world, but communicated by their author to the Royal Society of Edinburgh and published for the first time in the Transactions"'.[54] The Gunning Victoria Jubilee Prize was awarded to those with Scottish connections, 'in recognition of original work in Physics, Chemistry, or Pure or Applied Mathematics'.[55] Tait won the Keith Prize in 1867 for his paper, 'On the Rotation of a Rigid Body About a Fixed Point' and in 1871 for a paper entitled, 'First Approximation to a Thermoelectric Diagram'. The Gunning Victoria Jubilee Prize was awarded to Tait for his researches connected

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[53] [49] Published in *Proc. Roy. Soc. Edinburgh*, XXIV (1901–1902), pp. 2–4.

[54] [33, p153] The Keith Prize was awarded every two years (biological and physical sciences alternately); winners received a solid gold medal, which had been gifted to the Society by its first Treasurer, Alexander Keith of Dunottar. [Ibid.]

[55] [33, p152] The Gunning Victoria Jubilee Prize was a monetary award, presented every four years; it was founded in 1887 by His Excellency, Dr. R. H. Gunning (1818–1900). [Ibid.]

with the *Challenger* Expedition and in the wider context of his contribution to physical science.

1.2.4 Involvement in controversy

Aspects of Tait's character often led him into bitter disputes with other leading scientists. Knott believed that it was Tait's straight-forward approach and his loyalty to his friends and colleagues that often led to his involvement in controversy—'always on behalf of others'.[56] 'Tait was one who reacted sharply to anything he considered unfair or unjust either to himself or to his colleagues', writes Campbell and Smellie.[57] Following Tait's death in 1901, his personal friend, Professor Flint acknowledged:

> I am quite aware that great as he [Tait] was, he had his own limitations, and sometimes looked at things and persons from one-sided and exaggerated points of view, but the consequent aberrations of judgement were of a kind which did no one much harm and only made himself the more interesting. His strong likes and dislikes, although generally in essentials just, were apt to be too strong.[58]

Tait entered public disputes with the Irish natural philosopher, John Tyndall (1820–1893) over priority in energy physics and Forbes' glacier work.[59] Over Thomson's thermodynamic discoveries, he came into conflict with the Prussian, Rudolf Clausius (1822–1888). And his intense dislike of the vector calculus which threatened Hamilton's quaternions brought him into conflict with the Englishman, Oliver Heaviside (1850–1925) and the American, J. Willard Gibbs (1839–1903).[60]

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[56] [18, p37]

[57] [33, p69]

[58] [38, pp60–61]

[59] John Tyndall (1820–1893): Professor of Natural Philosophy at the Royal Institution; remembered for his work on glacial movement, light, sound, magnetism and radiant heat in the context of gases and vapours. [50]

[60] See Chris Pritchard's two papers, [51] and [52]. The first covers: Tait's initial interest in quaternions and his correspondence with Hamilton; the challenges faced by Tait during his campaign for

Tait's involvement in the controversy which arose in the 1860s over energy physics showed him in an especially bad light. Fiercely patriotic, Tait took to public forums to assert British priority in the development of the science of energy; in the discovery of the equivalence of work and heat, in particular. In defence of the Englishman, James Prescott Joule's (1818–1889) priority in the discovery, Tait criticized the contribution of the German, Julius von Mayer (1814–1878) and diminished his priority.[61] Understandably, accusations of chauvinism were levelled at Tait following his unreasonable conduct in the affair. His response was this:

> "I cannot pretend to absolute accuracy, but I have taken every means of ensuring it, to the best of my ability, though it is possible that circumstances may have led me to regard the question from a somewhat too British point of view. But, even supposing this to be the case, it appears to me that unless contemporary history be written with some little partiality, it will be impossible for the future historian to compile from the works of the present day a complete and unbiased statement. Are not both judge and jury greatly assisted to a correct verdict by the avowedly partial statements of rival pleaders? If not, where is the use of counsel?"[62]

1.2.5 Family life

Early home life

Peter Guthrie Tait was born in Dalkeith, Midlothian on 28 April 1831. His parents, John Tait (Private Secretary to the fifth Duke of Buccleuch, Walter Francis Scott) and Mary Ronaldson (daughter of John Ronaldson, a tenant farmer) were married on 27 June 1829 in Dalkeith. P.G.T. was the eldest of their three children: he had

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the acceptance of quaternions and Maxwell's position on quaternions. The second covers Tait's disputes with Arthur Cayley (over Cartesian versus quaternion methods) and with Heaviside and Gibbs.

[61] See [36] for an excellent account of the disputes which arose over energy physics during the nineteenth century.

[62] [42, pp43–44]

younger sisters, Anne Margaret and Mary.

P.G.T. lost both of his parents in his childhood: at the age of six he lost his father and at the age of fifteen he lost his mother. When his father died, the family moved to Warriston Crescent, Edinburgh. When his mother died, the children went to live in Somerset Cottage, Raeburn Place (also in Edinburgh); with Mary's bachelor brother, John Ronaldson who was a clerk in the National Bank of Scotland and Mary's maiden sister, Margaret Ronaldson. Presumably, John and Margaret assumed parental responsibility for the children. According to Knott, although John Ronaldson was a banker by profession, he had a keen interest in scientific investigation which influenced P.G.T.'s enthusiasm for science: Ronaldson spent much quality time with P.G.T.—on geological rambles, making astronomical observations and getting to grips with photography which had recently been invented.[63]

Wife and children

Tait married Margaret Archer Porter on 13 October 1857 in Shankill, Antrim. Margaret was sister to the Porter brothers, William Archer and James, who Tait had known at Cambridge.[64] Together, Tait and his wife Margaret had six children: Edith, John Guthrie, Mary Guthrie, William Archer, Frederick Guthrie and Alexander Guthrie. With the exception of Edith, who was born in Belfast, all the children were born in Edinburgh.

Edith married Harry Seymour Reid, later Bishop of Edinburgh.[65] Mary married Charles Walker Cathcart, an Edinburgh surgeon. Of all Tait's children, his eldest son, John Guthrie seems to have followed most closely in his father's footsteps. He was Cambridge educated (a student of Peterhouse) and became a Fellow of the R.S.E. Unlike Tait, however, after Cambridge he trained as a barrister and

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[63] [18, p3]

[64] James Porter (1827–1900) went on to become Master of Peterhouse, Cambridge (1876–1900) and Vice-Chancellor (1881–1884). [13] Tait grew close to the Porter family during his time in Belfast.

[65] H. S. Reid officiated, with the Revd. Canon Cowley Brown, at Tait's funeral which took place at St John's Episcopal Church, Edinburgh. [18, p40]

was called to the bar in 1888. Then in 1890 he left for India to take up a position in the government education department; and in 1908 he became Principal of the Central College, Bangalore, where previously he was Professor of Languages and Vice-Principal. In 1904 he married the daughter of his predecessor as Principal, John Cook who was originally a mathematics teacher of Arbroath, Scotland. William was a civil engineer who received his training at the University of Edinburgh and under Sir J. Wolfe Barry and H. M. Brunel, son of Isambard Kingdom Brunel (1806–1859). He was involved in building the Talla Reservoir in the Scottish Borders. He was also a Fellow of the R.S.E. and served as Vice-President between 1921 and 1924. Freddie (Figure 1.1, page 22) was a Lieutenant in the Black Watch. He was killed in action in the South African War, in the battle of Koodoosberg Drift: he was shot in the heart, leading his men in a reconnaissance mission under General Macdonald.[66] He is remembered today, especially in St Andrews, as Scotland's amateur golf champion.[67] More information on his military career and his golfing successes can be found in John Low's *F. G. Tait: A Record* [53], which is a most precious resource for those desiring an insight into Tait family life: excerpts from Freddie's letters home portray beautifully P.G.T. in his beloved role as head of the Tait family. Alexander, the youngest of Tait's children, was a glass merchant and spent a period of time in the port city of Liverpool. All Tait's boys were educated at the Edinburgh Academy and were noted for their sporting prowess: in golf, shooting and rugby. For further details about the lives of Tait's children see Appendix A.

A family tragedy

The end of Tait's long and remarkable career followed shortly after Freddie's death in February 1900. Tait was profoundly affected by the tragic event and from the

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[66]Following Freddie's death, a memorial fund was started in his name to put towards a new wing at the Cottage Hospital in St Andrews. [53, p226f] The new wing was opened by his mother Margaret, according to an exhibit in the St Andrews Preservation Trust Museum on North Street in St Andrews.

[67]Winner (1896, 1898); runner-up (1899).

time of his bereavement his own health began to fail. The change in him was noted by his life-long friend and collaborator, William Thomson in his obituary tribute to Tait:

> The cheerful brightness which I found on our first acquaintance forty-one years ago remained fresh during all these years, till first clouded when news came of the death in battle of his son Freddie in South Africa, on the day of his return to duty after recovery from wounds received at Magersfontein. The cheerful brightness never quite returned.[68]

Tait stopped teaching at the university in December 1900 and on 30 March 1901 he formally retired from the natural philosophy chair. He died less than four months later on 4 July 1901.

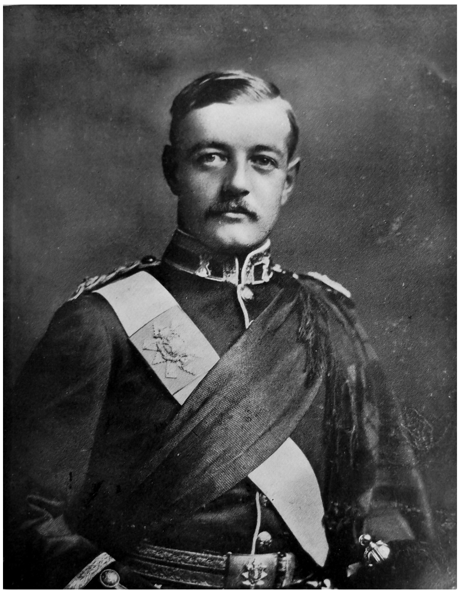

Figure 1.1: Lieutenant Freddie Tait, a photograph by Marshall Wane, 1896.[69]

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[68] [28, p369]

[69] [53, pfacing 78]

1.2.6 Spiritual life

Tait was known to have been a member of the Scottish Episcopal Church.[70] Other than this, little information is available on the spiritual aspect of his life. Knott, in his biography of Tait, offers the following insights. He writes:

> Tait was indeed a close student of the sacred records. The Revised Version of the New Testament always lay conveniently to hand on his study table; and frequently alongside of it lay the Rev. Edward White's book on Conditional Immortality. I am not aware that he distinctly avowed himself a believer in this doctrine, as Stokes did, but he often expressed the high opinion he held of Edward White and his writings. His reverence for the undoubted essentials of the Christian Faith was deep and unmovable; and nothing pained him so much as a flippant use of a quotation from the Gospel writings.[71]

Robert Flint, in his obituary tribute to Tait, described Tait as someone who had managed to reconcile, in his own life, the scientific and religious aspects:

> Our departed friend had no sympathy with theological dogmatism, and little with anti-religious scepticism, and consequently held in contempt discussions on the so-called incompatibility of religion and science. At the same time, he had a steady yet thoughtful faith in God, and in that universe which no mere eye of sense, aided by any material instrument, can see. This faith must have made his life richer, stronger, and happier than it would otherwise have been.[72]

Tait challenged 'the so-called incompatibility of religion and science' in *The Unseen Universe*, which is the subject of chapter 2.

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[70] Tait is known to have attended St John's Episcopal Church in Edinburgh, where his body is interred in the family grave. The Scottish Episcopal Church is a Scottish Christian Church in communion with, but historically distinct from, the Church of England; it is a member of the Anglican Communion; comprising seven diocese, each with a bishop.

[71] [18, pp36–37]

[72] [38, p62] Reproduced in [18, pp44–46].

CHAPTER 2

THE UNSEEN UNIVERSE (1875)

This chapter is about a remarkable book which Tait co-authored with the Scottish physicist and meteorologist, Balfour Stewart. In *The Unseen Universe*—which was originally penned anonymously—Tait and Stewart proposed hypotheses which they hoped would serve to unite science and Christian doctrine.

2.1 Introduction

In April 1875 an anonymous publication appeared, in which the possibility of immortality and the existence of an unseen universe were argued for on a scientific basis. The authors—revealed to be two leading figures in nineteenth century physical science—believed that they had found unity in the latest scientific theories and the established doctrines of Christianity.

The authors approached their task humbly, as 'reverent students of the Scriptures', hoping to communicate that science, properly understood, does not stand in opposition to religion.[73] Plainly, in their own words: 'Our object, in this present work, is to endeavour to show that the presumed incompatibility of Science and Religion does not exist'.[74] Guided by their fundamental Principle of Continuity, they believed that they could show science to be, in fact, the 'most efficient supporter' of Christian doctrines.[75] Their chief concern was to remove the scientific objection to immortality.

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[73] [54]

[74] [55, pxi]

[75] [55, p209]

They aimed their argument to those for whom scientific objections to man's immortality and the existence of an invisible world—raised by some in the scientific community—were proving to be insurmountable hurdles to a belief in these doctrines.

2.1.1 The anonymity game

The Unseen Universe; Or, Physical Speculations on a Future State ran through three anonymous editions before authorship was revealed officially in April 1876. With each edition came the critics' reviews and conjectures on the mysterious authorship issue.

Publishing anonymously using the third person plural prompted many to assume a joint or collaborative effort. Rumours of two distinguished physicists were supported by evidence from the book of the authors' 'thorough knowledge of the highest and most recent developments of natural philosophy'.[76] To some, however, the book's title and the nature of some of the authors' hypotheses were suggestive of spiritualist authors.

In May 1875 *The Spiritualist Newspaper* was able to quash rumours of spiritualist authors, by reporting the *Athenaeum*'s revelation of the authors' identities: 'Dr. Balfour Stewart, of Manchester, and Mr. P. G. Tait, Professor of Natural Philosophy at the University of Edinburgh.'[77] In the preface to their fifth edition, Tait and Stewart admonished the 'London "Weekly,"' who stated their names as facts, and did so without authorization, barely days after publication of the first edition.[78] Other critics played fairer and refused to publish names, in respect of the authors' desire to remain anonymous.

Despite such an early outing, which no doubt came as a blow to Tait and Stewart, there were pockets of individuals who remained unaware of the authors' true

[76] [56]

[77] [57]

[78] [58, pxxiii]

identities and the curious among them continued to search the text for clues left unwittingly by the authors. Naturally, those familiar with Tait and Stewart's scientific interests and their connections within the scientific community would have easily deduced authorship. There are references to William Rowan Hamilton and William Thomson, later Lord Kelvin, who were both associated with Tait. There are detailed references to sunspots and J. D. Forbes' work on glaciers, which both relate to Stewart. There are also many explicit references to Tait and Stewart's earlier publications, for instance: Tait's *Thermodynamics*; Tait and Thomson's *Natural Philosophy*; Tait and Steele's *Dynamics of a Particle* and Stewart's *Conservation of Energy.* References to the experiments carried out jointly by Tait and Stewart, on a disk rotating *in vacuo*, are also very suggestive.

As late on as December 1875, William Thomson's name remained associated with the book: as one of three authors (Tait, Thomson and Stewart) or as half of the already known collaborative team, Tait and Thomson. Tait responded to this succession of candidates by annotating a review published in *The Glasgow Herald*, with a reference in Greek to Book VI of Homer's *Iliad*: '$\phi\upsilon\lambda\lambda\omega\nu\ \gamma\epsilon\nu\epsilon\eta$'.[79] See Figure 2.1 (page 27). I quote from an 1866 translation by the English mathematician and astronomer, Sir John F. W. Herschel (1792–1871):

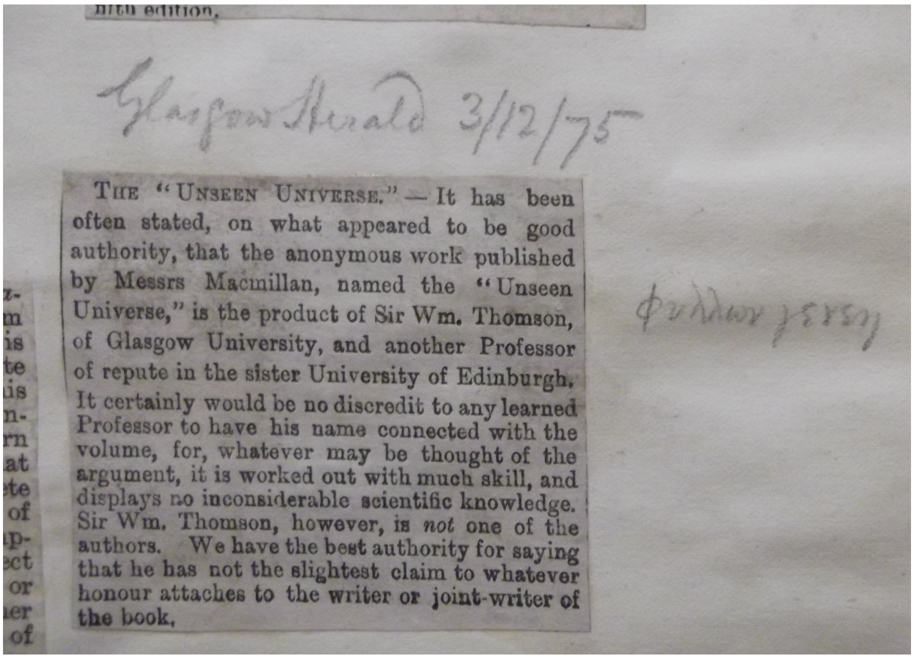
> Man's generations flourish and fall, like the leaves of the forest.
> Leaves on the earth by winds are strown, yet others succeed them,
> Ever renewed with returning spring. So fares it with mortals:
> One generation decays and its place is filled by another.[80]

The consensus view as reported by *The Glasgow Herald* was that Thomson and

[79] A quotation taken from line 146. I am grateful to Daniel Mintz (Ph.D. 2011, St Andrews, History of Mathematics) for recognizing the quotation's original source. Elsewhere in Tait's scrapbook, lines 146–149 are written out in full in the original Greek. Unfortunately, the entry is undated, however, it is possible that the entry dates from Tait's time at the Edinburgh Academy: the handwriting appears naive and, according to the Directors' *Report* for 1847 [59, p5], Tait would have studied Book VI of Homer's *Iliad* in his final year at the Edinburgh Academy.

[80] [60, p121] Lines 146–149 from Book VI of Homer's *Iliad*.

'another Professor of repute' had co-authored *The Unseen Universe*, which was contradicted by the newspaper, who claimed to have it on 'the best authority' that Thomson was not one of the authors.[81]



Glasgow Herald 3/12/75

THE "UNSEEN UNIVERSE."—It has been often stated, on what appeared to be good authority, that the anonymous work published by Messrs Macmillan, named the "Unseen Universe," is the product of Sir Wm. Thomson, of Glasgow University, and another Professor of repute in the sister University of Edinburgh. It certainly would be no discredit to any learned Professor to have his name connected with the volume, for, whatever may be thought of the argument, it is worked out with much skill, and displays no inconsiderable scientific knowledge. Sir Wm. Thomson, however, is *not* one of the authors. We have the best authority for saying that he has not the slightest claim to whatever honour attaches to the writer or joint-writer of the book.

φυλλων γενεη

Figure 2.1: Tait's annotation of a review published in *The Glasgow Herald*, 3 December 1875. Sourced from Tait's scrapbook. Reproduced with the kind permission of the J.C.M. Foundation. '$\phi\upsilon\lambda\lambda\omega\nu\ \gamma\epsilon\nu\epsilon\eta$' is a quotation from Book VI of Homer's *Iliad*, line 146.

It is interesting to speculate on Tait and Stewart's reasoning behind choosing anonymity. By withholding authorship, their readers would approach the work without preconceived ideas of what they might expect from a well known, named author: the book would be judged solely on the quality of the authors' speculations. Anonymity would also serve as a tactical measure, to promote discussion which would draw in a readership. Since Tait and Stewart invited criticism of their hypotheses and since they dealt with the criticism they received robustly and publicly, it is inconceivable that they opted for anonymity in order to shield themselves from

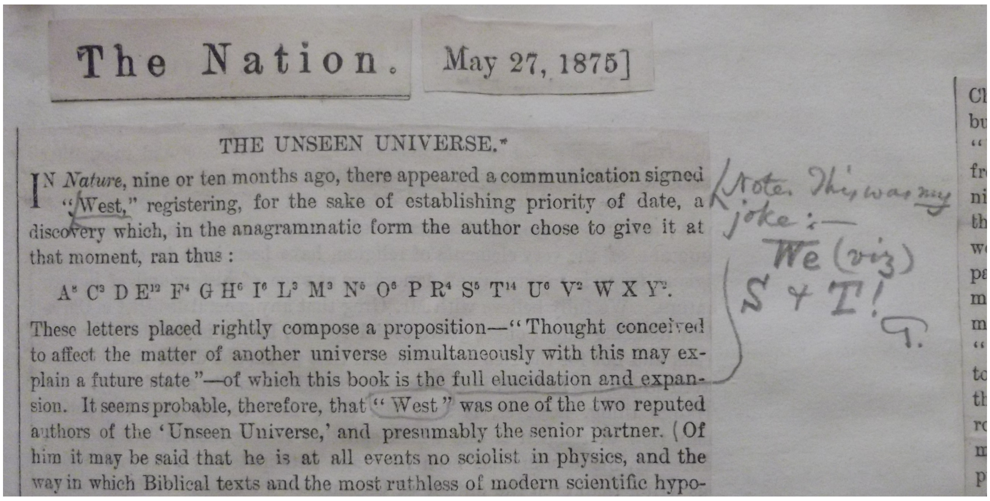
[81] [61]

harsh criticism and to protect their reputations. Tait, certainly, was not averse to the cut and thrust of public debate.

Evidence exists which suggests that Tait, in particular, was prepared to resist the draw of recognition so that he might enjoy participating in the intellectual speculations surrounding anonymity. In May 1875 a critic from *The Nation* newspaper brought to light a communication which had been published in *Nature* magazine on 15 October 1874. The communication, signed 'West', gave a proposition in the form of an anagram:

$$A^8\ C^3\ D\ E^{12}\ F^4\ G\ H^6\ I^6\ L^3\ M^3\ N^5\ O^6\ P\ R^4\ S^5\ T^{14}\ U^6\ V^2\ W\ X\ Y^2$$

With its letters correctly arranged it reads: '"Thought conceived to affect the matter of another universe simultaneously with this may explain a future state."' The critic recognized *The Unseen Universe* as 'the full elucidation and expansion' of this proposition and went on to reason that West was in fact 'one of the two reputed authors of "The Unseen Universe," and presumably the senior partner'.[82]



The Nation. May 27, 1875]

THE UNSEEN UNIVERSE.*

IN *Nature*, nine or ten months ago, there appeared a communication signed "West," registering, for the sake of establishing priority of date, a discovery which, in the anagrammatic form the author chose to give it at that moment, ran thus :

A⁸ C³ D E¹² F⁴ G H⁶ I⁶ L³ M³ N⁵ O⁶ P R⁴ S⁵ T¹⁴ U⁶ V² W X Y².

These letters placed rightly compose a proposition—"Thought conceived to affect the matter of another universe simultaneously with this may explain a future state"—of which this book is the full elucidation and expansion. It seems probable, therefore, that "West" was one of the two reputed authors of the 'Unseen Universe,' and presumably the senior partner. (Of him it may be said that he is at all events no sciolist in physics, and the way in which Biblical texts and the most ruthless of modern scientific hypo-

Note. This was my joke:—
We (viz) S & T!
T.

Figure 2.2: Tait's annotation of a review published in *The Nation*, 27 May 1875. Sourced from Tait's scrapbook. Reproduced with the kind permission of the J.C.M. Foundation.

[82] [62]

Again a cutting of the review is preserved in Tait's scrapbook. Figure 2.2 (page 28) is a photograph of the original. The name West has been circled and a handwritten annotation, signed by Tait, has been inserted: 'Note. This was my joke:– We (viz) S & T !' So the name West stood for both authors, "Stewart & Tait", who had written the communication in *Nature* so as to cryptically reveal authorship well in advance of publication and thereby establish priority when the first edition appeared.[83]

2.1.2 Balfour Stewart (1828–1887): a biographical sketch

Balfour Stewart (Figure 2.3), a distinguished Scottish physicist and meteorologist, is remembered for his dedication to observation and experimental research, and for his ability to make from these, incisive deductions in the areas of radiant heat and solar phenomena in particular.

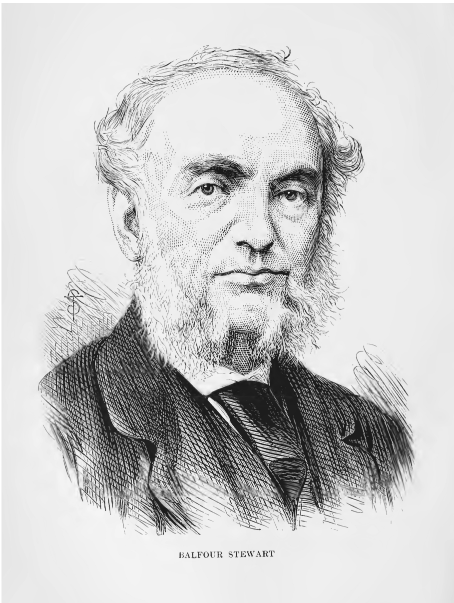


Figure 2.3: Balfour Stewart, a sketched portrait.[84]Original source unknown.

[83]To establish priority Tait and Stewart referred to the *Nature* article in the book. See [55, p159].

[84] [63, pfacing 257] The same source [63, pp359–361] gives a biographical sketch of Stewart.

Stewart and Tait became acquainted in 1861, when Stewart was appointed an Additional Examiner in Mathematics for the University of Edinburgh. They collaborated from 1866 onwards; initially, working together on experiments investigating the heating of a disk rotating *in vacuo*.[85]

Stewart was born in Edinburgh on 1 November 1828, to parents William Stewart, a tea merchant of Leith, and Jane Clouston, daughter of William Clouston (a minister of Stromness, Orkney). Stewart's university education began, at the age of thirteen, at St Andrews. From St Andrews he went on to Edinburgh (1845–1846) where he studied natural philosophy under Forbes. Encouraged by his parents, he embarked on a career as a merchant; but after a brief period spent in Leith and Australia, he returned to Edinburgh to resume his interest in physical science.

In February 1856 he was appointed as assistant observer to John Welsh (1824–1859) at Kew Observatory and in October of that year, assistant to Forbes at Edinburgh.[86] In 1859 he succeeded Welsh as Director of Kew Observatory. In 1862 he was elected a Fellow of the R.S.E.; honorary fellowship followed in 1878.[87] Stewart remained at Kew until 1870, when he took up a position as Professor of Physics at Owens College, Manchester.[88] He died in 1887, while on holiday in Drogheda, Ireland. He had married in 1863 and had three children.

Stewart might have had a distinguished career as a mathematician—he had shown promise as a mathematician under the influence of Kelland at Edinburgh,

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[85] For the results of Tait and Stewart's experiments see [64] and [65]. Follow-up papers on the subject were published in 1867, 1873 and 1878.

[86] [66] Stewart's biographer, Schuster has 1853 for the date Stewart became Forbes' assistant. See [67, p253].

[87] Stewart was proposed for fellowship in 1862 by Forbes. [29] Stewart also served as: Secretary to the Government Meteorological Committee (1867); President of the Society for Psychical Research (1885); President of the Manchester Literary and Philosophical Society (1887) and President of the Physical Society (1887). From the University of Edinburgh Stewart received the honorary degree of LL.D.

[88] Owens College was founded in 1851. In 1880 it became the Victoria University of Manchester and in 2004 it amalgamated with UMIST and became the University of Manchester.

and had published a paper on the theory of numbers—but he chose to devote himself to experimental science.[89] His interests—particularly in heat, meteorology and terrestrial magnetism—were influenced by his association with Forbes.[90]

Stewart's most important contribution was his work on radiant heat: in Forbes' laboratory in Edinburgh he undertook researches which led him to an extension of Prévost's Law of Exchanges. For these researches he was awarded the Rumford Medal by the Royal Society in 1868. In his obituary tribute to Stewart, Tait explains the significance of Stewart's contribution:

> His paper (which was published in the Transactions of the Royal Society of Edinburgh) contained the greatest step which had been taken in the subject since the early days of Melloni and Forbes. The fact that radiation is not a mere surface phenomenon, but takes place like absorption throughout the interior of bodies, was seen to be an immediate consequence of the new mode in which Stewart viewed the subject.[91]

Stewart was the author of a number of other noteworthy papers, including those in which he presented the results of his experiments on: radiant light emitted through glass, tourmaline and uniaxial crystals; and, with Tait, the heating of a disk rotating *in vacuo*.[92]

At Kew Stewart was actively engaged in research into meteorology and terrestrial magnetism. Under his guidance Kew became a centre for the standardization and

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[89] Stewart published one paper on mathematics: 'On a Proposition in the Theory of Numbers'. See [68].

[90] [69, pix]

[91] [Ibid.]

[92] Amongst Stewart's principal publications were the textbooks: *Treatise on Heat* (1866); *Lessons in Elementary Practical Physics* (1870) with W. W. Haldane Gee; *Conservation of Energy* (1872) and *Primer in Physics* (1872). For the ninth edition of the *Encyclopaedia Britannica* he contributed an article, 'Terrestrial Magnetism'. He also presented papers on a variety of subjects to the Royal Societies of London and Edinburgh. An extensive summary of his work is given by his former student and successor at Owens College, Arthur Schuster (1851–1934) in [67].

testing of the instruments used in experiments in these fields.[93] In the area of solar physics he did much to establish the connection between sun spots, planetary configurations and terrestrial meteorology.

At Owens College Stewart had as his students, the Nobel Prize winner and discoverer of the electron, J. J. Thomson (1856–1940) and the distinguished physicist, John H. Poynting (1852–1914).

Little information is available on the religious aspect of Stewart's life. Baptismal records reveal that he was baptised into the Scottish Presbyterian Church.[94] Otherwise, the best there is in this regard is the following description of him:

> A devoted and fervent Churchman, who in later years was a member of a committee appointed by Lambeth Conference to promote interchange of views between scientific men of orthodox opinions in religious matters, he [Stewart] maintained throughout his career a deep interest in the more mysterious problems of existence, and became one of the founders of the Society for Psychical Research, over which he presided from the year 1885 until his death.[95]

2.2 The Christian man of science

As Christian men, the authors of *The Unseen Universe* believed that they encountered God both in the study of nature and in the Scriptures. For them there was no conflict between these two sources: they believed them to be of the same divine origin and to constitute the Two Books of Revelation. Referring to the Book of Nature metaphor, Tait and Stewart write: 'In fine, the physical properties of matter form the alphabet which is put into our hands by God, the study of which will, if

[93] [66]

[94] Stewart was baptised at the Tron Church in Edinburgh in 1828. The Scottish Presbyterian Church is not a member of the Anglican Communion and is different from the Scottish Episcopal Church in its administration in that it does not have bishops.

[95] [70, p163]

properly conducted, enable us more perfectly to read that Great Book which we call the Universe.'[96]

Drawing on these two sources, Tait and Stewart would inevitably face criticism. Their appeal to Scriptures to unite physical theories and the spiritual was regarded by many as unscientific; they were criticized on points of religious licence and accused of venturing beyond their own areas of expertise. One critic applied the warning of the German botanist, Schleiden (1804–1881) who was one of the first to accept Darwin:

> The first rule which the exact investigator of nature should observe is, that he should not allow himself to pronounce an opinion, either in affirmation or negation, on subjects which do not fall and cannot fall within the sphere of his observation and experience. [...] If the natural philosopher comes, not in his special capacity, but in that of man merely, to speak of these matters (as every man has a right to do), then he must have before his eyes the second rule, which is, that he must not pass opinion, form his judgement, nor utter it, upon matters of any science to the present level of which he has not brought himself.[97]

To those that accused Tait and Stewart of 'invading the province of religion' they responded:

> we do not write for those who are so assured of the truth of their religion that they are unable to entertain the smallest objection to it. We write for honest inquirers—for honest doubters, it may be, who desire to know what science, when allowed perfect liberty of thought and loyally followed, has to say upon those points which so much concern us all.[98]

Consistent with Christian teachings, our authors had faith in a Creator who created us in His own image so that we might be capable of knowing Him. This doctrine encourages an honest inquiry into those fundamental questions which demand both

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96 [55, 185]

97 [71] Originally *Ueber den Materialismus* (Leipzig : W.S.W., 1863).

98 [55, p162]

a scientific and theological approach. With an intimate knowledge of Christian doctrine, acquired through personal faith, and with a profound understanding of contemporary science, few would have been better placed than Tait and Stewart to ponder these questions and to identify the 'connecting link between Revelation and Science'.[99] Their scientific investigations, functioning as an inlet to Revelation, might even be considered to constitute religious service.

Tait and Stewart expected opposition to the theories which they were putting forward, yet they invited reactions from within both the scientific and religious camps: 'Entertaining these views we shall welcome with sincere pleasure any remarks or criticism on these speculations of ours, whether by the leaders of scientific thought or by those of religious inquiry.'[100] A bold invitation indeed.

2.2.1 The science versus religion debate: Tait and Stewart's contribution

The Unseen Universe was regarded, by some at the time, as a model of how to engage in the science versus religion debate, which was at its height in that era—'according to the properly exacting conditions of Science'.[101] In their own minds, the authors' particular contribution to the debate was to identify, once and for all, the true antagonists.

Tait and Stewart's hypotheses were intended to demonstrate that there is nothing inherent in science which cannot accommodate Christian doctrines. Accomplishing this, materialism might be exposed as the true foe of religion. Tait, in particular, expressed a fervent dislike of materialists, referring in his address to Section A of the B.A.A.S. in 1871, to the 'ignorance' which shows itself in the 'pernicious nonsense of the Materialist'.[102] Clearly, he thought materialism abhorrent and a dangerous

[99] [72, p419]

[100] [55, p210]

[101] [73]

[102] [74, p7]

deviation from real truth.

On this basis, it was inevitable that readers of *The Unseen Universe* would make the connection with John Tyndall's Belfast address, which Tyndall had delivered in August 1874, as President of the B.A.A.S. Although Tait and Stewart made no reference to the address and no explicit references to Tyndall, it was determined that *The Unseen Universe* had been written as a refutation of Tyndall's address.

2.2.2 Tyndall's Belfast address

John Tyndall was a self-professed materialist, who argued, with characteristic directness, for the freedom of scientific inquiry from religious authority and an end to religious intrusion into the domain of science; maintaining the 'superior authority of science over religious or non-rationalist explanations'.[103] His philosophy was shared by fellow members of the X-Club; a group of friends and eminent scientists, who formed themselves into a London society. One of its members, the mathematician, Thomas A. Hirst (1830–1892) explained: 'the bond that united us was devotion to science, pure and free, untrammelled by religious dogmas'.[104]

The theme of Tyndall's address was the historical development of man's intellect, with an emphasis on how well he realizes the natural impulse, inherent in men, to consider the 'sources of natural phenomena'.[105] Primeval man, rightly and naturally, drew on his experiences but erring in his focus—looking to man and not to nature—his theories took on an 'anthropomorphic form' so that 'supersensual beings [...] were handed over the rule and governance of natural phenomena'.[106] From the relationship between these capricious gods and mankind developed a sub-theme in

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[103] [75]

[104] [76, p311] Members of the X-Club, in addition to Tyndall and Hirst: Joseph Hooker (1817–1911), Thomas Huxley (1825–1895), Herbert Spencer (1820–1903), Edward Frankland (1825–1899), George Busk (1807–1886), John Lubbock (1834–1913) and William Spottiswoode (1825–1883). The X-Club met for the first time in November 1864. [76, p307]

[105] [77, p1]

[106] [Ibid.]

Tyndall's address—the association of religion with fear, superstition and restricted freedom. Whenever in the course of the history given by Tyndall, scientific development is found to be slow, halting or practically non-existent, the Christian influence is blamed.

Some modern commentators have preferred to see Tyndall as a misunderstood pantheist rather than a materialist.[107] They have associated with him a love of nature and a belief in a Power which is 'immanent in or identical with the universe', according to the definition of pantheism.[108] Whatever the label, it is clear from his address that Tyndall's views would have challenged the traditional doctrines of the Christian faith; for instance, his insistence on a materialistic explanation for the universe and the origin of life.

Giving an account of Darwin's theory of evolution, Tyndall considered the implications of Darwin's primordial germ—the common origin of all life. His conclusion: if we are to abandon the idea of creative acts and say instead that the primordial germ developed from matter, then a new and very different understanding of matter is required, since the traditional conception cannot admit of life coming out of matter. It was a real frustration for Tyndall that science was unable to prove experimentally that life can develop from anything other than life. Keen to apportion blame for science's inability to understand the relationship between matter and life, he found fault with those who were the first to define matter—the mathematicians.

While Tyndall insisted on the necessity of materialism, he accepted its insufficiency: Understanding alone cannot satisfy man and for this reason, 'physical science cannot cover all the demands of his nature'.[109] Therefore, Understanding must be supplemented, with: passion, 'Awe, Reverence, Wonder'; 'love of the beautiful, physical, and moral, in Nature, Poetry, and Art' and 'religious sentiment'.[110] Tyndall's

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[107]Barton, for instance, in [78].

[108] [79]

[109] [77, pp6–7]

[110] [77, p60]

use of the term 'religious sentiment', which is loaded with offence, no doubt served to distinguish between faith, and philosophical or scientific reasoning.

Although Tyndall was prepared to tolerate religious sentiment, he mistrusted its development into doctrine and religious practice; but they too might even be permitted, so long as they adapted to fall in line with all other evolving forms of knowledge: 'The facts of religious feeling are as certain to me as the facts of physics. But the world, I hold, will have to distinguish between the feeling and its forms, and to vary the latter in accordance with the intellectual condition of the age.'[111]

Undoubtedly, Tyndall's address had the potential to offend and antagonize Christian scientists who maintained traditional beliefs. The biographers of William Robertson Smith (1846–1894), who Tait and Stewart consulted in the course of putting the book together, testify to this. Smith was a Scottish theologian and Semitic scholar and, between 1868 and 1870, assistant to Tait at Edinburgh.[112] His involvement in *The Unseen Universe* is evidenced in a series of letters he received from Tait.[113] His role was that of consultant: he was asked to give his opinion on proofs and to suggest improvements. His biographers recalled:

> Tyndall's address created a sensation both in the theological and in the scientific world which was quite out of proportion to its importance as a serious attack on the orthodox position. It gave special offence to a distinguished group of scientific men who, like Lord Kelvin and Clerk Maxwell and their great predecessor, Faraday, were staunch upholders of the truths of revealed religion. This feeling of irritation was probably the immediate occasion of *The Unseen Universe*, a work of some celebrity in its day, which may be regarded as an elaborate counterblast to Dr. Tyndall's provocative manifesto.[114]

A further connection between Tyndall's address and *The Unseen Universe* is the

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[111] [77, pvi]

[112] For a biographical note on Smith written by Tait see [18, pp291–292]; originally *Nature*, 12 April 1894.

[113] See [70, pp162–166].

[114] [70, pp162–163]

correspondence in themes. Both touch upon: the continuity of nature; the practice of reaching beyond the bounds of the senses with intellect; a history of atomic theory; the mind-body problem; molecular processes and consciousness; and the mysterious link between matter and life, and energy and life.

Tait's biographer, Knott established the chronological link between the two:

> In the winter of 1874, a few months after the delivery by Tyndall of his famous presidential address before the B.A.A.S. at Belfast, it began to be whispered among the students of Edinburgh University that Tait was engaged on a book which was to overthrow materialism by a purely scientific argument. When, in the succeeding spring, *The Unseen Universe* appeared it was at once accepted as the fulfilment of this rumour.[115]

All this suggests that *The Unseen Universe* was written as a refutation of Tyndall's address. But while we might admit that the book was hastily compiled in order to produce a timely response to Tyndall, we should realize that much of the substance of *The Unseen Universe* is present in the authors' earlier work.[116] Tait and Stewart themselves anticipated accusations alleging a hasty, ill-thought-out contribution. In the preface to their first edition: 'We may state that the ideas here developed—very imperfectly, of course, as must always be the case in matters of the kind—are not the result of hasty guessing, but have been pressed on us by the reflections and discussions of several years.'[117]

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[115] [18, p236]

[116] Stewart's earlier writings in the periodicals are discussed by Gooday in [80]; together with Stewart's further work in the periodicals post publication of *The Unseen Universe*, which he produced largely in response to criticism of the book. I am grateful to Isobel Falconer for this reference.

[117] [55, pxii]

2.3 The Principle of Continuity

The book's most difficult concept to understand is the Principle of Continuity—the thread by which all the arguments hang.[118] The authors promise to define it but never do so explicitly—an illustration in terms of astronomy has to suffice—and each time the Principle is invoked there is some re-modelling of it. So we appeal to the critics for clarification, whom, in their frustration, went to great efforts to formulate a precise definition.

The interpretation which is closest to what the authors appear to have had in mind offers a dual definition. In the spirit of science, the Principle constitutes a belief in the uniformity of the laws of nature: 'The government of the universe has proceeded on a certain plan, ruled by certain fixed laws, we may therefore infer that it will continue to be so'.[119] In the spirit of religion, the Principle is an expression of trust: 'God has endowed us with certain capacities which enable us to dwell safely in the world and serve Him according to His laws. He will not distress or alarm His children by capriciously suspending or setting aside the laws which guide His universe.'[120] Another critic, in effect, unites the two definitions by describing the uniformity of natural law as 'the steady expression of the unchanging Will of the Creator'.[121]

In the preface to the fourth edition, the authors (who felt some clarification was warranted) asserted that the Principle 'has solely reference to the intellectual faculties'.[122] So perhaps the best interpretation of the Principle is as some sort of intellectual process—a process which makes sense of a 'continuous chain of cause

[118] According to [81], the Principle of Continuity was 'first enunciated by Sir William Grove in his inaugural address to the B.A.A.S. at Nottingham'.

[119] [72, p422] Heimann discusses various philosophies of the uniformity of nature held in mid-Victorian Britain in [82].

[120] [72, p422]

[121] [81]

[122] [83, pvi]

and effect, of antecedent and consequent'.[123]

Tait and Stewart were more explicit about what constitutes a breach of continuity: 'Continuity, in fine, does not preclude the occurrence of strange, abrupt, unforeseen events in the history of the universe, but only of such events as must finally and for ever put to confusion the intelligent beings who regard them.'[124]

We will encounter the authors' application of the Principle as we follow their arguments. Tait and Stewart proposed the following working hypotheses on the basis of the latest scientific investigations and discoveries. These hypotheses attempt to explain the beginning and end of the universe, and the production of life, and to realize the possibility of immortality.

2.4 The authors' hypotheses

2.4.1 The Great First Cause, the beginning of the universe and the origin of life

From the outset, Tait and Stewart identify themselves as believers in a Creator God: 'Let us begin by stating at once that we assume, as absolutely self-evident, the existence of a Deity who is the Creator of all things.'[125] Still, they are men of science and as such they believe that they are required to adopt a particular philosophy when speculating on the origins of natural phenomena:

> We think it is not so much the right or privilege as the bounden duty of the man of science to put back the direct interference of the Great First Cause—the unconditioned—as far as he possibly can in time. This is the intellectual or rather theoretical work which he is called upon to do—the post that has been assigned to him in the economy of the universe.
>
> If, then, two possible theories of the production of any phenomenon are presented

[123] [81]

[124] [55, p60]

[125] [55, p47]

> to the man of science, one of these implying the immediate operation of the unconditioned, and the other the operation of some cause existing in the universe, we conceive that he is called upon by the most profound obligations of his nature to choose the second in preference to the first.[126]

Maintaining this philosophy throughout, Tait and Stewart consider first, the beginning of the universe, and the origins and evolution of life.

Following the development hypothesis of Kant and Laplace they explain how the universe was formed: initially there was a 'diffused or chaotic' state; then, when the gravitating matter condensed and coalesced, potential energy was converted into heat and visible motion; and at the centre of what would become our solar system, a swirling mass threw out satellites and 'planetary attendants' as it cooled.[127] Thus, the process of evolution had two principal elements—the integration of matter and the dissipation of energy.[128]

When the earth had matured sufficiently to produce favourable conditions, the first forms of life appeared and from these, complex biological life-forms developed. In formulating their hypotheses on the origins of life, Tait and Stewart do not offer any scientific objections to Darwin's primordial germ: they cannot, of course, deny the staggering explanatory power of Darwin's hypotheses. Still they have to account for the germ's existence.

They assume that this first form of life requires a living antecedent, for life cannot develop from anything other than life and the act of its creation would constitute a breach of continuity. And this antecedent must be conditioned, for the Principle of Continuity requires 'an endless development of the conditioned'.[129] To be 'conditioned' is to be subject to the laws of the universe—'laws according to which the beings of the universe are conditioned by the Governor thereof, as regards

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[126] [55, pp131–132]

[127] [55, p125]

[128] [Ibid.]

[129] [55, p169]

time, place and sensation'.[130]

Rejecting the hypotheses of abiogenesis and spontaneous generation, Tait and Stewart turn to Scripture for the conditioned, living antecedent of Darwin's primordial germ:[131]

> If we now turn once more to the Christian system, we shall find that it recognises such an antecedent as an agent in the universe. He is styled the Lord and Giver of Life. The third Person of the Trinity is regarded in this system as working in the universe, and therefore in some sense as conditioned, and as distributing and developing this principle of life, which we are forced to regard as one of the things of the universe, in the same manner as the second Person of the Trinity is regarded as developing that other phenomenon, the energy of the universe.[132]

In their eagerness to provide cohesive and comprehensive hypotheses, Tait and Stewart are somewhat presumptuous in their interpretations of the Christian God and the specifics of Trinitarian doctrine. Having said this, the routes to this hypothesis are well sign-posted: the Holy Ghost is the Spirit residing in the souls of the faithful, who works in preparation for everlasting life; the Son is the developer of the Will of the Father which is expressed in the laws of the universe in which energy plays a fundamental role.

In accounting for the variety of species, Tait and Stewart rule out separate acts of creation—maintaining the scientific man's philosophy. Informed by Darwin, Huxley and Wallace, they cite natural variation, and natural and artificial selection as the probable causes. Still, the initial creative force remains the same: 'We have driven the creative operation of the Great First Cause into the durational depths of the

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[130] [55, p47]

[131] *Abiogenesis*: the hypothesis that living matter can be produced from non-living matter; a term used first by T. H. Huxley in 1870. [84] *Spontaneous generation*: 'the development of living organisms without the agency of pre-existing living matter, usually considered as resulting from changes taking place in some inorganic substance'. [85]

[132] [55, p179]

universe,—into the eternity of the past,—but for all that we have not got rid of God.'[133]

2.4.2 The end of the visible universe

> Scripture and Science both point to a coming catastrophe, the one in language a child can understand, the other in the wordless eloquence of Nature's changeless laws.[134]

Central to the science behind Tait and Stewart's grand scheme is the objective existence of matter and energy. The objective existence of matter is a conviction based on the conservation of matter; a law which enshrines the 'experimental truth' that matter is not susceptible to changes in quantity.[135] Tait and Stewart reason that if we admit the objective existence of matter, then we must afford an objective existence to anything which is conserved '*in the same sense*' as matter.[136] Examining a number of possibilities from abstract dynamics, they find that energy is alone conserved in this very particular sense: while energy may undergo transformation into a variety of forms, the sum of all the various energies in a closed system remains constant, according to the conservation of energy.

The characteristic natures of matter and energy are described in a novel fashion in the following quotation from the book:

> matter is always the same, though it may be masked in various combinations, energy is constantly changing the form in which it presents itself. The one is like the eternal, unchangeable Fate or *Necessitas* of the ancients; the other is Proteus himself in the variety and rapidity of its transformations.[137]

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133 [55, pp185–186]

134 [72, p428]

135 [55, p72]

136 [55, p73]

137 [55, p81]

Now, if in the universe there exists only matter and energy, and if matter is merely passive, we must conclude, Tait and Stewart argue, that all physical changes, including the thoughts and actions of living things, are transformations of energy. Accordingly, the following question is 'of the very utmost importance': '*Are all forms of energy equally susceptible of transformation*?'[138] For if there exists some grade of energy which is less capable of transformation, after successive transformations, while the quantity of energy in the system remains the same, the majority will have degraded into the form which is least susceptible of transformation and will be unusable. Heat is the least transformable form of energy and unless a temperature gradient exists no work can be obtained from it. So while energy may be present in a system in the form of heat, none may be available for transformation.

The process of transforming heat into work takes place in the thermodynamic operations of Carnot's perfect heat engine.[139] Tait and Stewart explain that the whole purpose of such an engine—which operates on the reversible Carnot cycle—is to transform heat into work and that its operations, being reversible, enable the greatest amount of work to be obtained from a given amount of heat. The process involves taking the heat which is not converted into work to the condenser and then reinvesting this heat, together with the heat-equivalent of the work done, back into the boiler.

But even a perfect heat engine cannot transform all of the heat which passes through the system into useful work, for the condition which would enable this efficiency—that the temperature of the condenser is absolute zero—can never be achieved. So at each conversion attempt, only a portion of heat is available for transformation into work, the remainder is degraded. And with each successive

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[138] [55, p82]

[139] The perfect heat engine was a concept developed by the French engineer, Sadi Nicolas Léonard Carnot (1796–1832) in *Réflexions sur la puissance motrice du feu et sur les machines propres à développer cette puissance* (1824), a book on the thermodynamic workings of the steam engine. S. N. L. Carnot was the eldest son of Lazare Nicolas Marguérite Carnot (1753–1823), author of *Géométrie de position* (1803).

transformation the heat becomes more degraded, or more dissipated, and so less and less is available for transformation into work.

The analogy of the heat engine helps Tait and Stewart to explain why the present visible universe will one day come to an end. The sun functions as the furnace in the vast heat engine which we call the present visible universe. It radiates energy: a portion reaches earth, supplying life-giving energy; a larger portion still is released into the universe in the form of heat. The loss of the sun's heat causes it to cool. From the analogy of the heat engine we know that the availability of energy in the universe will continue to diminish.

Other events are also taking place. Tait and Stewart propose that one day the planets will be drawn into the sun by something like 'ethereal friction'.[140] They will lose their orbital energy, spin into the sun and merge into its mass. Upon impact, the power of the cooling sun will temporarily be restored as visible energy is converted into heat. The sun will then resume cooling, until the restoration of the next collision, and so on . . .

Still, within this process is the possibility of the formation of new solar systems, formed from the nebulous dust surrounding some of the new coalesced masses. But the process will not continue indefinitely: in general, the total number of masses is still decreasing. A time will come when the matter of the universe is but a solitary cooling mass. It will exhibit no visible motion and will amount to nothing more than a useless store of energy—heat at a uniform temperature.

Thus, according to Tait and Stewart, a combination of processes will affect the end of the present visible universe—the dissipation and degradation of energy, and the aggregation of masses.[141]

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[140] [55, p91]

[141] In [36, pp253–255] Crosbie Smith puts the *The Unseen Universe* in the context of developments in the science of energy during the nineteenth century. See also [86, pp63–66].

2.4.3 The existence of an unseen universe

Guided by the Principle of Continuity, Tait and Stewart arrive at the existence of something other than the visible—'an invisible order of things'.[142] They cannot suppose that only the visible universe exists, for both its beginning and end would constitute breaks of continuity. Instead, they must conclude that 'the visible system is not the whole universe, but only, it may be, a very small part of it'.[143]

Tait and Stewart propose that the visible, both in matter and energy, evolved out of an unseen universe and that into this unseen universe it will ultimately retreat. Though the two universes are currently independent, they remain 'intimately connected', with exchanges of energy taking place between them, facilitated by a less-than-perfectly-transparent ether.[144] The energy that is stored in the unseen is held for the purposes of new creation. This conception of the unseen provides a novel way of coming to terms with the wasteful loss of the sun's energy through dissipation: the law of conservation of energy applies to the whole system.

The scientific theory which is put forward to explain how the visible evolved out of the invisible is that of William Thomson's vortex atom. The theory, which was communicated to the R.S.E. in February 1867, was the latest speculation on the theory of matter. It regarded the universe's primordial atoms as vortices developed from a pre-existing perfect fluid filling all space. The idea that motion could be a basis for matter was a well-established one, with the first real contribution being the theory of vortex motion suggested by the German physicist, Hermann von Helmholtz (1821–1894).[145]

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[142] [55, p157]

[143] [Ibid.]

[144] [55, p158]

[145] Hermann von Helmholtz (1821–1894): initially, he was a doctor in the Prussian army; then, successively, Professor of Physiology at Könisberg, Bonn and Heidelberg; in the late 1860s he made the transition from physiology to physics and in 1871 he was appointed to the Chair of Physics at Berlin. For Tait, the appeal of Hamilton's quaternionic theory was the promise of its application in the theory of vortex motion which Tait had encountered in Helmholtz's 1858

Tait and Stewart's only real objection to Thomson's vortex atom theory was that Thomson had assumed a perfect fluid. In a perfect fluid there is no viscosity and in the absence of viscosity there can be no rotation without outside influence. Therefore, a perfect fluid necessitates a creative act in time which constitutes a break in continuity. Hence, in order to have both unbroken continuity and a vortex atom theory, the invisible universe cannot be a perfect fluid.

In a perfect fluid, as Helmholtz proved, the rotating portions of fluid are arranged in knotted filaments which once set in motion forever '*maintain their identity*'.[146] In contrast, in a fluid which is not perfect, the permanence of the structures is no longer guaranteed. Our authors saw in this, evidence of the non-permanence of the visible order of things.

Thomson's theory impressed and inspired Tait and Stewart. They claimed that Thomson had 'gone further than any one else' in accounting for the origin of life.[147] Indeed, during the 1870s Tait began to teach Thomson's vortex atom theory in his Edinburgh lectures. In his address to the B.A.A.S. in 1871, Tait shared his hopes for the promising theory: 'Our President's [Thomson's] splendid suggestion of Vortex-atoms, if it be correct, will enable us thoroughly to understand matter, and mathematically to investigate all its properties.'[148] Tait was to invest much of himself in these mathematical investigations during the latter years of his career. He was chasing a full classification of the forms of knotted vortex rings, believing in the existence of a unique form of vortex ring for each of the elements. Eventually, Tait admitted, in *Properties of Matter* (1885), that 'the discovery of the ultimate nature of matter is probably beyond the range of human intelligence'.[149]

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paper [87]. Captivated by quaternions, Tait maintained a life-long fascination with them. For Tait's translation of Helmholt'z paper into English see [88].

[146] [89, p20]

[147] [55, p186]

[148] [74, p6]

[149] [89, p15]

Pursuing the chain of continuity in a backwards direction, our authors are led to the possibility of an endless number of invisible universes. In form, their model (Figure 2.4) resembles the Ptolemaic model. Each universe is represented by one of the concentric circles in the diagram.

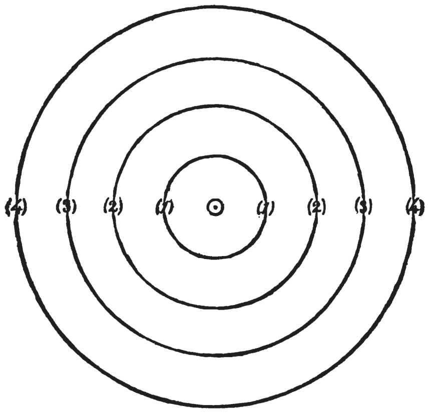

Figure 2.4: Tait and Stewart's concentric model of the Great Whole.[150]

At the centre of the model is an evanescent smoke-ring; produced here on earth, by a smoke box for instance, as in Tait's experiments in 1867. Tait's experiments with smoke rings were conducted for the benefit of William Thomson in order to verify Helmholtz's claims regarding the interaction of vortex atoms. The permanence of form of the smoke rings suggested the vortex atom theory to Thomson.

The molecules of this smoke ring—being part of the visible universe—are vortex atoms developed from the unseen. And the entities which constitute the invisible universe are themselves vortex atoms developed from another invisible universe and so on. Pursuing the chain of continuity infinitely far back, Tait and Stewart reach a universe of infinite energy, which has an intelligent developing agency of infinite energy. Together these universes form 'the Great Whole'—a self-contained system, which is 'infinite in energy, and will last from eternity to eternity'.[151]

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[150] [55, p171]

[151] [55, p172]

2.4.4 Superior and angelic intelligences

Our authors do not rely on their multiverse theory, as is often done today with the modern equivalent, as a means of explaining how special our universe is.[152] For them our universe has not been endowed with such fruitfulness on account of the laws of probability but on account of the generosity of the Creator.

Tait and Stewart appreciate the 'delicacy of construction'.[153] In the development of complexity from that which is simple: be that the development of compounds from rudimentary elements, or human life from Darwin's primordial germ. In the delicate balance of unstable forces: be that the regularity of the planets, or the abrupt meteorological changes of the sun and earth. And in the uniformity of the construction of atoms: evidence, for Herschel and Clerk Maxwell, that atoms are 'manufactured articles'.[154]

The possibility that other intelligences besides humankind may exist in the universe is not ruled out, which is surprising. Tait and Stewart are open, for example, to the possibility of life on Mars—in agreement with astronomers and physicists of the time, according to Tait and Stewart—though they understand that relevant knowledge will not be forthcoming in their own lifetime.

A discussion on the reality of angelic intelligences is not entered into. Tait and Stewart say only that such beings would not belong to the present visible universe; for we cannot perceive them, nor do we imagine that their fate depends on that of the visible universe.

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[152] In the multiverse theory, there exists a multitude of unobservable universes but, owing to the fine-tuning of natural constants, only one with suitable conditions for our survival. The historical development of the theory is given in [90]. I am grateful to Mark McCartney for this reference.

[153] A phrase used throughout *The Unseen Universe*.

[154] [55, p167]

2.4.5 Immortality and the spiritual body

Tait and Stewart do not speculate on the likelihood of personal immortality, having nothing similar to the Principle of Continuity to apply to it, but they do cite two sources in favour of immortality: statements about Christ and man's 'intense longing for immortality'.[155] Whether the latter, being need-driven, constitutes real evidence is questioned by one critic, who regards a materialist accepting this as evidence as 'a creature of the imagination'.[156]

Speculations on immortality are put into three categories of doctrine: (i) the ethereal state, (ii) the bodily state and (iii) the inconceivability and/or impossibility of a future state. The authors' view is grounded in Scriptural revelation concerning Christ's death and resurrection and the specific nature of His physical transformation. Thus, the nature of our future physical state is bodily but spiritual, or angelic, rather than natural.

Tait and Stewart suppose that immortality is a transference. They propose the following three suppositions:

> It may be regarded as a transference from one grade of being to another in the present visible universe; or secondly, as a transference from the visible universe to some other order of things intimately connected with it; or lastly, we may conceive it to represent a transference from the present visible universe to an order of things entirely unconnected with it.[157]

To these suppositions they apply the Principle of Continuity. The first supposition is discarded on the grounds that the visible universe had its beginning in time and will eventually come to an end. The third supposition is discounted for the following reason. Our authors state two conditions of continued intelligent existence. An individual must: (1) possess an organ of memory, in order to maintain a

[155] [55, p166]

[156] [91]

[157] [55, pp66–67]

connection with the past, and (2) be capable of action, or varied movement, in the present. If such an individual, with a connection to the past in one order of things, was to enter an entirely unconnected order of things, they would suffer permanent intellectual confusion which would constitute a breach of continuity. Therefore, only the second supposition remains; hence, immortality must represent a 'transference from the visible universe to some other order of things intimately connected with it'.

Regarding the nature of the transference, the authors' hypotheses follow from their discussion on the latest theories—those of Huxley, especially—on the connections between mind and matter, in terms of brain traces and the physical foundations of memory. Tait and Stewart construct a 'frame' for each individual so that they might be connected with the unseen—a 'spiritual body' which receives into it the molecular displacements of the brain.[158] In this conception, the meaning of the anagram in *Nature*, noted earlier, becomes clear.

In the conception of the spiritual body both conditions of continued intelligent existence are fulfilled. The second condition, the capability of action or movement, follows from the author's premise that the unseen is to be full of energy when the visible universe comes to an end. In satisfying both conditions in the conception of the spiritual body, Tait and Stewart feel that they have demonstrated the possibility of the continuance of life beyond the death of the perishable material body.

Tait and Stewart admit that their conception of the spiritual body, as an instrument for personal immortality, may be 'detached from all conceptions regarding the Divine essence'.[159] They maintain, however, that we are logically bound to accept some kind of spiritual body if we accept both the doctrine of immortality and the Principle of Continuity.

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158 [55, p159]

159 [55, p198]

2.4.6 Divine action: miracles, the incarnation and the resurrection

Tait and Stewart believe that in the light of their work there need no longer be a scientific objection to miracles. They are to be regarded as 'transmutations of energy from the one universe into the other'; 'the result of a peculiar action of the invisible upon the visible universe'.[160] The incarnation of Christ presents no problem either: there is no breach of the Principle of Continuity because, in traditional Christian doctrine, Christ submitted Himself to the laws of the universe which are an expression of the Will of the Father. Tait and Stewart suppose that the resurrection of Christ could also have been accomplished without a break in continuity, by an infinite intelligent agency who is capable of developing the visible universe from the unseen.

2.4.7 The authors' practical conclusion

The analogy of Carnot's perfect heat engine inspires the practical conclusion of Tait and Stewart's hypotheses:

> And just as reversibility is the stamp of perfection in the inanimate engine, so a similar reversibility may be the stamp of perfection in the living man. He ought to live for the unseen—to carry into it something which may not be wholly unacceptable. But, in order to enable him to do this, the unseen must also work upon him, and its influences must pervade his spiritual nature. Thus a life *for* the unseen *through* the unseen is to be regarded as the only perfect life.[161]

2.5 Reception of *The Unseen Universe*

Early editions of *The Unseen Universe* were widely known and subject to unusually close scrutiny: the authors were successful in securing a readership that would

[160] [55, p189]

[161] [55, p192]

normally have turned away from an inquiry into unseen worlds.[162] Despite being of reduced significance in the scientific world, having incorporated religious doctrine, *The Unseen Universe* made a notable impact on the 'enlightened portion of the public'.[163] Its appeal: the authors' 'real and intimate' knowledge of the latest scientific theories.[164] The dialogue which continued through subsequent responsive editions ensured the longevity of its appeal.

Demand led to a sequel, *Paradoxical Philosophy* (1878), in which the hypotheses of *The Unseen Universe* were reworked into the form of a dialogue conversation between a German mathematician and a select group of the religious and social orthodoxy. It was dedicated to the members of the Paradoxical Society.

Reactions to *The Unseen Universe* were truly diverse. One described the work as: '212 pages of the most hardened and impenitent nonsense that ever called itself "original speculation"'.[165] Another declared: 'nothing as original, and, so far as we judge, satisfactory, whether as regards respect for science, or as giving a logical basis for the belief in man's immortality and the divine rule, has been written this century'.[166]

Criticism targeted many areas. (i) The authors' ability to reason philosophically: the book's philosophical sections were weaker than its scientific sections, making for a 'curious compound of severe science and third-rate poetry'.[167] (ii) The legitimacy of their scientific inferences: critics recognized illegitimate and unscientific speculations and cautioned against reliance on analogy.[168] (iii) The effective communication of their hypotheses: the unscientific reader would understand little of the involved

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[162] [73]

[163] [92]

[164] [81]

[165] [93] Quoted by Tait and Stewart in the preface to the second edition, [55, pvi].

[166] [94]

[167] [95, p3]

[168] [96, p1426]

science. (iv) Their authority to speak on matters of faith. (v) Issues of religious licence: in order to tie up a philosophical and theological Trinity, Tait and Stewart had produced a Trinity unconnected with the orthodox and 'without warrant from Scripture'.[169] And (vi) their application of science: inappropriately, for the purposes of authenticating the Bible and validating the teachings of Scripture.

Issues with the authors' conception of a spiritual body were numerous and complex, and thoroughly understandable. On a practical level: How is it that spiritual bodies do not interact with one another? Can consciousness exist in two places at once? Does the physical body retire entirely into the spiritual body upon death? ... The English mathematician and philosopher, Professor William K. Clifford (1845–1879), writing in *The Fortnightly Review*, seemed set to demolish the integrity of *The Unseen Universe*, having particular issue with the concept of a spiritual body.[170]

Tait and Stewart dealt robustly with the criticism they received. Later editions were revised and enlarged to make use of criticism and reply to objections; however, they refused to recall any statements. In the preface to the second edition they listed the charges brought against them:

> Some call us infidels, while others represent us as very much too orthodoxly credulous; some call us pantheists, some materialists, others spiritualists. As we cannot belong at once to all these varied categories, the presumption is that we belong to none of them. This, by the way, is our own opinion.[171]

Braced for an attack from religious leaders, Tait and Stewart were 'delightfully perplexed', and encouraged, by their response: the two parties agreed on many points and where there were differences of opinion, they were pointed out 'with the utmost courtesy' so as to safeguard the Church's independence and show due regard

[169] [97, p416]

[170] A much-edited version of Clifford's review appears in [98]. For Tait and Stewart's robust response to the review see the preface to their second edition, [55, ppviii–x].

[171] [55, pv]

for the authors.[172]

Inevitably, Tait and Stewart were accused of treating others badly in the course of their inquiry, particularly theologians, materialists and spiritualists. It is of little doubt that the work 'wields a very heavy blow at materialism', in the words of one commentator, and that the views expressed about contemporary spiritualists are extremely provocative; however, the claim has no real foundation as regards theologians.[173]

Another challenge faced by the authors was the misrepresentation of their work. Commentators evidently felt qualified, and at liberty, to report on the book's scientific content. They lifted extracts from the book with no regard for context, often inserting terms of their own invention. This careless malpractice infuriated the authors, who addressed the issue publicly in new editions.[174]

Commentators who reacted positively to the book praised its boldness and originality, and applauded the authors' novel ideas which were laid out, without prejudice, in an honest search for the truth and for a noble purpose. Critics recognized an 'earnest religious spirit' which ran through the work.[175] The authors were commended for their applications of 'sound scientific reasoning'.[176] And for their 'many clear expositions of scientific truth'.[177] A number of commentators hoped and believed that there may be some truth in their major propositions.

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[172] [83, pv]

[173] [92]

[174] For instance, in [83, pviii] Tait and Stewart object to the term 'luminiferous force' which was misquoted by *The Christian Treasury* in [97, p414].

[175] [56]

[176] [99]

[177] [100]

2.6 String theory and M-theory anticipated

In *The Unseen Universe*, Tait and Stewart went far beyond attempting to reconcile science and religion on a few points of conflict: they had proposed a theory of everything; and they were, without question, overambitious in attempting to work out every detail of the unity they saw.

While we cannot remark on the verity of their hypotheses regarding immortality, we might note that some of the scientific hypotheses that they advanced have since reappeared, in modern physics' approaches to a single scientific theory of everything: the concept of an infinite number of universes features in the modern multiverse theory; and Thomson's vortex atom, which was finally abandoned following the acceptance of the theory of relativity around 1910, appears recast in modern string theory.[178]

Having said this, expect to find in the book evidence of imperfections in the contemporary knowledge of science, recognized today as false theories and antiquated notions. The long-forgotten ether is probably the best example but even this has value again, reinterpreted in quantum theory as the energy of the vacuum. Addressing the B.A.A.S. in 1871, Tait spoke of the value of "incorrect" scientific hypotheses: 'in science nothing of value can ever be lost; it is certain to become a stepping-stone on the way to further truth'.[179]

2.7 Closing remarks

Tait and Stewart—men of both scientific perception and religious instinct—maintained that science and religion are complementary aspects of the same truth. They were

[178] Knotted strings are at the heart of both Thomson's vortex-atom theory and modern string theory. In modern string theory, vibrating strings are considered to be the fundamental building blocks of matter; the resonant frequencies of the strings determining the type of particle produced. Both theories are discussed in their historical contexts in [90].

[179] [74, p5]

described by one critic as modern day examples of Newton and Faraday: ‘princes in science, and yet humble, believing Christian men’.[180] Still, they recognized as two distinct groups, those who study the How of the universe and those who study the Why:

> A division as old as Aristotle separates speculators into two great classes,—those who study the How of the Universe, and those who study the Why. All men of science are embraced in the former of these, all men of religion in the latter. The former regard the Universe as a huge machine, and their object is to study the laws which regulate its working; the latter again speculate about the object of the machine, and what sort of work it is intended to produce.[181]

In our final quotation from the book, Tait and Stewart express their understanding of their roles as men of science, accepting humbly the limits of their own human intellect:

> the position of the scientific man is to clear a space before him from which all mystery shall be driven away, and in which there shall be nothing but matter and certain definite laws which he can comprehend. There are however three great mysteries (a trinity of mysteries) which elude, and will for ever elude his grasp [...]—they are the mystery of matter and energy; the mystery of life; and the mystery of God,—and these three are one.[182]

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[180] [101]

[181] [55, p2]

[182] [55, p183]

CHAPTER 3

TAIT'S STATISTICAL MODELS

This chapter reveals Tait's surprising involvement in statistics.

3.1 Introduction

3.1.1 Tait's pocket notebook

In January 2011 a pocket notebook once belonging to Tait (Figure 3.1) came into my possession. It had been retained prior by the Edinburgh Mathematical Society. I believe few are aware of the existence of the notebook and perhaps none have made a careful study of its contents. Although Tait utilized only a very small number of pages in the notebook, its contents reveal much about the scope of this remarkable polymath's expertise and interests.

Figure 3.1: Tait's pocket notebook.

Occupying the first three pages of the notebook is Tait's quaternion version of Green's theorem.[183] There is also a reminder to add in some references—presumably to a draft of Tait's address to Section A of the B.A.A.S., 1871—and a draft of an unpublished poem by Tait (transcribed in Appendix B).

But the most significant find in the notebook (Figure 3.2) comprises just six lines:

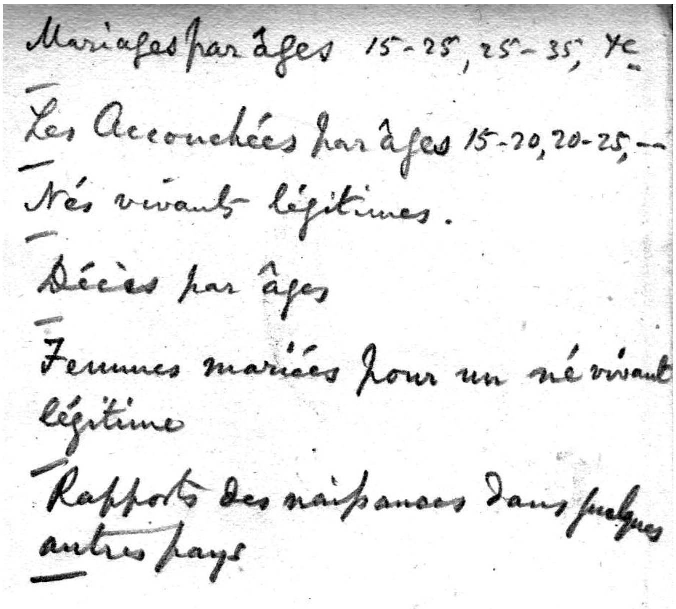
Mariages par âges 15-25, 25-35, &c
Les Accouchées par âges 15-20, 20-25, —
Nés vivants légitimes.
Décès par âges
Femmes mariées pour un né vivant
légitime
Rapports des naissances dans quelques
autres pays

Figure 3.2: An entry in French in Tait's pocket notebook.

Mariages par âges 15–25, 25–35, &c., . . .

Les accouchées par âges 15–20, 20–25, . . .

Nés vivants légitimes

Décès par âges

Femmes mariées pour un né vivant légitime

Rapports des naissances dans quelques autres pays

And so it appears that Tait, at some stage, was concerned with the collection and/or

[183]This entry may relate to Tait's paper, 'On Green's and Other Allied Theorems', *Trans. Roy. Soc. Edinburgh* XXVI (1870).

analysis of data; data on the number of children born to women of various ages, taking into account the age at which they were married. Immediately, the following questions spring upon us: What was the nature and circumstance of Tait's involvement in statistics? Did he contribute anything significant, including published papers? Who was (and is) aware of his involvement? And why was he writing in French? I aim to address each of these questions during the course of this chapter.

3.1.2 Tait's contribution to statistics and probability

As a preliminary step in trying to understand the significance of this notebook entry, I compiled a list from Tait's published work of contributions which might come under the umbrella of statistics and looked for a correspondence with the notebook entry. One paper listed in Knott's bibliography of Tait's works in [18] looked particularly promising: 'Note on the Formulae Representing the Fecundity and Fertility of Women' [102], published in the R.S.E.'s *Transactions* in 1867. On this paper, Knott made the following intriguing remark: 'Fecundity is found to depend linearly and fertility parabolically on age [...] The formulae are known to Statisticians as Tait's Laws.'[184] From this I inferred that in his paper on fecundity and fertility, Tait had modelled the regularity he had observed in some data and that his results were well known.

An extended version of this paper appeared as part VI of James Matthews Duncan's book, *Fecundity, Fertility, Sterility and Allied Topics* (1866) [103]. Comparing the notebook entry with Tait's contribution to Duncan's book, it was immediately obvious that the two corresponded; in relation to data categories and the grouping of ages. And there was a further correspondence; between dates recorded in the notebook (August, October and November 1870) and the time when Tait would be preparing for the second edition of the book, published in 1871.

Further investigation revealed the full extent of Tait's contribution to probability and statistics:

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[184] [18, p353]

1865 'Probability' [104]: an article published in the first edition of *Chambers' Encyclopaedia.*[185] And 'On the Law of Frequency of Error' [105]: a paper in which Tait devised an intuitive and probabilistic approach to deriving the error law.[186]

1866 Part VI of James Matthews Duncan's *Fecundity, Fertility, Sterility and Allied Topics* [103]: Part VI is an extended version of Tait's paper, 'Note on the Formulae Representing the Fecundity and Fertility of Women' [102]. In 1871 a second edition of Duncan's book [106] was published, with additions, amendments and a review of the first edition.

1873 'On a Question of Arrangement and Probabilities' [107]: a paper on a combinatorial problem related to the game of golf.[187]

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[185] In this article, Tait provides an introduction to the elementary definitions and principles behind the mathematical theory, using familiar examples from games of chance: coins, dice, balls drawn from a bag, horse racing odds, etc. An indication of the applications of the theory is also given: to life assurance; in the application of the method of least squares; and in ascertaining the value of evidence given in a trial and the probability of correctness of a jury's verdict.

[186] It occurred to Tait, while working on his probability article, that the best approach to the concept of error is to take, as the logical foundation for your investigations, something simple and intuitive; thereby, avoiding the 'unnecessarily elaborate analysis' of Laplace and Poisson. [105, p139] Adopting this philosophy, Tait bases his investigations in his paper on error on the following problem: '*To find the relative probabilities of different combinations of mutually exclusive simple events in the course of a large number of trials.*' [Ibid.] He looks at the possible combinations of black and white balls drawn from a bag. The error associated with a particular combination is its deviation from the most probable combination. The law of error $y = Ae^{-M^2x^2}$ is the ratio of the probability of that particular combination to the probability of the most probable combination.

[187] The problem Tait investigates in this paper is suggested by the game of golf: 'When a player is x holes "up," and y "to play," in how many ways may he win?' [107, p37] (a reincarnation of the Problem of Points). Tait considers both a geometrical and analytical approach to the problem. His fundamental equation is the recurrence relation $P_{x+1,y+1} = P_{x+2,y} + P_{x+1,y} + P_{x,y}$, where $P_{x,y}$ is the number of ways a player may win when x up, with y to play. At any given hole he may win, draw (halve) or lose the hole, so that the number of holes he is ahead may increase by 1, stay the same or decrease by 1. Chris Pritchard has published on this paper: see [108]. He

1885 Five papers on the kinetic theory of gases in which Tait applied probability to
–1892 the behaviour of particles.[188]

It was a delight to realize, after some research, that Tait's contributions to probability and statistics do in fact serve to sign-post developments in these areas up until around 1900. It was in reading of the progress of probability and statistics that I came to understand how natural Tait's involvement was and the significance of his contributions in their proper context.[189] In this chapter, the focus is Tait's contribution to Duncan's book, *Fecundity, Fertility, Sterility and Allied Topics.*

3.1.3 James Matthews Duncan (1826–1890): a biographical sketch

James Matthews Duncan M.A. M.D. (Aberdeen) (Figure 3.3, page 63) was a leading physician, obstetrician and surgeon. He was born in Aberdeen, in April 1826. He was educated at the grammar school in Aberdeen and at Marischal College, and studied medicine at Edinburgh and Paris. He was regarded as the most promising student of Sir James Simpson (1811–1870), the internationally celebrated physician and obstetrician, with whom he had worked on the introduction of chloroform as an anaesthetic.

Duncan established himself in Edinburgh during the first half of his career: he was instrumental in the founding of Edinburgh's Sick Children's Hospital (1860); he served on the ward for diseases of women at the Edinburgh Royal Infirmary; and he lectured on midwifery and was an examiner on the subject in various universi-

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gives some historical background to the Problem of Points, explains how Tait solves the problem by analysing the coefficients of a trinomial expansion $(a + 1 + a^{-1})^n$ and devises an extension of the problem, assigning unequal probabilities to the three possible results at each hole.

[188] Tait's papers on the kinetic theory of gases were reprinted in volume II of his *Scientific Papers.*

[189] See Porter's *The Rise of Statistical Thinking 1820–1900* [109] for example.

ties and colleges, including St Andrews.[190] From Edinburgh he went to London, as Physician Accoucheur and lecturer on midwifery at St Bartholomew's Hospital (1877). In 1883 he delivered the Goulstonian lectures, 'On Sterility in Women' to the Royal College of Physicians of London. These lectures were based on *Fecundity, Fertility and Sterility.*[191] Duncan's approach—which was founded on extensive statistical work—was 'widely regarded as a breakthrough in the study of fecundity and sterility'.[192]

Figure 3.3: Dr. Matthews Duncan, a newspaper clipping of a sketched portrait from *The Modern Athenian*, 13 October 1877. Sourced from Tait's scrapbook.

Duncan was a Fellow of the Royal College of Physicians of Edinburgh (1851), the Royal Societies of Edinburgh (1863) and London (1883), and the Royal College

[190] University College, Dundee was founded in 1881. In 1897 it became a constituent college of the University of St Andrews. In 1954 University College was renamed Queen's College; becoming the University of Dundee in 1967, when it gained independence from St Andrews. Pre-1967, midwifery was taught on the Dundee site.

[191] The lectures were reproduced in the *British Medical Journal* in March 1883.

[192] [110]

of Physicians of London (1883). He served as President of the Obstetrical Societies of Edinburgh (1873–1875) and London (1883). He received honorary degrees from the Universities of Edinburgh and Cambridge (LL.D.), and the University of Dublin (M.D.).

Duncan married in 1860 and the couple had at least ten children. He died on 1 September 1890 at Baden-Baden, aged sixty-four.[193]

Association with Tait

Duncan and Tait were associated through the R.S.E. and through an Edinburgh society called the Evening Club. It is possible that they also knew one another through Duncan's Edinburgh practice.

—*Through the Royal Society of Edinburgh.* Around the time of collaboration, both Tait and Duncan were prominent members of the R.S.E.: Tait had been elected a Fellow in 1861 and as one of the Secretaries to the Ordinary Meetings in 1863; Duncan had been elected a Fellow in 1863 and during the periods 1866–1868 and 1875–1877 he served the Society as Councillor.[194]
—*Through membership of the Evening Club.* Both men were among the founding members of an Edinburgh society called the Evening Club. The society was modelled on London's Century and Cosmopolitan Clubs, and was a thriving society for some twenty-five years. According to Knott, it could boast of such members as: 'prominent Edinburgh lawyers, artists, physicians, clergymen, teachers both in college and school, bankers, commercial men, publishers, engineers etc.'.[195] From Knott we also learn that Tait personally introduced some seventy guests to the Club between 1870 and 1884, and that members of the society would 'recall Tait as one of the great personalities, taking his full share in the talk, and enjoying the relaxation

[193] Uncited sources of biographical information on Duncan: [29] and [111].

[194] [29]

[195] [18, p348] According to the same source [18, pp347–348] Mr. A. Findlater, editor of *Chambers' Encyclopaedia*, also enjoyed membership at the Evening Club.

from the hard thinking in which he usually passed his evenings'.[196] The group met, generally twice weekly, 'for purely social intercourse, cards and serious subjects of debate being taboo'.[197] Unfortunately, Duncan and Tait's collaboration pre-dates their association at the Evening Club which was founded in 1869.

—*Through Duncan's private practice in Edinburgh.* It is possible that Tait, aware of Duncan's reputation as a leading obstetrician, had him to tend to his wife Margaret during her pregnancies: Duncan had a private practice in Edinburgh between 1851 and 1877, and Tait's children were born in Edinburgh between 1861 and 1873, excluding his eldest child, Edith who was born in Belfast in 1860.

3.2 *Fecundity, Fertility and Sterility* (1866)

Tait opens his chapter in *Fecundity, Fertility and Sterility* with the following statement, in which he explains his objective and qualifies the extent of his involvement:

> Dr. Matthews Duncan having requested me to point out to him some simple method of comparing the fertility of different races, I endeavoured, as a preliminary step, to represent by formulae some of the chief results which he has obtained in his very lucid and elaborate papers recently read to this Society [R.S.E.] and printed in their Transactions for 1863–4 and for the present session. Some of the formulae which I have obtained are so simple, and accord so well with the tables, that I have thought them worth bringing before the Society. Of course it must be understood that I advocate no theory, and pretend to no physiological knowledge of the question. I merely try to present, in a simple analytical form, the contents of some of Mr. Duncan's tables.[198]

Tait proposes to work towards his objective by considering: (i) the fertility and fecundity of the mass of wives; (ii) the fertility and fecundity of the average individual; and (iii) the relative fertility and fecundity of different races. He defines

[196] [18, p349]

[197] [18, p348]

[198] [103, p207]

fecundity and fertility as follows:[199]

> By *fecundity* at a given age we mean the probability that during the lapse of one year of married life, at that age, pregnancy, producing a living child, will ensue.
>
> By *fertility*, at any age, we mean the number of children which a married woman of that age is likely to have during the rest of her life, or some numerical multiple of it.

Note, Tait's definition of fecundity is easy to misinterpret: the event that Tait is concerned with is pregnancy, rather than the birth itself.

3.2.1 Fertility and fecundity of the mass of wives

Tait's data sets I:

- Data set A—Fecundity values for the wives of Edinburgh and Glasgow, as a whole— [103, p19]
- Data set B—Fertility values for the mass of wives in one district in Edinburgh, St George's-in-the-East— [103, p139]

Linear model of fecundity

Working with his definitions, Tait formulates an expression for fertility at age t. It is given as the sum of the fecundities from age t until the onset of sterility, which Tait assumes takes place when a woman reaches 50. Thus

$$F_t = f_t + f_{t+1} + \cdots + f_{49} = \Sigma_{i=t}^{49} f_i \tag{3.1}$$

To verify the formula, Tait takes the fecundity values in data set A and, using (3.1), calculates fertility values for the Edinburgh and Glasgow wives. He then compares these values with the observed fertility values from data set B for married women in one district in Edinburgh. Mindful of problems with the data—including the non-comparability of the two data sets in terms of age groupings— Tait is pleased

[199] [103, pp208, 208]

with the comparison, writing: 'These numbers agree as well as could possibly be expected.'[200]

To derive his linear model of fecundity (3.2), Tait fits a straight line to the continuous curve (Figure 3.4, page 68) comprising the fecundity values from data set A. The formula for the straight line is easily deduced once the value of the x-intercept has been estimated. This is Tait's linear model of fecundity: he has fecundity at age t

$$f_t = k(50 - t) \tag{3.2}$$

where k is a constant to be found.

Tait describes his linear model as 'a simple formula very closely representing the tabulated results'.[201] He remarks on the closeness of the fit between ages 17 and 40 (Figure 3.4) and suggests, in places where the curve clearly departs from the line, that inaccuracies and omissions in the data are to blame. Analysis of the same data (Table LXXVII [103, p209]) using the statistical package R suggests that Tait provided very precise estimates of the y-intercept and slope of the fitted line.[202]

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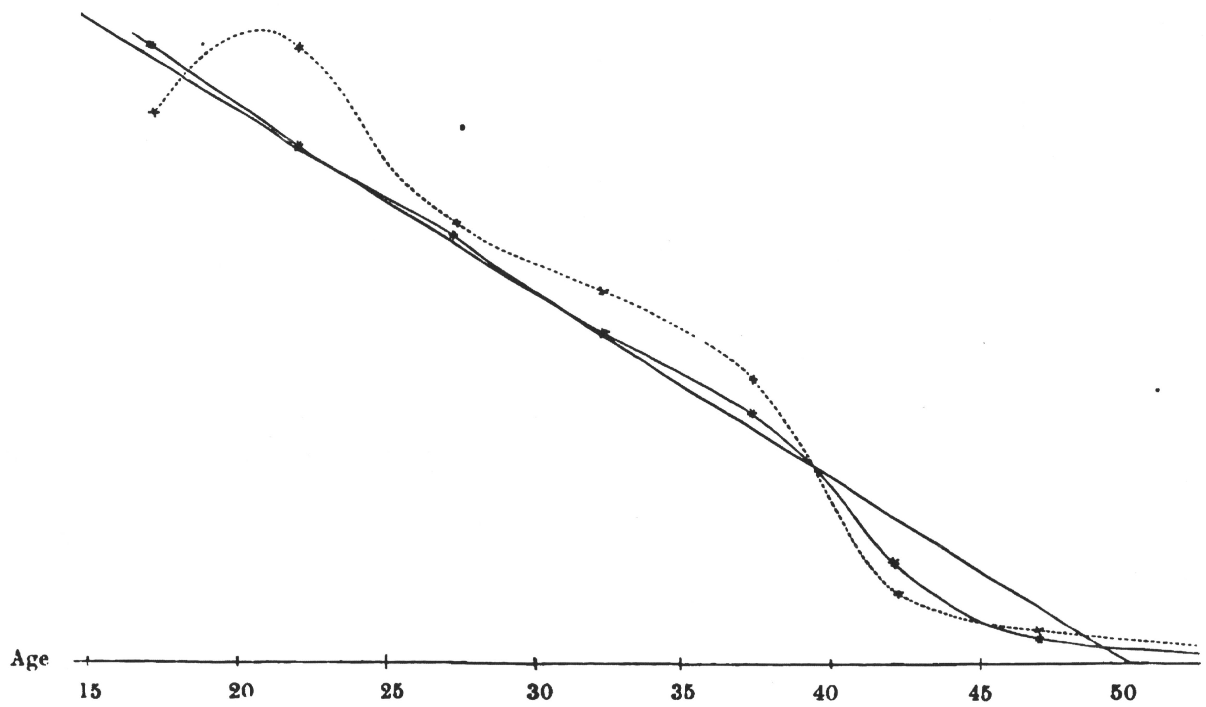
[200] [103, p211]

[201] [103, p213]

[202] The coefficient of determination (the R^2 statistic) is 0.9944, meaning that 99% of the total variation in fecundity is explained by age. R produces the following linear regression model: $E(\text{fecundity}) = 78.2 - 1.63(\text{age})$ (3 s.f.). Tait has $f_t = \frac{3}{2}(50 - t)$. Tait's values for the y-intercept and slope fall within the 95% confidence interval R generates for the parameters, i.e. a range of values for β_1 and β_2 such that 95% of the data is explained by the model $E(Y) = \beta_1 + \beta_2 x$.



Fecundity at Different Ages.

The continuous curve is founded on Edinburgh and Glasgow Tables. The dotted curve on Tables of Sterility. The straight line from the formula $k\,(50-t)$

Figure 3.4: Tait's graph of 'fecundity at different ages'. [103, p212] Note the absence of a vertical axis.

Quadratic model of fertility

Tait derives a quadratic model of fertility (3.3) by substituting his linear model (3.2) into (3.1) for each fecundity. He has fertility at age t

$$F_t = \frac{1}{2}k(50-t)(51-t) \approx \frac{1}{2}k(50-t)^2 \tag{3.3}$$

which is a reasonable approximation to make.

Once again, a tabular comparison follows; this time between observed and model-generated fertility values. Putting the quadratic formula to use, Tait estimates the total number of children a woman can expect to have (i.e. her *completed family size*) based on the number of children she has had to date.[203]

Taking age at marriage into account

Tait finds evidence within Duncan's tables to suggest that there is a relationship between the age at which a woman is no longer able to have children and her age at marriage: 'the age of sterility is *uniformly* later as the age at marriage is greater'.[204] See Table 3.1 (page 70) for the data.

Tait adjusts his linear and quadratic models accordingly. His linear model (3.2) becomes

$$f_t = k(C-t)$$

where C is the age at which sterility arrives. However, since the value of the parameter C is dependent on the age at marriage, Tait has, in effect,

$$f_{ta} = k(C_a - t)$$

where f_{ta} is the fecundity at age t for women married at age a and C_a is the corresponding age of sterility for women married at age a.

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[203] See [103, p214].

[204] [103, p216]

Age at Marriage	Age of Sterility
15–19	43
20–24	46
25–29	47.5
30–34	48.5

Table 3.1: 'showing the age at marriage, and, of the advent of sterility'. Reproduced from Table LXXXIV [103, p216].

Age at Marriage	Whole Fertility	F_t
15–19	10	10
20–24	7.7	7.4
25–29	5.5	5.0
30–34	3.4	3.1

Table 3.2: 'showing the influence of the advent of sterility upon the whole fertility of marriage'. Reproduced from Table LXXXV [103, p217]. The values in the third column are those produced by the original quadratic model (3.3) in which age at marriage is not taken into account.

Adjusting his quadratic model of fertility (3.3), Tait derives an expression for what he terms "whole fertility". He has

$$\text{whole fertility} = F_a = \frac{1}{2}k(C_a - a)^2$$

where a is the age at marriage and C_a is the corresponding age of sterility.[205] We

[205]Tait uses t for both the current age of a woman and her age at marriage. To avoid confusion, I

can interpret whole fertility as the total number of children any woman marrying at age a can expect to have before the onset of sterility which is determined by her age at marriage; that is, her *completed family size* as determined by her age at marriage. The figures are given in Table 3.2 (page 70).

Geometric distribution?

Of particular interest is a remark Tait makes in connection with his interpretation of the expression $f_{17} = \frac{10}{13}$. He writes: 'It may be well to notice that the interpretation of the expression $f_{17} = \frac{10}{13}$ is, that *a wife of* 15–19 *will, on the average, become pregnant at 1.3 years after marriage*—that is, she will have a child within about two years of marriage.'[206] It appears that Tait had an intuitive understanding of the geometric distribution. He has given, in effect, the expected value $\frac{1}{p}$ of a geometrically distributed random variable, p being the probability of success in each trial. In this case, the random variable is the time interval between marriage and the first success, i.e. pregnancy resulting in a living child.

3.2.2 Fertility and fecundity of the average individual

Tait's distinction between "for the mass of women" and "for the average individual" is unclear. I can only think that for the average individual he is considering age-at-marriage subgroups.

Tait's data sets II:

- Data set C—Tables of sterility, giving the percentages of sterile and not-sterile wives from Edinburgh and Glasgow, taken as a whole— [103, p193]

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have chosen to use a to denote age at marriage.

[206] [103, p217] Tait is working with midpoint values because the data for each separate age is unavailable.

Testing the linear and quadratic models against new data

This new data set C provides Tait with a further opportunity to test his models. He supposes that for each five-year age group, the figures for the percentage of not-sterile wives will be proportional to the average fecundities and so will form a straight line when plotted.[207] Instead, however, they produce the dotted curve in Figure 3.4 (page 68). Tait suspects problems with the data.

For the depressions in the curve between the ages 25–30 and 40–45, Tait offers a remarkable explanation, citing 'the loose way in which women from 30 to 40 call themselves 30, and those from 40 to 50 call themselves 40'.[208] To substantiate his bold claim, Tait quotes some figures from the 1851 Census Report. He calculates that 140,000 women aged less than ten years in the period between the censuses of 1841 and 1851. This is evidence to Tait of 'how strong is the desire [amongst women] to be considered as remaining under the magic limit of thirty years of age'.[209] Of course, possible alternative explanations might include: (i) the questionable reliability of the early censuses and (ii) immigration, remembering, especially, the displacement of people by the potato famine in Ireland during the period 1845–1852.

A tabular comparison follows, comparing (i) the observed values of percentages of not-sterile women with the model-generated fecundity values produced by the linear model (3.2) for a suitable k (Table LXXXVII [103, p220]) and (ii) fertility values, derived from equation (3.1) using observed percentages of not-sterile women, and model-generated fertility values obtained from the quadratic model (3.3) (Table LXXXVIII [103, p221]).

Foremost in Tait's mind is the importance of deriving a simple model. He is willing to sacrifice a little in terms of the amount of data explained by the model for the sake of its simplicity: 'It is easy, of course, to construct a formula to represent

[207]In this context, sterility means an inability *thus far* to produce a living child.

[208] [103, p219]

[209][Ibid.]

any series of numbers, but unless it be simple it is of little use'.[210]

Tait is pleased with the results of the tabular comparisons, bearing in mind the following problems with the sterility data. First, there are insufficient numbers of women in the age groups 40–44, 45–49. Second, there is an issue with the 15–19 year old group: women in this group may have been married for such a short time that pregnancy and children will fall within the next age group; and this effect is amplified within the first age group as there is no group below (of younger women) feeding into it, i.e. the 15–19 group will not include mothers married at 14. Third, plural births are not eliminated. Here is a simple example to illustrate the problem with plural births. Consider three women A, B, C. Woman A gives birth to two children, woman B gives birth to one child and woman C is childless. Without due care, it may appear that three women have each given birth to one child, which will result in a 0% sterility figure. Tait notes the implausibility of a 0% sterility figure, an *outlier*, for the 20–24 age group (Table LXXXVII [103, p220]).[211]

3.2.3 Relative fertility and fecundity of different races

Applying the above results, Tait is able to compare the fertility of different races.

Tait's data sets III:

- Data set D—The Registrar General's Reports for England and Scotland (1866), extracted into Tables LXXXIX and XC— [103, p224]

Tait's ratio

Working towards a means of comparing the fertility of two given races, Tait has to assume a number of 'postulates' (or assumptions). First, that for a period of ten or fifteen years, the number of marriages at any age remains the same—a *stationary variable* in modern terminology—so that we can represent the number of births in a

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[210] [103, p220]

[211] See [Ibid.].

single year by the total fertility of those married in that year. Second, that the two countries obey the same law of fertility; or, the ratio of the fertility of the second race to that of the first at every age $t, t+1, \ldots$ is a constant e. Tait has

$$e = \frac{\beta'}{\beta}\frac{\Sigma \mu_t F_t}{\Sigma \mu'_t F_t}$$

for t from 15 to 49, where μ_t is the number of marriages of women at t years of age in any one year, β is the number of legitimate births in a year and F_t is the fertility at age t. Dashes distinguish between the nations.[212]

Comparing the fertilities of England and Scotland, Tait finds $e = 0.812$ or that the fertility in England is $\frac{8}{10}$ of that in Scotland. On this result, he makes the following remark:

> It is to be observed that if the insinuations we sometimes hear about Scottish marriages have any foundation in fact, their consideration would tend to make the difference in fertility between the two countries even greater than that just given; for legitimation *per subsequens matrimonium* does not put a child's name on the Registrar's books.[213]

By *per subsequens matrimonium*, Tait means the legitimization upon marriage of children born before their parents married: it had been a law in Scotland for centuries before it was introduced in England in the Legitimacy Act of 1926.[214]

Tait's fertility values for England and Scotland, taking age at marriage into account, are those shown in Table 3.3 (page 75).

[212] For details of how Tait forms his ratio see [103, pp222–224].

[213] [103, p225]

[214] [112, pp5–6]

	15–19	20–24	25–29	30–34	35–39	40–44	45–49
Scotland	7.44	5.54	3.77	2.30	1.24	0.37	0.06
England	6.04	4.49	3.02	1.87	1.01	0.30	0.05

Table 3.3: 'showing comparative fertility of a mass of wives in England and Scotland, taking account of the age at marriage'. Reproduced from Table XCI [103, p226].

3.2.4 Tait's appreciation of good data

Throughout, Tait has been plagued by issues associated with poor quality data. In his concluding remarks to Part VI, he alludes to the ineffectiveness of contemporary information gathering which, of course, has a bearing on the validity of the conclusions drawn from analysis of the data. He writes:

> As in all questions of average, the value of our deductions in this matter is mainly dependent on the extent and accuracy of our data, and it is sad to think that the enormous blue-books which load our shelves contain so much painfully-elaborated information which is of no use, and so little of those precious statistics which would at once be easy of acquirement and invaluable to physiologists.[215]

3.2.5 The second edition (1871)

The first edition of *Fecundity, Fertility and Sterility* was so well received that a second edition [106] was published in 1871. Included in this revised and enlarged second edition is an anonymous review of the first edition.

Regarding the critic's identity, we know only that he is not known to Duncan, for the critic remarks on Duncan's kindness in sending his tables to him, who is 'a complete stranger'.[216] I discovered the identity of the critic in an unusual context—in

[215] [103, p227]

[216] [106, p256]

Memoir of Fleeming Jenkin (1912) by Robert Louis Stevenson (1850–1894). Before becoming a celebrated author, Stevenson had studied engineering at the University of Edinburgh under Professor Fleeming Jenkin (1833–1885).[217] Stevenson reveals in the *Memoir* that it was Jenkin who had reviewed Duncan and Tait's first edition and that Duncan had included the review in the second edition.[218]

Fleeming Jenkin's review

Jenkin's position is that he believes that factual information on fertility and fecundity issues ought to be available to ordinary people, so that they might evaluate the risks associated with pregnancy and childbirth, and make informed decisions about the size of their families. Here is a brief summary of some of the key points he makes in the review, in relation to Tait's contribution especially.

— Duncan and Tait's approach is straightforward. They have presented their information honestly and intelligently, without looking to manipulate their findings or draw inferences.

— Insufficient data was available to Duncan and Tait.

— Duncan has been unable to produce information on the risks associated with bearing children in rapid succession. An optimum interval between births needs to be ascertained.

— Duncan and Tait's definitions of fecundity and fertility are inconsistent. The difference in their interpretation of fecundity is responsible for the difference

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[217] [5, p186] For a period Stevenson had worked on scientific investigations in Tait's laboratory. [33, p47] See [113] for Stevenson's recollections. Jenkin and Stevenson were both former pupils at the Edinburgh Academy: Henry Charles Fleeming Jenkin (1833–1885) was a classmate of Tait's during the 1840s; Robert Louis Stevenson (1850–1894) attended the Academy between 1861 and 1863. [4, pp117, 263–264]

[218] See [114, p91].

in the conclusions they draw from the data. When Jenkin redefines the terms for himself he follows Tait more closely than Duncan.[219]

— Tait's linear model (3.2) is a 'general law of great importance'.[220] It produces fecundity values which are close to the observed figures. Its simplicity is impressive: 'It is very rare to find a very simple result derived from complex elements'.[221] From Tait's linear model we understand that women are possessed of different degrees of fecundity (high, low and intermediate), the levels exhibiting themselves in the number and spacing of children produced. An awareness of the degrees of fecundity may lead to an explanation of what has been observed without the need to consider, for instance, sterility as depending on age at marriage. However, Tait's linear model has been 'proved for a mass of women only' and so the law must be re-interpreted as follows: 'The average number of children per annum born to a mass of women of any age is proportional to the difference between that age and 50.'[222] It is likely the law will hold for an individual but to prove so would mean verifying that the law holds for large subgroups of women, all married at the same age.

— Tait's application of his formulae to the comparison of nations is worthwhile. Similar analyses should be carried out on the 'inferior races'.[223]

Comparing the fertility of England, Scotland, Ireland and Sweden

In the second edition—with access to new data from Sweden and from the Registrar-General's Reports for England, Scotland and Ireland—Tait compares the fertility of

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[219] See [106, p245].

[220] [106, p246]

[221] [106, p247]

[222] [106, p246]

[223] [106, p261] Surely a reference to the developing nations.

these four nations. The results of his comparison are presented in Table XCV [106, p239] and in Table 3.4 where the average fertility of married women of 15–19 in each of the four countries is given.

Nation	Average fertility of married women of 15–19	Fertility relative to Scotland
England	6.21	86%
Scotland	7.23	100%
Ireland	7.14	98%
Sweden	8.48	117%

Table 3.4: Average fertility of married women of 15–19 for four nations. Reproduced from Table XCVI [106, p240] with an additional third column showing how Tait interprets each nation's figures as a percentage of Scotland's figure.

It is likely that Tait chose to include Sweden in his cross-national comparison because of the wealth of high-quality demographic data available in that country. Indeed, professional demographers have since made frequent use of Sweden's demographic records, either to demonstrate new methodologies or to test new models.[224] Sweden boasted an unusually complete set of parish registers, recording vital statistics (births, marriages and deaths) from the sixteenth century onwards. Yet further improvements in the quality of Swedish demographic data took place because of a reorganization of data-collection and data-recording procedures, following the establishment of a central statistical commission and office in 1858. The central statistical office had exclusive charge of the statistics of population. Its statistical library contained some 4000 volumes c.1860. In 1860 Dr. Berg—Vice-President of the commission and head of the statistical office who worked on examining and tabulating extracts of Sweden's parish registers—attended the International Statistical

[224] [115, pp8–9]

Congress in London as the official delegate for Sweden.[225] He presented a detailed report on the present state of statistical inquiry in Sweden.[226]

3.2.6 Why Tait was writing in French

The further work done by Tait in the second edition provides a possible explanation for why he was writing in French in his pocket notebook. The data that he had access to from Sweden was made available to him by Dr. Berg, Chief of the Statistical Bureau in Stockholm.[227] In correspondence with Dr. Berg, I believe that Tait would have used French as a common language.[228] Therefore, I suggest that Tait's French entry in his pocket notebook was perhaps a draft list of the data he wished to obtain from Dr. Berg. I have already remarked (page 60) that the dates recorded in the notebook correspond with the time when he would have been preparing for the second edition of Duncan's book. Of course, it is possible that Tait was corresponding with someone else, attempting to get hold of some data from France perhaps. What is less likely is that Tait was writing in French purely as an intellectual exercise, just for fun or to keep the contents of his notebook private.

3.3 Concluding remarks

Ultimately, Tait's "digression" into probability and statistics is not surprising if we bear in mind the following:

— Tait's credentials. (i) Being a "combinatorial man" Tait would have had no

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[225] [116, p384]

[226] [117, pp45–49] The preceding information on Sweden has come from this source.

[227] [106, p237]

[228] French was the language of the International Statistical Congress, which was held in London in 1860: German was permitted at the Congress but English was not. [118, p377] It is very unlikely that Tait attended the International Statistical Congress in 1860. He certainly does not appear on the list of delegates.

difficulty in overcoming the logic of the subject.[229] (ii) He had experience in handling data and would treat Duncan's data as he would any data coming from experiment or observation.

— Tait could be counted amongst those driving forward statistical developments in the nineteenth century—not mathematicians, especially, at this stage—but social scientists, biologists, physicists and generalists.[230]

3.3.1 Influence of Tait's Laws

Earlier in this chapter (page 60) I included a quote from Knott in which he referred to "Tait's Laws". Evidence suggests that the first of Tait's Laws (his linear model) was of influence well into the twentieth century.

In 1906 the statistician, G. Udny Yule (1871–1951) referred to Tait's first law in a paper [119] published in the *Journal of the Royal Statistical Society* entitled, 'On the Changes in the Marriage- and Birth- Rates in England and Wales During the Past Half Century'.[231] Yule used Tait's linear model of fecundity on age to calculate expected birth-rates and his coefficient k as a measure of fertility. It seems that Yule's 1906 paper led to a number of subsequent authors citing Tait's Laws.[232]

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[229]For evidence of Tait's remarkable intuition when tackling combinatorial problems, consider his work on knot enumeration: without the benefit of rigorous methods, Tait was able to determine whether or not two knot diagrams were equivalent; and in this way, he was able to produce the very first knot tables.

[230] [109, pxi]

[231]G. Udny Yule (1871–1951) was a Fellow of the Royal Statistical Society of London, the Royal Society and St John's College, Cambridge. He came to statistics, through the influence of Karl Pearson (1857–1936), from a background of engineering and experimental physics. His area of expertise was correlation and regression. [120]

[232]Including the American biophysicist, Alfred J. Lotka (1880–1949) who is remembered, in particular, for having developed—in 1925, independently of Vito Volterra—the Lotka–Volterra predator-prey model. He refers to "'Tait's Law" of linear fertility decrease with age' in [121]: see page 160f for the reference to Tait.

References to Tait's Law(s) continued until the 1970s.[233] Yule referred to Tait's Laws again in 1920.[234]

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[233] From the 1970s onwards, an 'enriched environment for research on populations' developed; due to 'technological advances in the publication of census information, as well as vital statistics' and the 'growth in the range and frequency of population surveys'. [122, p27] This in turn led to a period of statistical innovation, which included the development of new methodologies in data processing and analysis, model building and parameter estimation. [115, p1] In particular, the methods of exploratory data analysis (E.D.A.), developed by Tukey and Mosteller, enabled fertility models to admit the influence of: natural fertility; patterns of marriage, divorce and widowhood; the pace/timing of childbearing and efforts to limit fertility. For more information see [115].

[234] Yule refers to Tait's Laws again in [123].

CHAPTER 4

TAIT'S SCHOOLBOY INTRODUCTION TO COMPLEX NUMBERS

This chapter describes Tait's first encounter with complex numbers and their geometrical representation.

4.1 Introduction

From age ten to sixteen, Tait was educated at the celebrated Edinburgh Academy. He was a pupil in Dr. Cumming's class during his first four years; thereafter, the class came under the care of the Rector, Archdeacon John Williams (1792–1858).[235] Dr. James Cumming (1800–1875) instructed the boys in classics; the three R's was taught by Robert Hamilton ("Hammy"); mathematics by James Gloag and French by François Senébier ("Snibby").

During his time at the Edinburgh Academy, Tait won a number of prizes and medals, including Dux annually throughout his school career.[236] For the full list of his schoolboy achievements see Appendix C (pages 232–233). Lieutenant-Colonel

[235] In 1850 the surviving members of Dr. Cumming's class formed themselves into the Cumming Club, with the intention of maintaining or re-establishing those early friendships and sharing in their mutual affection for Dr. Cumming. Dr. James Cumming had a reputation as a first-class educator and as a master who was popular with the boys: he was known to be fair, good-humoured and kind. [5, p146] For a book on the Cumming Club, written by a classmate of Tait's, see [32].

[236] Tait's school medals are on display at 14 India Street, Edinburgh. A gold medal was awarded to the Dux in the Rector's class and a silver medal to the Dux in all other classes; winners of lower prizes received maps and books. [5, p95] The gold Dux medal was engraved with the Louvre bust of Virgil; the silver medal featured the Townley bust of Homer from the British Museum. [5, p106]

Alexander Fergusson (1831–1892)—a classmate of Tait's and author of *Chronicles of the Cumming Club*—recalled that such sustained success won for Tait a reputation amongst his schoolfellows as 'Our permanent Dux'.[237] He writes: 'Through all the classes, from the First to the Sixth, when he left the Academy, Tait was easily our leader.'[238]

4.1.1 Schoolboy association with Maxwell

When Tait was in the fourth class at the Academy, he became friends with fellow student, James Clerk Maxwell. This association is well known.[239] Both boys were the same age but Maxwell was placed in the class above Tait (Mr. Carmichael's class) because Dr. Cumming's class was full when Maxwell came to enrol. Tait remembered his early acquaintance with Maxwell in a short biographical note he wrote for the R.S.E. upon Maxwell's death:

> When I first made Clerk-Maxwell's acquaintance about thirty-five years ago, at the Edinburgh Academy, he was a year before me, being in the fifth class while I was in the fourth.
>
> At school he was at first regarded as shy and rather dull; he made no friendships, and he spent his occasional holidays in reading old ballads, drawing curious diagrams, and making rude mechanical models. His absorption in such pursuits, totally unintelligible to his schoolfellows (who were then quite innocent of mathematics), of course procured him a not very complimentary nickname ["Dafty"[240]], which I know is still remembered by many Fellows of this Society. About the middle of his school career, however, he surprised his companions by suddenly becoming one of the most brilliant among them, gaining high, and sometimes the highest, prizes for Scholarship,

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[237] [32, p22]

[238] [32, p202]

[239] Indeed, for some, Tait's association with Maxwell is all they know of Tait. Amongst those who know of Tait through Maxwell, it seems that the majority prefer Maxwell: they think of him, in later years, as a more remarkable talent, as being more modest and as having a less aggressive personality than Tait.

[240] See [5, p149] and [32, pp25–26].

Mathematics, and English verse composition. From this time forward I became very intimate with him, and we discussed together, with school-boy enthusiasm, numerous curious problems, among which I remember particularly the various plane sections of a ring or *tore*, and the form of a cylindrical mirror which should show one his own image *unperverted.* I still posses some of the MSS. which we exchanged in 1846 and early in 1847. Those by Maxwell are on "The Conical Pendulum," "Descartes' Ovals," "Meloid and Apioid," and "Trifocal Curves." All are drawn up in strict geometrical form and divided into consecutive propositions.[241] The three latter are connected with the first published paper, communicated by Forbes to this Society and printed in our "Proceedings," vol. ii, under the title "On the Description of Oval Curves, and those having a plurality of foci" (1846).

At the time when these papers were written he had received no instruction in Mathematics beyond a few books of Euclid, and the merest elements of Algebra.[242]

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[241] For Maxwell's manuscripts see *The Scientific Letters and Papers of James Clerk Maxwell* edited by P. M. Harman: "The Conical Pendulum" (text 6 [124, pp64–67]); "Descartes' Ovals", and "Meloid and Apioid" (texts 3(1) [124, pp47–54] and 3(2) [124, pp55–61]); and "Trifocal Curves" (text 2 [124, pp43–46]). Text 2 has been transcribed from the original in the University Library, Cambridge. Texts 3 and 6 have been reproduced from Campbell and Garnett's *Life of Maxwell* (1882). Harman explains [124, p43f(1)]:

> The manuscript 'On Trifocal Curves', [Number 2] which is annotated in pencil by Tait, and dated 'March 1847', is clearly the paper referred to by Tait. The manuscript on the conical pendulum dated 25 May 1847 (Number 6), and the propositions on 'Oval' and 'Meloid and Apioid' (Number 3) which are reproduced from the *Life of Maxwell*, are possibly the other papers referred to, or are drafts of these papers. The papers 'On Trifocal curves', 'Oval' and 'Meloid and Apioid' are closely related in content; the order in which they are reproduced here [in Harman (1990)] may not be the chronological order of their composition.

[242] [8, p332]

4.1.2 The Tait–Maxwell school-book

Tait copied the 'MSS. which [he and Maxwell] exchanged in 1846 and early in 1847'—referred to in the above extract—into his school-book. It seems that the school-book was originally intended as a fair-copy book: some entries are written carefully in ink, and are signed and dated; however, there is also an abundance of rough pencil work, with workings-out and schoolboy sketches subsequently fitted into available space.

Entries in the school-book include: (factually dubious) notes on the history of enumeration; a table recording the positions of the satellites of Jupiter, as observed by Tait at the age of thirteen; a number of problems and solutions on the mensuration of heights and distances, which I have traced to a contemporary textbook [3]; and copies of the MSS. which Tait and Maxwell exchanged; but the bulk of the school-book is taken up with Tait's notes which he abstracted from the *Encyclopaedia Britannica* (7th edition, 1842).[243] Tait made copious notes on a range of mathematical topics covered in the articles, 'Algebra' and 'Fluxions'.

From the article on algebra Tait made notes on the arithmetic of sines, producing a comprehensive list of trigonometric formulae. The article on fluxions is in two parts: Part I details the direct method of fluxions; Part II explains the inverse method of fluxions, otherwise known as the integral calculus. Tait's notes on Part I cover: successive differentiation, Taylor's theorem, Maclaurin's theorem, differentiation of equations of two variables, vanishing fractions, the greatest and least values of a function, determination of tangents to curves, generation of curves by evolution and contact of curves. Part II covers: integration of rational functions involving one variable, integrals of irrational fractions, integration of binomial differentials and integration of angular or circular functions. Tait's approach was to copy down key results/formulae and some illustrative examples. Perhaps this wider reading was in preparation for university or perhaps Tait did not find the mathematics syl-

[243] I imagine that Tait would have had a set of encyclopaedias at home.

labus at school challenging enough.[244] Certainly, the material extracted from the encyclopaedia ventures beyond the Academy's syllabus.[245]

4.1.3 Gloag's influence

For an indicator of the nature of the mathematical instruction given at the Academy, there is none better than a profile of Tait's mathematics master, James Gloag.[246] Knott writes: 'Gloag was a teacher of strenuous character and quaint originality [...] With him mathematics was a mental and moral discipline'.[247] By all accounts, Gloag was strict but fair, and the boys, once over their initial terror, seemed to progress well under his tutelage:

> The fact that he [Gloag] was much the strictest disciplinarian in the school, and deadly with a tawse withal, was offset by his indisputable fairness and impartiality, a gruff kindliness under the irascibility, and the growing evidence that the boys learned well under him—once they had got over their initial terror of him.[248]

On Gloag's technique with the tawse, Tait is reported to have said: "To use a well-known cricketing phrase, Gloag could get "more work" on the tawse than any of the other masters. His secret was in great part a dynamical one."[249]

It was in Gloag's lessons that young Maxwell's abilities in mathematics became apparent: growing in confidence, as a result of Gloag's influence, Maxwell's performance improved, not only in mathematics but across the board.[250] Indeed, at the

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[244]It is known that Tait entered the higher division of Prof. Forbes' natural philosophy class at Edinburgh. According to [124, p4] the higher division course required knowledge of the calculus.

[245]See Table C.1 (page 231) for the syllabus for classes 6–7 in 1846–1847.

[246]James Gloag: Master of the Arithmetical and Geometrical School at the Edinburgh Academy (1824–1864). [4, pxlix]

[247] [18, pp4–5]

[248] [5, p99]

[249] [5, p102]

[250] [5, p150]

age of fourteen, Maxwell achieved the rare distinction of having an academic paper read before the R.S.E. The paper on ovals—which is referred to in Tait's biographical note (page 84)—was communicated to the Society by Professor J. D. Forbes and published in the *Proceedings*.[251] Gloag revelled in his pupils' success and in 1852 when news of Tait's achievements at Cambridge reached the Academy, he took great pride in the part he had played early on.[252]

4.1.4 Significance of a particular school-book entry

On 18 January 1847 Charles Hughes Terrot, Bishop of the Scottish Episcopal Diocese of Edinburgh, read a paper [126] before the R.S.E. on the geometrical representation of complex numbers.[253] On 29 January Terrot's paper was handed over to Professor Kelland who was to produce a report on it and on 12 February, based on Kelland's report, it was ordered that the paper be printed in the Society's *Transactions*.[254] The paper was published in the *Transactions* with the title, 'An Attempt to Elucidate and Apply the Principles of Goniometry, as published by Mr Warren, in his Treatise on the Square Roots of Negative Quantities'.[255]

On 27 May Tait entered into his school-book an abstract of Terrot's published paper under the heading, 'On the imaginary roots of negative quantities. By the Right Reverend Bishop Terrot. 1847'. In May 1847 Tait was sixteen, in his final year at the Academy, and headed for the University of Edinburgh in the autumn.

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[251]For more on Maxwell's paper on ovals see: [125, pp74–79] and [124, p2–3].

[252] [32, pp85–86] To celebrate Tait's success at Cambridge, members of the Cumming Club arranged a banquet to be held in Tait's honour. The event took place at Archers' Hall in Edinburgh on 22 March 1852, with Dr. Cumming, Dr. Gloag, Mr. Hamilton and M. Senébier in attendance. See [32, pp84–88] for Fergusson's account of the high-spirited event.

[253]See [127] for the record of the communication in the *Proceedings*.

[254]According to the R.S.E.'s Council minute books: National Library of Scotland, Acc.10000/19, Nov. 1846 – Feb. 1859.

[255]*goniometry*: the measurement of angles; derived from the word *goniometer*, which is 'an instrument used for measuring angles'. [128]

Terrot's paper was surely Tait's first introduction to complex numbers and their geometrical representation. Quadratic equations were on the syllabus at the Academy for classes 6–7; however, it is extremely unlikely that the schoolboys, during the normal course of their lessons, would have been exposed to anything other than real roots.[256] In Davidson's *System of Practical Mathematics* [3]—a contemporary textbook from which Tait's worked examples in the school-book on the mensuration of heights and distances are taken—only real roots are covered, questions having been carefully selected to avoid complex roots.

The fact that the Bishop's paper was Tait's first introduction to complex numbers and their geometrical representation makes the find in the school-book significant but further research has added weight to this significance: it will be through this particular entry in the school-book that we will come to learn something further about the discovery of quaternions and extend the scope of the history of the Argand diagram, going beyond Wessel, Argand and Gauss.

First, an introduction to the Right Revd. Bishop Terrot.

4.2 Bishop Terrot

4.2.1 Charles Hughes Terrot (1790–1872): a biographical sketch

Charles Hughes Terrot (1790–1872) (Figure 4.1, page 90) was born at Cuddalore, India on 19 September 1790. He was the son of Elias Terrot, a Captain of the 52nd Regiment in the Indian Army. His great grandfather, Monsieur de Terotte—a protestant (Huguenot) exile from France—had fled from La Rochelle and sought refuge in England on the revocation of the Edict of Nantes in 1685.[257]

Upon Elias' death at the siege of Bangalore in 1790, Terrot's mother, Mary Fonteneau left India with her infant son and settled in Berwick. Young Terrot was

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[256] Again, see Table C.1 (page 231) for the syllabus for classes 6–7 in 1846–1847.

[257] A visual indication of Terrot's ancestry is given in [129, p227].

placed under the care of the Revd. John Fawcett of Carlisle and was educated at Carlisle Grammar School.

It was at Cambridge that Terrot earned a reputation for scholarship, particularly in mathematics. He entered Trinity College in 1808, gaining his B.A. in 1812, graduating with mathematical honours, despite disappointing Tripos examination results. The problem had not been a lack of intellect, rather an unwillingness to apply it in tedious activity. From the R.S.E.'s *Proceedings*, with information provided by Prof. Kelland:

> The fact is that Terrot's mind revolted at the drudgery of acquiring branches of the science [mathematics] towards which he felt no inclination. It was characteristic of him to tread a small circle, but to tread it well; and he was constitutionally unfitted for stowing away in his memory, for temporary purposes, facts and figures in which he took no interest.[258]

Fortunately for Terrot, he had had ample opportunity outwith the Tripos examinations to prove his abilities and on this basis he was elected a Fellow of Trinity College in 1813. The same year he was ordained a deacon, with ordination to the priesthood following in 1814. In 1815 he settled in Haddington, taking up a position held previously by his uncle, the Revd. William Terrot, as Minister of the Episcopal congregation.

Terrot's move to Edinburgh took place in 1817: he was to assist the Revd. James Walker at St Peter's in Roxburgh Place. In 1833 Terrot joined two other clergy at St Paul's in York Place. During the next twenty years, his appointments grew in prestige: he was appointed Dean of Edinburgh and Fife in 1837; Rector of St Paul's in 1839; Pantonian Professor at the theological college, and Bishop of Edinburgh, in 1841 and Primus of the Scottish Episcopal Church in 1857. He remained as Primus until 1862, when a paralytic stroke forced his resignation. He married twice: his first wife, with whom he had fourteen children, left him a widower.[259] He died in

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[258] [130, p9]

[259] According to [131] Terrot's eldest daughter accompanied Florence Nightingale to the Crimea and for her service there she was awarded a Royal Red Cross.

Stockbridge on 2 April 1872, aged eighty-two.

Figure 4.1: Charles Hughes Terrot by Mason & Co (Robert Hindry Mason), *albumen carte-de-visite*, 1860s. Reproduced with the kind permission of the National Portrait Gallery, London. The handwritten text reads '$\mathrm{B^p}$ [Bishop] of Edinburgh'. The size of this reproduction is representative of the typical size of a *carte-de-visite*, 54mm × 89mm mounted on a card 64mm × 100mm.

4.2.2 Devotion to mathematics

Understandably, Terrot sought relief from the heavy burden of his responsibilities. As Kelland recalled, Terrot turned to mathematics for refuge: 'To mathematics, when harassed by the cares and vexations incident to his position, he had recourse as a retreat from irritating thoughts. His passion for the science was strong enough to take possession of his mind, and soothing enough to settle it down to repose.'[260]

[260] [130, p11]

The extent of Terrot's reliance on mathematics and its implications, both beneficial and otherwise, were discussed by the Revd. Walker—whom Terrot had assisted at St Peter's—in his biographical work, *Three Churchmen* (1893). Walker writes:

> Absorbed in the depths of original research, the bishop found that which can, it is said, be always found in the depths of the ocean, viz., calm, in the midst of storm.[261]

> Readers will probably take very different views of Bishop Terrot's occasionally ardent devotion to mathematical study. Some will think that whenever he had any spare time for investigation and research it ought to have been devoted exclusively to professional subjects, such as the theological and biblical problems of the day. Others will hold that the one study was a help rather than a hindrance to the other; the occasional subjection of the mind to the vigorous mathematical discipline being the best corrective of loose thinking and illogical reasoning. Those who take this view will believe that the bishop's addiction to mathematical research was an advantage to the Church as well as to himself; not only assuring him an occasional refuge from worry, but also maintaining in him that judicial frame of mind which never deserted him in the hottest controversies, and which extorted the admiration of his opponents.[262]

> The only evil effect of the bishop's mathematics was probably a little intolerance of the loose talk and inconsequential reasoning which often prevail in general society.[263]

Apart from mathematics, Terrot spent time writing poetry and much of his leisure time while at Haddington was devoted to it. His poem 'Hezekiah and Sennacherib, Or the Destruction of Sennacherib's Host'—about the destruction of Sennacherib's army before Jerusalem—won him the Seatonian Prize in 1816.[264] He also had a love of architecture and enjoyed membership of the Architectural Society of Scotland.

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[261] [132, p156]

[262] [132, p161]

[263] [132, p163]

[264] The Seatonian Prize has been awarded by the University of Cambridge annually since 1750, for a poem written in English on a sacred subject.

4.2.3 Reputation in Edinburgh, contribution to the Royal Society of Edinburgh

Terrot was elected a Fellow of the R.S.E. in 1840, proposed for fellowship by J. D. Forbes. Terrot's role in the Society was described by Kelland thus:

> For many years of his life he was one of the regular attendants at our meetings; and when not actively engaged in the work going on, he was an active listener, and, when occasion called for it, and unsparing critic. He had a real love for the Society. As he left the building for the last time, he expressed himself to the effect, that henceforth his heart would be with us, but that the work of his hands was done. The only part of the proceedings which he did not relish was the tea-drinking after the meeting.[265]

Terrot served as a Councillor for the Society (1841–1844) and as their Vice-President between 1844 and 1860. Looking through the *Proceedings*, there are numerous instances of the 'Right Rev. Bishop Terrot, Vice-President, in the Chair'.

Terrot delighted in good conversation and, according to the Revd. Walker, in Edinburgh he had developed a 'very high reputation as a talker of the Johnsonian type'.[266] By this, Walker meant:

> precision of thought and language, ready wit, repartee and love of argument—all set off to advantage by a distinct voice and deliberate utterance. He [Terrot] was also almost as impatient as Johnson himself was of twaddle and of pretence—"humdrum and humbug"—and thus to weak reasoners and pretentious talkers he appeared to be, and doubtless sometimes was, severe and sarcastic. But to men of like mind with himself—deep and just thinkers and earnest talkers—his conversation was very highly prized, and his society much courted.[267]

Terrot's contributions to the R.S.E. comprise the following papers (ordered by date of communication to the Society):

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[265] [130, pp12–13]

[266] [132, p167] Dr. Samuel Johnson (1709–1784): author and lexicographer; remembered as the compiler of Johnson's *Dictionary*, published in two volumes in 1755.

[267] [132, p182]

1845 'On the Sums of the Digits of Numbers'

1847 'An Attempt to Elucidate and Apply Mr Warren's Doctrine Respecting the Square Root of Negative Quantities'

1848 'On Algebraical Symbolism'

1849 'An Attempt to Compare the Exact and Popular Estimates of Probability'

1850 'On Probable Inference'

1853 'On the Summation of a Compound Series, and its Application to a Problem in Probabilities'

1856 'On the Possibility of Combining Two or More Independent Probabilities of the Same Event, so as to Form One Definite Probability'

1858 'On the Average Value of Human Testimony'

According to Kelland in [130], Terrot's 1856 paper was his 'best contribution to mathematical science': it had inspired Boole's paper, 'On the Application of the Theory of Probabilities to the Question of the Combination of Testimonies or Judgements', for which Boole was awarded the R.S.E.'s Keith Prize in 1858.[268]

4.2.4 Associations with the Edinburgh Academy

An obituary in the *British Medical Journal* suggests that Terrot, at some stage, taught at the Academy: 'Spencer Thomson, MD., Torquay [...] Educated at the Edinburgh Academy, under the late Rev. Dr. Terrot, afterwards Bishop of Edinburgh'.[269] Yet there is no evidence in the Academy's *Register* [4] to support such a theory: the three references to 'Terrot' in the *Register* reveal only that Terrot had educated his children at the Academy between 1825 and 1842.[270] It seems unlikely,

[268] [130, p12]

[269] [133, p442]

[270] Terrot's children educated at the Edinburgh Academy: Charles Samuel John, Elias Charles and William H. [4, pp35,63,84]

therefore, that Terrot was ever a member of staff at the Academy.

Perhaps the obituarist had misinterpreted Terrot's influence on young Thomson. I suggest that Terrot may have acted as a mentor or tutor to the boy since Thomson was in the same class at the Academy as one of Terrot's sons, Elias Charles, and he resided in the same street as the Terrot family, Northumberland Street.[271] In Cassell's *Old and New Edinburgh*, a description is given of Northumberland Street and Bishop Terrot, as one of its residents:

> In the narrow and somewhat sombre thoroughfare named Northumberland Street have dwelt some people who were of note in their time. [...] No. 19 in the same street was for some years the residence of the Right Rev. Charles Hughes Terrot, D.D., elected in 1857 *Primus* of the Scottish Episcopal Church, and whose quaint little figure, with shovel-hat and knee-breeches, was long familiar in the streets of Edinburgh. [...] few men were more esteemed and respected by others than Dr. Terrot of the Episcopal Church.[272]

It is possible that Terrot provided Thomson with private tuition in mathematics. From the time of his fellowship at Trinity, Terrot had supplemented his income by taking on pupils for private tuition.[273] Indeed, a substantial increase in salary had to be arranged so that Terrot no longer needed to tutor and could devote himself to his ecclesiastical duties: 'The minute of the vestry states that "so large an advance has been at once made to Mr. Terrot's salary, making it higher than any clergyman of the chapel ever received before with the view of securing his undivided attention to his duties of minister of the chapel."'[274]

Terrot was known to have had a close association with Archdeacon William's successor as Rector, Dr. John Hannah (1818–1888) who was a former Fellow of Lincoln

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[271] [4, p35]

[272] [134, p198]

[273] [132, pp104–105]

[274] [132, pp110–111]

College, Oxford.[275] The Revd. Walker describes Dr. Hannah as one of the Bishop's 'most intimate friends'.[276] The two men lived near to each other and Dr. Hannah took great pleasure in familiarizing Terrot with 'Oxford forms of thought' at a time when the philosopher, Sir William Hamilton (1805–1865) was promoting communication between Edinburgh and Oxford.[277] The two were also connected through the R.S.E.: Hannah was elected a Fellow in March 1848, having been proposed by Terrot in the January.[278,279]

4.3 Bishop Terrot's 1847 paper

A summary of the Bishop's paper now follows. For a transcription of Tait's notes on the paper see Appendix D.

4.3.1 Summary of the paper

Terrot begins his paper by explaining that while $\sqrt{-1}$ is called "impossible" or "imaginary", since any number squared must be positive, with a geometrical interpretation $\sqrt{-1}$ is no more impossible than $+1$ or -1, for each is capable of being represented by a directed line in the plane. A line of length a, fixed at one end to the origin and inclined at an angle ϑ, he symbolizes by $a^{\frac{\vartheta}{2r\pi}}$.

He then asks us to consider lines of equal length, or radii of a circle expressed as $R \times 1^{\frac{\vartheta}{2r\pi}}$, where R is the length of the radius and $1^{\frac{\vartheta}{2r\pi}}$ is the "coefficient of direction". He explains that the radii of a circle, with equal angles between them, actually represent the nth roots of unity and he sets out a method which enables us

[275] Archdeacon John Williams was Rector at the Academy during the periods 1824–1828 and 1829–1847, and Dr. John Hannah was Rector at the Academy between 1847 and 1854. [4, pxliii]

[276] [132, p178]

[277] [132, p168]

[278] According to Hannah's entry in [29].

[279] Uncited sources of biographical information on Terrot: [13] and [29].

to properly order the roots: the nth roots of unity appear in pairs, $a+\sqrt{-b}$ with $a-\sqrt{-b}$, either side of the original radius, with equal angles between them and the original radius. Note, Terrot does not use the term *complex conjugates*.

Keen to show how to multiply lines together, Terrot then introduces an alternative expression for $1^{\frac{\vartheta}{2r\pi}}$; written in polar form as $\cos(\vartheta)+\sqrt{-1}\sin(\vartheta)$. Similarly, a line inclined at an angle $p\vartheta$ he expresses in the form $\cos(p\vartheta)+\sqrt{-1}\sin(p\vartheta)$. By thinking of a line inclined at an angle $p\vartheta$ as the result of p rotations through an angle ϑ, he establishes De Moivre's theorem. As a corollary, he gives a means of obtaining the quadratic factors of the polynomial $x^p-1=0$, by multiplying together conjugate pairs of the pth roots.

Bishop Terrot also explains in his paper that from the algebraic expression of a line $a^{\frac{\vartheta}{2r\pi}}$ we know the line's length and its direction; and, therefore, we might think of $a^{\frac{\vartheta}{2r\pi}}$ as representing the transference of a point in space, moving from A to C say. Within the triangle ABC there are two routes from A to C: the direct route; or, via B, journeying a distance of $|AB|$ in the direction of AB and then a distance of $|BC|$ in the direction BC. Vectors in all but notation. Terrot establishes one side of the triangle as a line in an original position and expresses all other sides as rotations of this original. Note, Terrot uses round brackets to indicate the length of a line.

Having applied his symbolism to various elementary propositions in plane trigonometry, Terrot then uses it in a proof of Cotes' theorem. In the R.S.E.'s *Proceedings* Terrot's proof is described as 'a new demonstration of Cotes' properties of the circle'.[280] Terrot's proof of Cotes' theorem is the summit of his paper.

Theorem: Cotes' Properties of the Circle.[281] *Let the circumference of the circle be divided into n equal parts; and to the extremities of these let lines be drawn from the centre* [Figure 4.2, page 97], *as OP_1, OP_2, &c., and from any other point C in the diameter. Then*

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[280] [127, p111]

[281] Wording abstracted from Terrot's paper [126, p353].

$$|CP_1| \times |CP_2| \times |CP_3| \cdots |CP_n| = |OA|^n - |OC|^n$$

[*and variants of this, depending on the position of* C.]

Proof. For Terrot's proof of the theorem see [126, pp353–354] and page 242 of this thesis.

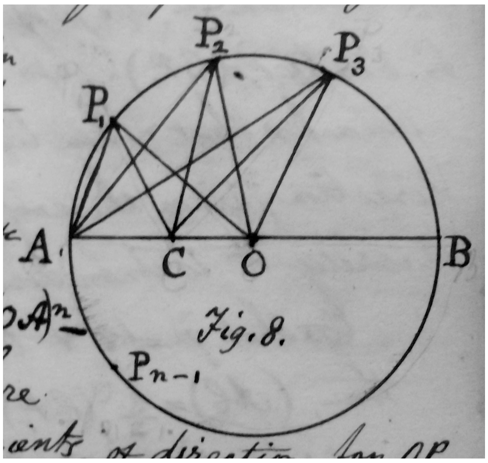

Figure 4.2: Tait's drawing of the figure to accompany Cotes' theorem. Sourced from the Tait–Maxwell school-book. Reproduced with the kind permission of the J.C.M. Foundation.

A remark on Wessel's proof of Cotes' theorem

The above theorem was formulated by Roger Cotes (1682–1716) in 1716 and was published posthumously in *Harmonia Mensurarum* in 1722; however, Cotes left no demonstration of the theorem. In 1797 the Norwegian surveyor, Caspar Wessel (1745–1818) gave a proof of the theorem in his report to the Royal Academy of Denmark. The report was published in the Academy's memoirs in 1799.

In his proof of Cotes' theorem [135, p111] Wessel recognizes the vertices of the regular n-gon as the solutions of the cyclotomic equation $z^n - r^n = 0$. He writes each of the CP_i (Terrot's notation, not Wessel's) in the form ($z-$a root), so that their product is given by $z^n - r^n$. He calls on a previous result to show it is legitimate to take the modulus of both sides.

In Terrot's proof, the product of the CP_i is written as a sum where each term is a product: multiplied together are the product of the coefficients of direction for the OP_i (since $CP_i = OP_i - OC$) and the product of the lengths. In the first term there are n OP_i taken together; in the second, $n-1$ OP_i are taken together, etc. Recognizing the coefficients of direction as roots of $x^n - 1 = 0$ Terrot reasons on the values of the various products of these coefficients: all but the first and the last terms in the sum disappear. He has then only to divide through by the product of the coefficients of direction so as to consider length alone.

4.3.2 Tait's notes on Bishop Terrot's paper

In summarizing the Bishop's paper, Tait copies down only what constitutes essential information: he sifts out and records the key mathematical concepts and results; notably, he chooses to omit the Bishop's references to the Revd. John Warren's earlier work on the subject, to Peacock's *Algebra* and to a paper on symbolical geometry by Sir William Rowan Hamilton published in the *Cambridge and Dublin Mathematical Journal*.[282]

It is interesting to note that Tait's preference is to use mathematical notation as a shorthand where possible, employing the symbols $\because$ for 'because', $\therefore$ for 'therefore' and $\angle$ for 'angle'. He translates Terrot's words into the equivalent mathematical expressions; for instance, from Terrot's 'Let BAC represent a right-angled triangle ...', Tait writes '$BCA = 90°$'.[283] Another point of departure from Terrot's paper is

[282] References to Warren: Terrot explains how he will take Warren's work further and he gives some of Warren's key definitions and propositions. See [126, pp346,348,351]. References to Peacock's *Treatise on Algebra*: Terrot writes that Peacock dealt with coefficients of direction in his treatise; Terrot takes his unordered list of roots of $x^6 = 1$ as one of his examples from the same source; and he remarks that he has derived the trig. formulae for $\sin(A+B)$ and $\cos(A+B)$ in a similar fashion to Peacock. See [126, pp346,350]. Reference to Hamilton: Terrot writes that Hamilton had written something similar to him with respect to the symbolical sum of lines. See [126, p348f].

[283] [126, p349]

that Tait uses Arabic numerals 1, 2, 3, ... to denote his sections, where the Bishop opts for Roman numerals I, II, III, ...

As one would expect, Tait writes his summary in language which is more familiar to him—which he is more comfortable with. Thus 'procure' becomes 'obtain', 'extremity' becomes 'end' and 'algebraic' becomes 'algebraical'. Now and then he shows real flair, when, for instance, he adds in phrases such as 'described on the radius AD'. As he gets further into the Bishop's paper, Tait begins to follow Terrot's wording more closely. This is, of course, the natural tendency when summarizing an extended piece of writing but it is true that the mathematics is more involved later in the paper.

Tait is methodical. He makes neat copies of Terrot's figures and places them near to where they are referenced in the text, unlike Terrot who—presumably because of type-setting constraints—sometimes has them positioned in less sensible places. Tait clearly understands Terrot's paper. Indeed, there are numerous instances when Tait corrects typographical errors which appear in Terrot's printed mathematics, for instance, Terrot has: (i) $\sin(A+B) = A \times \cos B + \cos A \times \sin B$ and (ii) (CD_1^2) which should be $(CD_1)^2$.[284]

4.3.3 Tait's interest in the paper

So why did Tait take a special interest in this particular paper? He copied no other papers into the school-book. In actual fact, it was Cotes' theorem which drew Tait to Bishop Terrot's paper. The evidence for this exists elsewhere in the school-book: within Tait's notes on the section, 'Arithmetic of sines' from the article on algebra in the *Encyclopaedia Britannica*, Tait has added in a cross-reference to Terrot's paper. He writes: 'These are the analytical expressions of the Theorem of Cotes. (See proceeding paper.)' and the paper follows after Tait has finished recording his notes on that particular section. Presumably, Tait's cross-reference was added in at

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[284] [126, pp352,353]

a later date.[285]

So Tait, wanting to learn more about Cotes' theorem, had consulted Bishop Terrot's paper. But how did Tait know that Terrot had published a paper in which he had given a proof of the theorem? There are a number of possible scenarios, one of which is this: that someone, aware of Tait's interest in Cotes' theorem and with knowledge of the Bishop's paper, suggested to Tait that he read the Bishop's paper; perhaps they also furnished him with a copy of the *Transactions*. Tait's uncle, John Ronaldson—though not a Fellow of the R.S.E.—took an active interest in science. Archdeacon John Williams, Rector of the Edinburgh Academy, was a Fellow of the R.S.E.[286] There was also Maxwell's father, John Clerk Maxwell (c.1790–1856).[287] He was a Fellow of the R.S.E. and on a number of occasions he had taken young James along to the R.S.E. meetings.[288] Indeed, it was John who had written out a fair copy of James' paper on ovals so that it could be presented to the Society.[289] Perhaps Tait had mentioned his interest in Cotes' theorem to Maxwell and through Maxwell he learned of the Bishop's paper and got a copy of the *Transactions*.

However events unfolded, this episode is of interest because it is evidence of Tait's early engagement with R.S.E. publications and the *Encyclopaedia Britannica*. Tait, later in life, would become a prolific contributor to the R.S.E.'s *Proceedings* and *Transactions* and the author of a number of articles for the *Encyclopaedia Britannica* on a variety of scientific subjects and on the life and work of Sir William Rowan Hamilton in a biographical piece.

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[285] Visual indications suggest that this is the case: the text appears bolder and some of the lettering goes through a horizontal line which Tait had drawn in to mark the end of the section.

[286] Archdeacon John Williams was elected a Fellow of the R.S.E. on 7 February 1825. He served the Society as Councillor between 1835 and 1838. [29]

[287] John Clerk Maxwell was elected a Fellow of the R.S.E. on 5 February 1821. [29]

[288] According to Campbell and Garnett in [125, p73].

[289] According to Harman in [124, p35f(2)].

4.3.4 On priority: John Warren's influence

Unfortunately, Bishop Terrot has no claim to priority as the first discoverer of the geometrical representation of complex numbers. This honour falls to Caspar Wessel, whose 1797 report, referred to above (pages 97–98), serves to establish his priority. Therein, Wessel has the geometrical representation of complex numbers framed within a wider system of vector analysis. The regrettable fact that Wessel's contribution went unnoticed by the mathematical community for close to a century enabled subsequent workers to rediscover his results and claim for themselves some share in the priority of the discovery.[290] I am thinking of Jean Robert Argand (1768–1822), a Swiss book-keeper resident in Paris, who presented his results in a privately-published pamphlet in 1806; the Revd. John Warren (1796–1852), a Fellow of Jesus College, Cambridge, who published his treatise [136] in 1828; and the German, Johann Carl Friedrich Gauss (1777–1855), who had the geometrical representation of complex numbers explicit for the first time in his 1848 proof of the fundamental theorem of algebra (F.T.A.).[291] In 1843, of course, there came the pivotal discovery of Hamilton's quaternions.[292,293]

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[290] Wessel's work was rediscovered ninety-eight years later and republished in French in 1897, on its one-hundredth anniversary, at the request of the Danish Academy.

[291] Gauss invented the term "complex numbers". He was also the first to use the letter i to represent $\sqrt{-1}$; and the first to represent a complex number in the form $a + ib$, in *Theoria residuorum biquadraticorum* (1831).

[292] In fact, even as early as 1833 Hamilton had, in effect, removed the conceptual difficulties surrounding imaginaries by expressing complex numbers $a+ib$ as algebraic couples or ordered pairs of real numbers (a, b).

[293] For more information on the historical developments which preceded the discovery of quaternions, and on the adoption of the modern system of vector analysis, see [137]: a timeline of the developments is given on pages 256–259; the contributions of Wessel, Buée, Argand, Mourey, Warren and Gauss are discussed on pages 5–11.

John Warren's influence on Terrot

For his *Treatise on the Geometrical Representation of the Square Roots of Negative Quantities* (1828), the Revd. John Warren is regarded as the English source of the Argand diagram. Warren was, like Terrot, Cambridge educated; admitted to Jesus College in 1814 and graduating (B.A.) as fifth Wrangler in 1818 (M.A. in 1821). He remained, until 1829, a Fellow and tutor of the College and during the period 1825–1826 he functioned as moderator and examiner. As the son of the Dean of Bangor, perhaps it was inevitable that Warren would be ordained, firstly as a deacon in 1819 and as a priest the following year. He served the communities of Caldecott, Huntingdonshire between 1822 and 1852, and Graveley, Cambridgeshire between 1828 and 1852. Warren married in 1835 but had no children. He was elected a Fellow of the Royal Society of London in 1830.[294]

Comparing Terrot's paper with Warren's 1828 treatise, it is clear that Terrot's work was heavily influenced by Warren: chapter 1 of Warren's treatise provided Terrot with the fundamentals of his theory; and in chapter 2, Terrot was given all the tools he needed for his work on nth roots.[295] For specific examples of Terrot's use of material from Warren's treatise see Table 4.1 (page 104). Tait made no mention of Warren in his notes but Terrot did readily acknowledge Warren's contribution. In his paper Terrot stated that his role was to explain Warren's results in greater detail and explore new applications of Warren's theory. He writes:

> On some points, however, Mr Warren has been too sparing of his words, and has thus apparently used the common symbols of algebra in a sense very different from their ordinary acceptation. In the following paper I have endeavoured to supply this deficiency of explanation; and then to apply the system of symbols so established to some important problems of goniometry to which, as far as I know, it has not yet been applied.[296]

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[294] Sources of biographical information on Warren: [138] and [13].

[295] Chapters 3 and 4 of Warren's 1828 treatise bear little resemblance to the material covered in Terrot's paper.

[296] [126, p346]

So while Warren should be credited with having established the fundamentals, the Bishop surely deserves some recognition: he seems to have had a sound grasp of the mathematics involved, which he was able to apply in valid and novel ways, and it is through his paper that Tait had his first introduction to a relatively new mathematical discovery of huge import. Certainly, the R.S.E. held Terrot's mathematical researches in this area in the highest regard:

> The subject [...] had been floating somewhat dimly before the eyes of mathematicians for half a century, and was just then beginning to assume a living form in the mind, and a living exponent, though a somewhat obscure one, in the writings of Sir W. R. Hamilton. It was not until six year later that the doctrine of Quaternions of the great master, as developed in his "Lectures," swallowed up in its vast amplitude all that had preceded it. Terrot accordingly must be considered as one of the pioneers of the science.[297]

The Revd. Walker believed that Terrot might have had a remarkable career in mathematics had he chosen to commit himself to it exclusively:

> had he [Terrot] cared to devote himself to "research," living chiefly on his fellowship, he might have made important discoveries in some branches of the higher mathematics, probably anticipating Sir William Rowan Hamilton in his discovery of Quaternions. But he had other views; and doubtless he took the wiser course.[298]

[297] [130, pp11–12]

[298] [132, p102]

Chapter 1 of Warren's 1828 treatise	
Arts. 1, 2	A line has both direction and length. It is drawn in the plane, anchored to the origin, inclined at an angle.
Art. 3	The sum of two lines is the diagonal of their parallelogram.
Art. 4	The subtraction of lines is the reverse process of addition.
Art. 8	Positive and negative lines are opposite in direction.
Arts. 51, 52	Two propositions relating to the addition of angles when multiplying lines together.
Chapter 2 of Warren's 1828 treatise	
Art. 54	A definition of the nth root of a quantity.
Art. 58 corr.	The nth roots of a quantity are co-equal in length.
Art. 61	There are n nth roots of a quantity.
Arts. 105–107, 110	A method of calculating, and representing geometrically, the 4th roots of unity.
Arts. 109, 113	A proof that 'Any quantity may be expressed in the form $\pm a \pm b\sqrt{-1}$ where a and b are positive quantities'; with an example, expressing the 6th roots of unity in this form.
Art. 119	The length of a complex number given by $a = \pm b \pm c\sqrt{-1}$ is $\sqrt{b^2 + c^2}$. [Used in Terrot's proof of Pythagoras' theorem.]

Table 4.1: Examples of Terrot's use of material from Warren's 1828 treatise [136].

4.4 Associated historical insights

4.4.1 The discovery of quaternions

As well as being a source of inspiration for the Bishop's mathematical researches, John Warren's treatise of 1828 was of influence to Sir William Rowan Hamilton in his discovery of quaternions.

Hamilton wrote to the editors of the *Philosophical Magazine* on 20 November 1844. He enclosed a copy of a letter he had sent to the mathematician and jurist, John T. Graves (1806–1870) in October 1843.[299] In this letter to Graves, Hamilton had reported a break-through in the theory of quaternions and given an account of the thought processes which had led him to the discovery. Some of the letter had been published in the July and October editions of the magazine; but Hamilton hoped publication in full would provide an opportunity to publicly acknowledge John Warren's contribution and might encourage Graves—with whom Hamilton had been in fruitful correspondence for many years—to make public how he had been able to extend Hamilton's work and bring to light his own results.

Hamilton and Graves had maintained a close friendship from the time they began their studies together at Trinity College, Dublin in 1823:

> For many years Graves had been Hamilton's sympathetic friend and mathematical confidant, and the two men maintained an active correspondence, in which they competed with each other in their attempts to produce a full and coherent interpretation of imaginaries. Graves worked at perfecting algebraic language; Hamilton had the higher object of arriving at the meaning of the science and its operations.[300]

It was Graves' work on imaginary logarithms that eventually led Hamilton to his discovery of quaternions: from conjugate functions to the theory of triplets; onto the sets of moments, steps and numbers; all building to the discovery of quaternions. This source of inspiration was acknowledged by Hamilton but Graves 'modestly

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[299] John T. Graves was the brother of R. P. Graves, Hamilton's chief biographer.

[300] [139]

disclaimed the credit of suggestion'.[301]

Hamilton's letter to the editors ran as follows:[302]

To the Editors of the Philosophical Magazine and Journal

Gentlemen,

I have been induced to think that the account contained in the following letter [H to Graves (17 Oct. 1843)], of the considerations which led me to conceive that theory of quaternions, a part of which you have done me the honour to publish in two recent Numbers (for July and October) of your Magazine, might not be without interest to some of your readers. Should you think proper to insert it, a public acknowledgement (very pleasing to my own feelings) will have been rendered, on the one hand to the Rev. Mr. Warren, whose work on the Geometrical Representation of the Square Roots of Negative Quantities (printed at Cambridge in 1828) long since attracted my attention and influenced my thoughts; and on the other hand to the gentleman (John T. Graves, Esq.) to whom the letter was addressed, and with whom I had been engaged, at intervals, for many years in a correspondence, very instructive and suggestive to me, on subjects connected therewith. Nor am I without hope that Mr. Graves may thus be led to communicate through you to mathematicians some of the extensions which he has made of results of mine, with some of those other speculations which are still more fully his own. On some future occasion I may perhaps be allowed to mention any other quarters from which I may be conscious of having derived more recent assistance, in my investigations on the same mathematical subject, many of which are hitherto unpublished.

I have the honour to be, Gentlemen,
Your obedient Servant,
William Rowan Hamilton

Observatory of Trinity College, Dublin,
November 20, 1844

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[301] [Ibid.]

[302] H to *Phil. Mag.* (20 Nov. 1844) in [140, pp489–490].

Clearly, Hamilton owed a debt to Warren. In fact, Hamilton had not known about the geometrical representation of complex numbers until 1829 when Graves had encouraged him to read Warren's treatise.[303] Hamilton generously acknowledged his debt to Warren in the preface to his *Lectures on Quaternions*, writing: 'To suggestions from that Treatise [(Warren 1828)] I gladly acknowledge myself to have been indebted, although the interpretation of the symbol $\sqrt{-1}$, employed in it, is entirely distinct from that which I have since come to adopt in the geometrical applications of the quaternions.'[304]

Hamilton's letter to Graves is of great value as it paints a very honest picture of the road to mathematical discovery.[305] Given our present focus, it is highly significant that therein Hamilton explained his investigations had sprung from geometrical considerations. The letter began:

> Observatory, October 17, 1843
>
> My Dear Graves,—A very curious train of mathematical speculation occurred to me yesterday, which I cannot but hope will prove of interest to you. You know that I have long wished, and I believe that you have felt the same desire, to possess a Theory of Triplets, analogous to my published Theory of Couplets, and also to Mr. Warren's geometrical representation of imaginary quantities. Now I think that I discovered yesterday a *theory of quaternions* which includes such a theory of *triplets.*
>
> My train of thought was of this kind. Since $\sqrt{-1}$ is in a certain well-known sense, a line perpendicular to the line 1, it seemed natural that there should be some other imaginary to express a line perpendicular to the former; and because the rotation from 1 to this also being doubled conducts to -1, it also ought to be a square root of negative unity, though not to be confounded with the former. Calling the old root, as the Germans often do, i, and the new one j, I inquired what laws ought to be assumed for multiplying together $a + ib + jc$ and $x + iy + jz$.[306]

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[303] [19, p262] Also page 191 of this thesis.

[304] [141, p(31)f]

[305] For more on Hamilton's discovery of quaternions see the largely historical preface to [141].

[306] H to Graves (17 Oct. 1843) in [140, p490].

The break-throughs which led Hamilton to the now familiar algebraic relations $i^2 = j^2 = k^2 = ijk = -1$ are worth noting. They were to come out of his reasoning on the ij that appeared in the product $(a+ib+jc)\cdot(x+iy+jz)$. Hamilton asked: 'but what are we to do with ij?'[307]

He began by considering the simplest case of a product, a square, and noted that the *law of multiplication of moduli* holds if we ignore the term in ij altogether.[308] Thus if

$$(a+ib+jc)\cdot(a+ib+jc) = [a^2-b^2-c^2] + [2ab]i + [2ac]j + \cancel{[2bc]ij}$$

then

$$(a^2+b^2+c^2)^2 = [a^2-b^2-c^2]^2 + [2ab]^2 + [2ac]^2$$

But to Hamilton it seemed 'odd and uncomfortable' to set $ij = 0$.[309] Instead, he set $ij = -ji = k$, with the value of k (zero or otherwise) still to be determined. This assumption achieved the same desired effect—the suppression of the ij term—but seemed 'less harsh' to Hamilton than requiring $ij = 0$.[310]

In order to find k, he considered the law of multiplication of moduli again, this time for the triplets $(a+ib+jc)$ and $(x+iy+jz)$; imagining that the term in k might, once again, require suppression for the law to hold. Thus if

$$(a+ib+jc)\cdot(x+iy+jz) = [ax-by-cz] + [ay+bx]i + [az+cx]j + \cancel{[bz-cy]k}$$

then ... but this time

$$(a^2+b^2+c^2)(x^2+y^2+z^2) \neq [ax-by-cz]^2 + [ay+bx]^2 + [az+cx]^2$$

as the right hand side is too small by $[bz-cy]^2$, which is the square of the coefficient of k. Seeing this, Hamilton realized that k cannot be written as a linear combination

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[307] [Ibid.]

[308] *Law of multiplication of moduli*: if $(a_1, a_2, a_3)\cdot(b_1, b_2, b_3) = (c_1, c_2, c_3)$ then $({a_1}^2 + {a_2}^2 + {a_3}^2)\cdot({b_1}^2 + {b_2}^2 + {b_3}^2) = ({c_1}^2 + {c_2}^2 + {c_3}^2)$.

[309] H to Graves (17 Oct. 1843) in [140, p491].

[310] [Ibid.]

of i and j, and that in order to work successfully with triplets he would need to work in four dimensions:

> And here there dawned on me the notion that we must admit, in some sense, *a fourth dimension* of space for the purpose of calculating with triplets; or transferring the paradox to algebra, must admit a *third* distinct imaginary symbol k, not to be confounded with either i or j, but equal to the product of the first as multiplier, and the second as multiplicand; and therefore was led to introduce *quaternions*, such as $a + ib + jc + kd$, or (a, b, c, d).[311]

The fact that Hamilton was open to the possibility of non-commutativity is remarkable. In not conforming absolutely to all the laws of algebra formerly thought inviolable, Hamilton had given himself room to manoeuvre and consequently made his discovery.

4.4.2 Warren's reference to Buée and Mourey

Warren followed his 1828 treatise with two related papers: in the first [142] he countered objections raised against the geometrical representation of imaginaries and in the second [143] he extended the treatise. It is in the first of these follow-up papers that Warren brought to light two contributions to the geometrical representation of complex numbers: Adrien-Quentin Buée's 1806 paper [144] and C.-V. Mourey's 1828 work [145].[312] Warren stated—for the sake of priority—that he became aware of these contributions only after his own treatise was written: Buée's in November 1827 and Mourey's in December 1828. Biographical information on l'abbé Buée and further information on his contribution is given in Appendix E, where Gergonne's 1811 conception of a two-dimensional table of real and imaginary magnitudes is also discussed. Mourey and his mathematics are discussed at length in Chapter 5.

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[311]H to Graves (17 Oct. 1843) in [140, pp491–492].

[312]See [142, pp251–254].

4.4.3 Tait's account of the developments

Some forty years after making his notes on the Bishop's paper, Tait—in an article entitled, 'Quaternions' in the ninth edition of the *Encyclopaedia Britannica*—gave a typically thorough, yet concise, history of developments in the geometrical representation of $\sqrt{-1}$ in the lead-up to the discovery of quaternions. His account is confined to the contributions of those whose interpretations had 'geometrical applications in view', as Hamilton's had.[313] Thus, his chronicle excludes anything that might be termed "the calculus of complex numbers". According to Tait, Hamilton 'was led to his great invention by keeping geometrical applications constantly before him while he endeavoured to give a real significance to $\sqrt{-1}$ '.[314] Tait recognizes the contributions of Wallis, Buée, Argand, Français, Warren, Mourey and, of course, Hamilton, who achieved that which the others could not—the ability to work in three dimensions. Note, Wessel's 1799 contribution is missing from Tait's account.

4.5 Coming full circle

With Tait's chronicle our record of these associations is complete. We began with Bishop Terrot's paper introducing Tait to the geometrical representation of complex numbers, whilst Tait was still a schoolboy at the Edinburgh Academy; the Bishop had been inspired by Warren, as had Hamilton, and it would be Hamilton's quaternions which would captivate Tait and inspire a lifetime of mathematical research.

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[313] [146, p445]

[314] [Ibid.]

CHAPTER 5

C.-V. MOUREY'S SINGLE SCIENCE OF ALGEBRA AND GEOMETRY

This chapter is on C.-V. Mourey's 1828 work, *La Vraie Théorie des quantités négatives et des quantités prétendues imaginaires.*

5.1 Introduction

In 1828 C.-V. Mourey shared his results relating to the difficulties presented by the theory of algebra, in a book published in Paris under the title, *La Vraie Théorie des quantités négatives et des quantités prétendues imaginaires.* Seeking algebraic reform, Mourey had set out to discover a brand new set of definitions and fundamental principles as a basis for algebra. To this end, he developed a theory of directed lines, which constituted a single science of algebra and geometry; and, as an application of the theory, he gave a proof of the fundamental theorem of algebra (F.T.A.).

In this chapter I reconsider Mourey's motivations, re-evaluate his mathematics and present new biographical information on Mourey who has remained an unknown to historians of mathematics for the past 186 years. There will also be a survey of the reception of Mourey's work, a highlight of which is the ten-year correspondence on Mourey between Hamilton and De Morgan.

Language is an obvious barrier to understanding so, where appropriate, I have provided bi-lingual quotations from Mourey's text. The original French appears in the footnotes. Currently, there is no English translation of the work.

A second and significant barrier to understanding is the over-abundance of Mourey's original terms and notations. Their sheer number confuses and distracts the reader and obscures the development of Mourey's approach. So, as a further aid to the reader, I have produced a look-up table in Appendix G which provides a summary of Mourey's original terms and notations. Its entries are given in the order in which they appear in Mourey (1861) so that it might be used as a companion-guide to the text. In itself the table provides a good summary of Mourey's approach. In the commentary occasional reference is made to terms in the table. These references are given within square brackets in the footnotes, e.g. [See *angle directif.*].

5.2 C.-V. Mourey: a biographical enigma

On the title page of Mourey's 1828 first edition (Figure 5.1, page 114) the author's address is given as 'Paris [...] rue des Quatre-Vents, no. 8'. This address constitutes the only biographical information on Mourey currently available.[315] In 1861 a reprint of the book was produced by the same publishers, Bachelier.[316] In this later edition no address is given for the author, which suggests that by 1861 Mourey was deceased.[317] I assume that Bachelier published the reprint at their own expense in response to demand for the first edition.

As early as 1846, an appeal for biographical information on Mourey was published in *Nouvelles annales de mathématiques* [147]. The editors write that copies of Mourey (1828) have become extremely rare and that to their knowledge, amongst mathematicians in Paris, only Lefébure de Fourcy (1787–1869) has a copy.[318] They ask that those with biographical information on Mourey make contact. Since there was no published response to the appeal—notably, Mourey himself failed to make contact with the editors—it seems likely that Mourey had left Paris, or died, a short time after the publication of his book.

It seems that Mourey was truly an unknown in Paris' academic circles. Cer-

[315]I have found no other author who has made reference to this address—perhaps owing to the rarity of the first edition.

[316]By then Mallet–Bachelier.

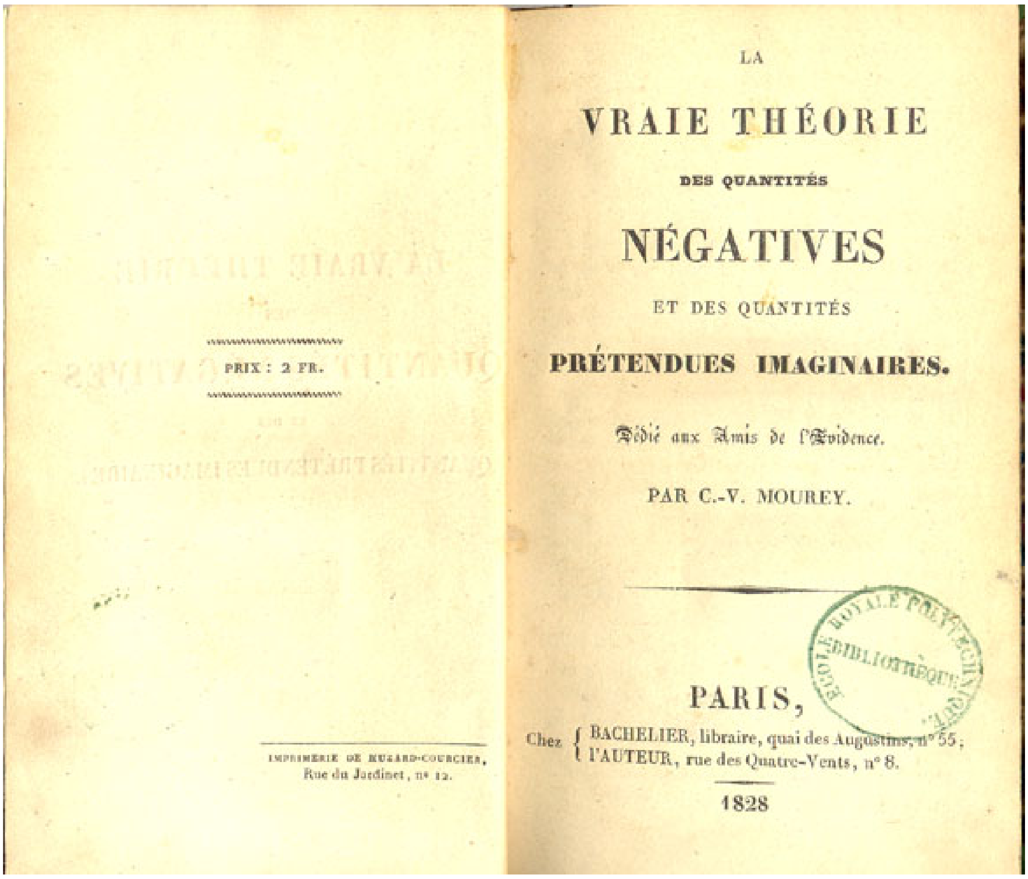
[317]It appears that no other additions or amendments were made to this second edition; notably, typographical errors, which presumably were in the first edition, remain uncorrected.

[318]At that time Lefébure was Examiner for Admissions at the École Polytechnique and Chair of Differential and Integral Calculus at the Faculté de Sciences.

tainly, there is no indication, in either edition of his work, of his affiliation with any academic or scientific institution. He is not remembered as a student at any of the well known educational establishments in the city and has never been referred to in connection with the great mathematicians who were known to have been in Paris during that period; thinking of the young talents of Abel and Galois, but also of Cauchy, Poisson, Legendre, Hachette, Dirichlet, Fourier and Lacroix. While the work is scholarly, it was not written for the purposes of instruction, which supports the theory that Mourey held no fixed teaching position in mathematics.

The text itself offers no information on its author; notably, it is without reference to Mourey's mathematical influences and in this respect it is similar to the work of the great Greek authors. The most probable reason for the absence of references is that Mourey simply had no instinct to cite other authors. Consequently, we have no information on which mathematics books Mourey had read. Likewise, we have no information on the nature of Mourey's mathematical training. Presumably, Mourey did receive some training in mathematics: the strong focus on trigonometry and mechanics in the book suggests that he did receive some training in the practical application of mathematics.[319]

[319]See page 138 of this thesis for Mourey's references to trigonometry and mechanics.



PRIX : 2 FR.

IMPRIMERIE DE HUZARD-COURCIER,
Rue du Jardinet, n° 12.

LA

VRAIE THÉORIE

DES QUANTITÉS

NÉGATIVES

ET DES QUANTITÉS

PRÉTENDUES IMAGINAIRES.

Dédié aux Amis de l'Évidence.

PAR C.-V. MOUREY.

PARIS,

Chez BACHELIER, libraire, quai des Augustins, n° 55;
l'AUTEUR, rue des Quatre-Vents, n° 8.

1828

Figure 5.1: Title page of the first edition of Mourey. Reproduced with the kind permission of Collections École Polytechnique, Palaiseau (France). The author's address is given as 'Paris [. . .] rue des Quatre-Vents, no. 8'. This first edition copy is now in very poor condition.

5.2.1 Self-funded publication

From the preface of his book, we learn that Mourey's 1828 publication was in fact an abridgement of a larger manuscript which Mourey had not been able to publish in full because of certain undisclosed circumstances. Mourey writes:

> Until now I have only dealt, as you understand, with the fundamental principles, and still I have written quite a considerable manuscript. However, as circumstances do not allow me now to have such a voluminous work printed, I have decided to publish this booklet first, which is but a small abstract.[320]

From this I infer that Mourey had published at his own expense and that he had restricted funds. The theory of a self-funded publication is also supported by the lack of evidence to suggest that Mourey's work was subject to peer review prior to publication. But if finances were an issue for Mourey, why did he not submit his work to be published in a journal, such as: Gergonne's *Annales*, *Memoire de l'Institut de France* or Ferrusac's *Bulletin*? There are a number of possible reasons.

From indications given by Mourey in his preface, it appears that the work published in 1828—his only publication—was the culmination of many years of private study: his mathematical researches had remained a private project until such a time when he considered his ideas to be sufficiently developed to share with others or until he had acquired the means to publish. Consequently, the impact of his work would have been a key consideration for Mourey. Had he published his results in a journal, he would have had to divide his work into a number of small contributions, on account of the amount of material, which might have had a negative bearing on impact.

Mourey may have also faced difficulties in getting his work published, as an unknown in academic circles with no-one to recommend him for publication. Mourey

[320]'Jusqu'ici je n'ai pu m'occuper, comme on le pense bien, que des principes fondamentaux, et cependant j'ai composé un manuscrit assez considérable. Mais, les circonstances ne me permettant pas de faire imprimer actuellement un ouvrage aussi volumineux, j'ai pris le parti de publier d'abord cet opuscule, qui n'en est qu'un faible abrégé.' [145, pix]

probably considered self-funded publication to be the quickest and easiest route to getting his work noticed by the mathematical community. Mourey dedicated his work to the 'Amis de l'Evidence' (or 'the friends of evidence'), which was probably not an actual group but a motto of Mourey's: he dedicated his work to all those who, like him, search for truth. See Figure 5.1 (page 114) for the dedication.

5.2.2 Mourey's identity

Taking everything into account, it seems reasonable to state the following null hypothesis regarding Mourey's identity:

Mourey was not a professional academic but a first-class amateur mathematician whose trade involved the practical application of mathematics.

And I have found a candidate who fits this profile perfectly:

Claude-Victor Mourey, mécanicien à Paris

Claude-Victor Mourey, *mécanicien à Paris*, held five-year patents for two inventions: a timber-profiling machine and a tree saw. The patents were granted on 12 July 1822 and 3 August 1822 respectively.[321] Two files containing documents relating to Mourey's patents are extant in the archives of the Institut National de la Propriété Industrielle.[322] Both files contain: a detailed specification of the invention and technical drawings (Figures 5.3–5.5, pages 120–122) which were submitted by Mourey; and the correspondence between Mourey and the administration, the Comité consultatif (advisory committee) des Arts et Manufactures. These documents are all handwritten. Mourey signs and dates the documents and gives his address. He signs himself 'Victor Mourey. Mécanicien' (Figure 5.2, page 119). The address he gives on 6 May 1822 is rue Saint-Honoré, no. 130, Seine, Paris. On 21 May, a

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[321] Mourey's patent for his tree saw lapsed in November 1824, possibly because the maintenance payments had not been kept up.

[322] I.N.P.I. file nos. 1BA1681 and 1BA1691. Available to view online: see [148] and [149].

different address: rue Saint-Maur, no. 84, Faubourg du Temple, Seine, Paris.[323] In some published lists of granted patents, Mourey's full name is used: 'Mourey (Claude-Victor), mécanicien à Paris'.[324]

MM. Hacks and Co.

The reports of the advisory committee, for both machines, indicate that before Mourey had submitted his patent applications, another *mécanicien* resident in Paris, M. Jean-Pierre Hacks, had designed and built the same or similar machines and had shown them to several members of the Société d'Encouragement pour l'Industrie Nationale. Hacks' own timber-profiling machine was given a favourable mention in the Society's general meeting on 3 October 1821. His tree saw was presented to the Society on 28 March 1822; was given a favourable report in a meeting on 15 May and featured in the June issue of the Society's *Bulletin*.[325] Hacks submitted a patent application for his timber-profiling machine on 17 April 1823 and a five-year patent was granted on 22 May.[326]

From indications given in a Paris directory of the period, it appears that Jean-Pierre Hacks was the proprietor of MM. Hacks et compagnie—manufacturers of machines to saw timber, trees etc., who operated out of a workshop on the grande rue du Faubourg Saint-Antoine, no. 47.[327] In July 1823 MM. Hacks et compagnie

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[323] See [150] for an 1830 map of Paris by Girard.

[324] For instance: [151] and [152, p209].

[325] For the feature on Hacks' saw in the Society's *Bulletin*, see [153] and [154]: the report by M. Molard (a member of the advisory committee) was approved in the meeting of the Society on 15 May 1822.

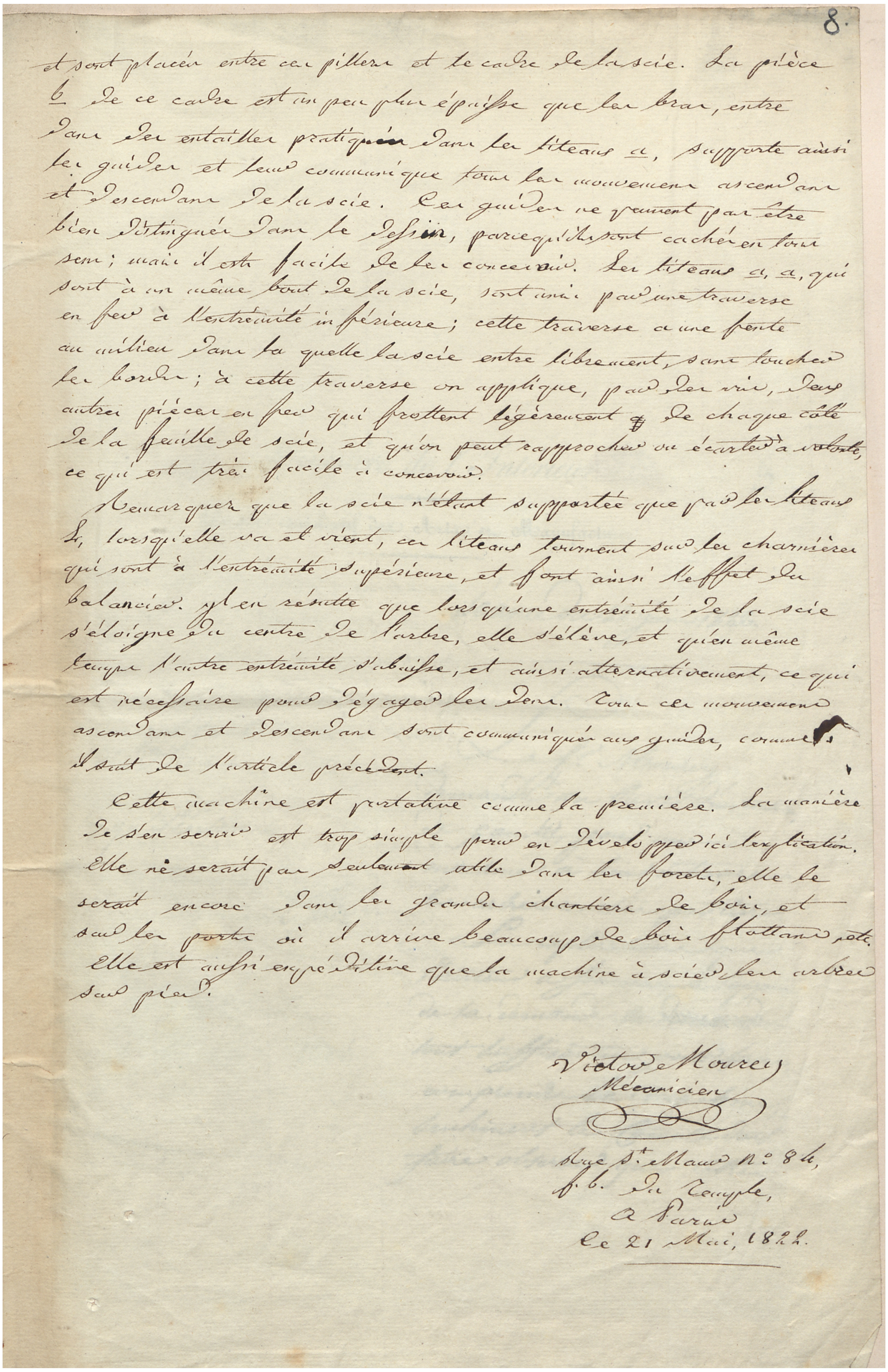
[326] I.N.P.I. file no. 1BA1799. Available to view online: see [155]. The documents relating to Hacks' patent application contain no reference to Mourey, however, the committee report is missing from the file.

[327] The directory featured top artists and manufacturers in Paris. See [156, pp259–260] for Hacks' entry in the directory, which includes a description of a number of Hacks' machines, including his saw for standing trees. Note, Mourey is not listed in the directory.

won a silver medal for mechanically-produced wooden mouldings at an exhibition of French products of industry that was held at the Louvre Palace.[328]

This series of coincidences strongly suggests an association between Mourey and Hacks. I suggest the possibility that Mourey was employed by Hacks and Co. as a draughtsman and that the designs for both machines were originally his. I have found no evidence to suggest that Mourey had his own workshop.

[328] See [157, p419] for the jury's report on Hacks' entry at the exhibition.


8.

et sont placés entre ces piliers et le cadre de la scie. La pièce b de ce cadre est un peu plus épaisse que les bras, entre dans des entailles pratiquées dans les liteaux a, supporte ainsi les guides et leur communique tous les mouvemens ascendans et descendans de la scie. Ces guides ne peuvent pas être bien distingués dans le dessin, parcequ'ils sont cachés en tous sens; mais il est facile de les concevoir. Les liteaux a, a, qui sont à un même bout de la scie, sont unis par une traverse en fer à l'extrémité inférieure; cette traverse a une fente au milieu dans laquelle la scie entre librement, sans toucher les bords; à cette traverse on applique, par des vis, deux autres pièces en fer qui frottent légèrement de chaque côté de la feuille de scie, et qu'on peut rapprocher ou écarter à volonté, ce qui est très facile à concevoir.

Remarquez que la scie n'étant supportée que par les liteaux a, lorsqu'elle va et vient, ces liteaux tournent sur les charnières qui sont à l'extrémité supérieure, et font ainsi l'effet du balancier. Il en résulte que lorsqu'une extrémité de la scie s'éloigne du centre de l'arbre, elle s'élève, et qu'en même temps l'autre extrémité s'abaisse, et ainsi alternativement, ce qui est nécessaire pour dégager les dents. Tous ces mouvemens ascendans et descendans sont communiqués aux guides, comme il suit de l'article précédent.

Cette machine est portative comme la première. La manière de s'en servir est trop simple pour en développer ici l'explication. Elle ne serait pas seulement utile dans les forêts, elle le serait encore dans les grands chantiers de bois, et sur les ports où il arrive beaucoup de bois flottant, etc. Elle est aussi expéditive que la machine à scier les arbres sur pied.

Victor Mourey
Mécanicien

Rue St. Maur N° 84,
f.b. du Temple,
à Paris
Le 21 Mai, 1822.

Figure 5.2: Mourey's signature: 'Victor Mourey. Mécanicien. Rue St. Maur no. 84 f.b. [Faubourg] du Temple, à Paris, le 21 Mai, 1822.' Reproduced from the original patent application for his tree saw. Source: Archives I.N.P.I.

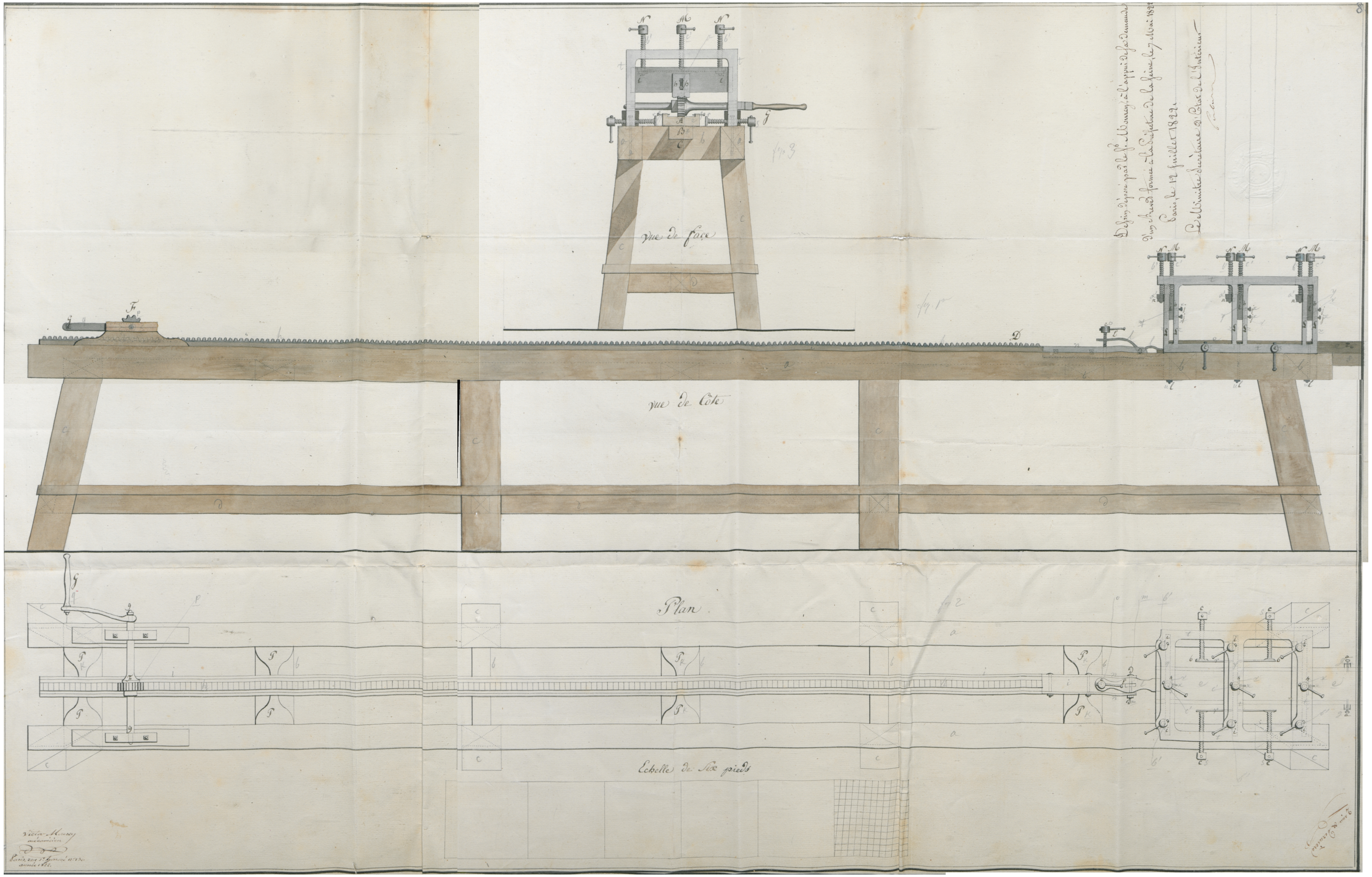


Figure 5.3: Mourey's technical drawings for his timber-profiling machine. Source: Archives I.N.P.I. It operates by feeding timber through the device, while the profile is cut into the wood by three height-adjustable iron cutters.

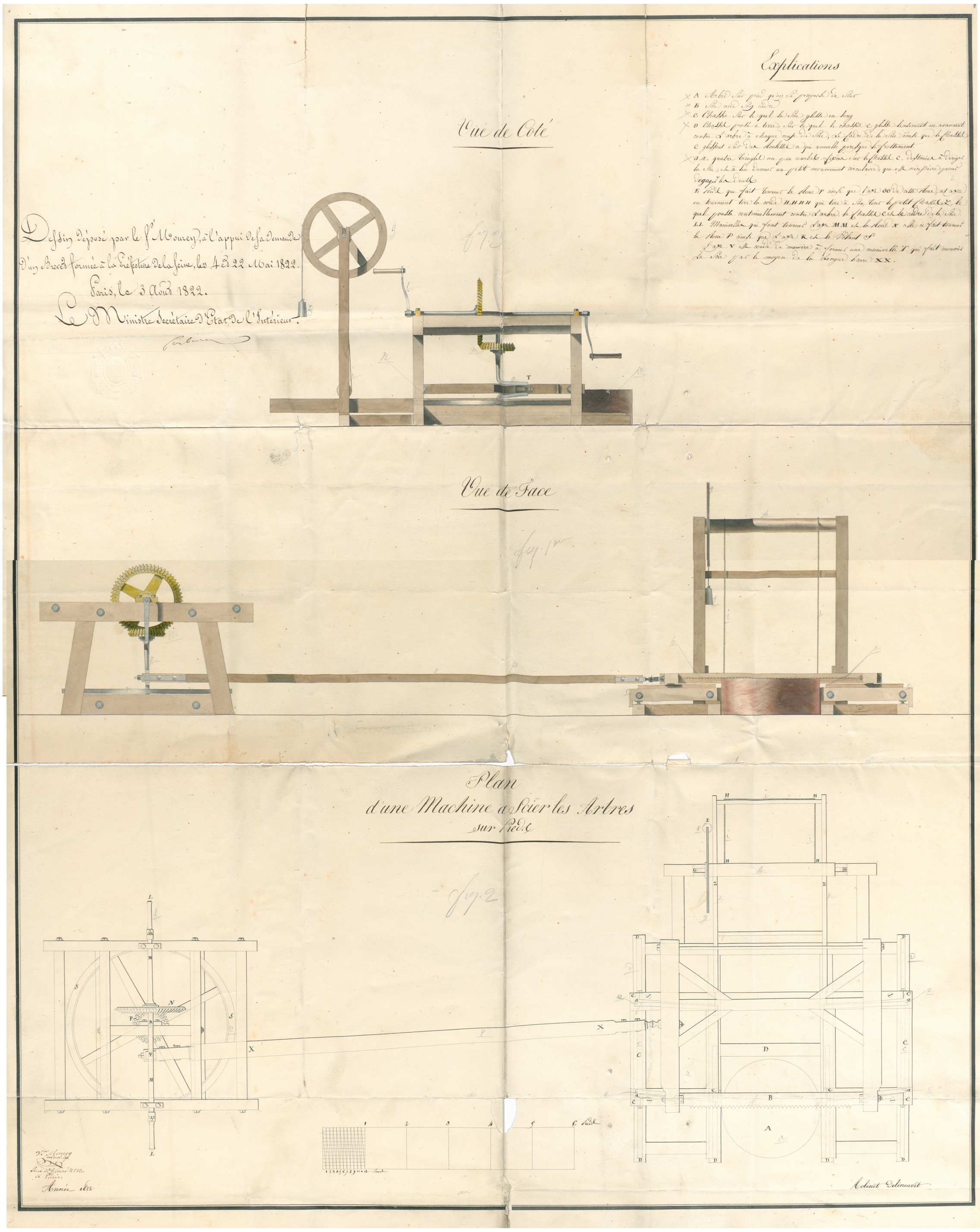


Figure 5.4: Technical drawings for Mourey's machine to saw standing trees. Source: Archives I.N.P.I.

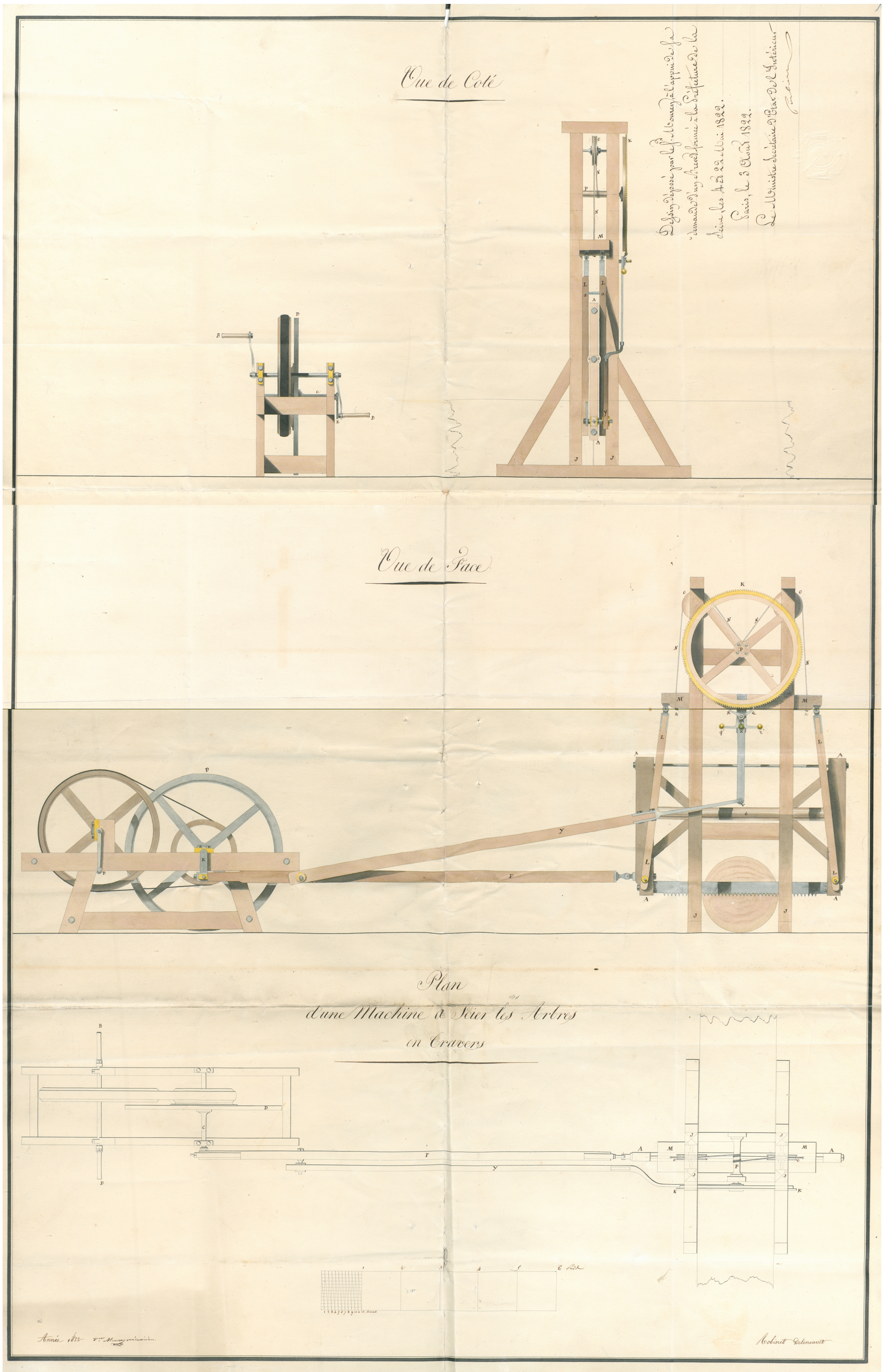


Figure 5.5: Technical drawings for Mourey's machine to saw felled trees, adapted from his designs for his machine to saw standing trees. Source: Archives I.N.P.I.

Mourey's Paris addresses

Looking to prove that Mourey the *mécanicien* was the author of the mathematics book in 1828, the only way forward was to follow up on the three addresses: from the title page of the first edition and the two addresses given in the patent applications. Fortunately, ledger volumes are extant in the Archives de Paris that contain information on property owners in Paris (their name, occupation etc.), together with property valuations and details of when properties changed hands and for what reason (e.g. through sale or succession).[329] The search for Mourey, through his Paris addresses, was made more difficult by the fact that the house number for each of Mourey's addresses had changed twice: each property has an old, current and new number. Unfortunately, following up on the address drew a blank: I found no reference at all to Mourey; however, the search did rule out the possibility that the address given in the first edition was a holding address for the publisher.

It is hard to believe that there is no information on record about Mourey. However, if Mourey was not born in Paris, for instance, then there would be no record of his birth in the Paris archives; and if he did not have the means to afford to purchase his own property but instead rented rooms as a private tenant—as is suggested by the fact that he moved to a number of different addresses—he would not appear in the records of property owners in Paris. In any case, there would be little chance of finding any official record of Mourey because almost all the civil records for Paris prior to 1860 were destroyed by a fire at the Hôtel de Ville (where the paper records were stored) in May 1871 during the Paris Commune uprising: between five and eight million records were destroyed.

Claude Mourey (1791–1830)

There are, however, two records extant in the Paris archives for a *Claude Mourey* (note the absence of a middle name): an *acte de décès* (death certificate) and an

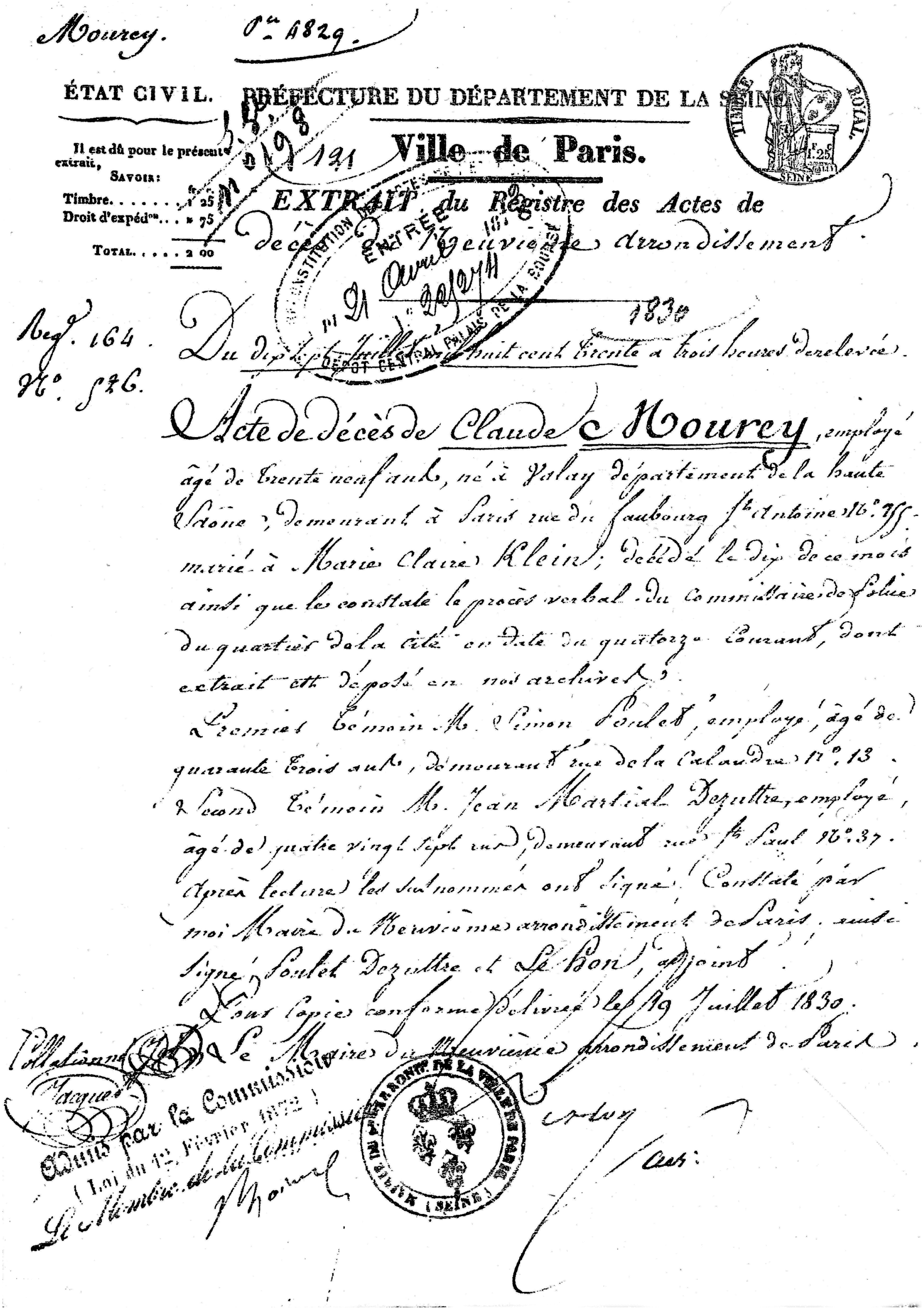
[329] One or two streets would normally be in each volume and each building would usually comprise around ten apartments.

acte de mariage (marriage certificate).[330] According to his death certificate (Figure 5.6, page 126) Claude Mourey was born in the Valay department in the Haute-Saône region in the east of France but was resident in Paris, at the time of his death, at rue du Faubourg, Saint-Antoine, no. 255; he was married to Marie Claire Klein and he died in Paris on 10 July 1830, aged thirty-nine. The term 'employée', which appears on his death certificate, indicates that Claude Mourey was not a gentleman; that is, he did not have the means to support himself without engaging in some sort of trade. The marriage certificates for Claude and his wife Marie Claire provide further information: the couple were married in Paris, in the parish of St Marguerite on 21 October 1829; Claude was the son of Claude Joseph Mourey and Anne Françoise Fontaine; and Marie Claire was the daughter of Henry Klein and Marie Françoise Grégorie.

It is quite possible that our three Moureys are in fact one and the same individual. Certainly, I have found no evidence which rules out the possibility: the chronology makes sense and the various indications given of Mourey's occupation, style of living etc., are all consistent. If indeed they are the same person, then assimilating all the biographical information available gives the overview of his life on page 125; but we move on now to consider Mourey's mathematics.

[330] Found amongst the microfilmed copies of the reconstituted civil records for the period between the C16th and 1859.

Claude-Victor Mourey (1791–1830), *mécanicien à Paris*, was born in the Valay department in the Haute-Saône region in the east of France; to parents Claude Joseph Mourey and Anne Françoise Fontaine. In 1822, having moved to Paris, he took out five-year patents for two machines he had invented: a timber-profiling machine and a tree saw. He may have been employed as a draughtsman by MM. Hacks et compagnie, whose workshops were on the grand rue du Faubourg, Saint-Antoine, no. 47. In 1828, aged thirty-seven, Mourey had made sufficient money from his inventions to enable him to publish a mathematics book entitled, *La Vraie Théorie des quantités négatives et des quantités prétendues imaginaires*: he was a first-class amateur mathematician. In Paris, on 21 October the following year, he married Marie Claire Klein, daughter of Henry Klein and Marie Françoise Grégorie. He died in Paris on 30 July 1830, aged thirty-nine; just two years after his book was published and after only nine months of marriage. By 1846 there were very few copies of his work in Paris and his identity, as its author, had fallen into obscurity.



Mourey. 8e 1829.

ÉTAT CIVIL. PRÉFECTURE DU DÉPARTEMENT DE LA SEINE.

Il est dû pour le présent extrait, Savoir:

Timbre .... f. 1 25
Droit d'expédon.... » 75
Total 2 00

Ville de Paris.

EXTRAIT du Registre des Actes de décès du Neuvième arrondissement.

Reg. 164. No. 576.

Du dix sept juillet mil huit cent trente à trois heures de relevée.

Acte de décès de Claude Mourey, employé âgé de trente neuf ans, né à Valay département de la haute Saône, demeurant à Paris rue du faubourg St Antoine No. 75, marié à Marie Claire Klein; décédé le dix de ce mois ainsi que le constate le procès verbal du Commissaire de police du quartier de la Cité en date du quatorze courant, dont extrait est déposé en nos archives.

Premier témoin M. Simon Poulet, employé, âgé de quarante trois ans, demeurant rue de la Calandre No. 13.

Second témoin M. Jean Martial Deputtre, employé, âgé de quatre vingt sept ans, demeurant rue St Paul No. 37.

Après lecture les susnommés ont signé. Constaté par moi Maire du Neuvième arrondissement de Paris, sous signé, Poulet Deputtre et Le Bon, adjoint.

Pour copie conforme délivrée le 19 Juillet 1830.

Le Maire du Neuvième arrondissement de Paris.

Visé par la Commission (Loi du 12 Février 1872)

Le Membre de la Commission

Figure 5.6: *Acte de décès* for Claude Mourey. Reproduced from microfilm. Source: Archives de Paris.

5.3 Mourey's mathematics

5.3.1 Motivated by algebra, seeking algebraic reform

Mourey explains in the opening to his preface that his motivation was algebra. He writes:

> Trained, almost from childhood, in analytical reasoning, I can resist no longer the desire to submit to the judgement of scholars some of the results that I have reached. These are the difficulties presented by the theory of Algebra, which have, for many years, been the principal subject of my reflections. Eager for clarity, my mind could not be satisfied with this science as it has been presented to this day.[331]

Dissatisfied with algebra—with the theories of negatives and imaginaries in particular—Mourey sought improvements. He was concerned over two problems in particular: (1) how the operation of subtraction would translate from arithmetical algebra into a more symbolical form of algebra and (2) the practice of treating imaginaries in the same way as reals without proof that it is legitimate to do so.

Problem 1 Mourey understood that persisting with the definition of subtraction in arithmetic was causing problems for algebraists who were dealing with unknown or arbitrary quantities represented by algebraic symbols: if both the magnitude and the sign of the quantities involved in the subtraction $a - b$ are unknown, it cannot be determined whether the operation is "possible" (for $b < a$) or "impossible" (for $b > a$). Mourey pronounced:

> it is impossible to express the difference [between two quantities] by means of the sign $-$, when one of the terms is unknown or arbitrary. [...]

[331]'Entraîné, presque dès l'enfance, dans les méditations analytiques, je ne puis résister plus longtemps au désir de soumettre au jugement des êtres pensants quelques-uns des résultats aux-quels je suis parvenu. Ce sont les difficultés que présente la théorie de l'Algèbre, qui ont fait, pendant de longues années, le sujet principal de mes réflexions. Avide de clarté, mon esprit ne pouvait être satisfait de cette science, telle qu'elle a été présentée jusqu'à ce jour.' [145, pv]

It follows that the sign $-$, considered to express subtraction, cannot be admitted in Algebra. Because Algebra is supposed to deal only with unknown or arbitrary quantities, it cannot admit of subtraction.[332]

And so Mourey resolved 'to find the means of expressing the difference between two quantities, without having recourse to subtraction'.[333]

Problem 2 Mourey acknowledged that it was encouraging that the practice of manipulating imaginaries as if they were reals had not been shown to lead to false results; yet, he demanded more of algebra. He wanted algebra to have a logical structure and a basis similar to that of geometry: 'It must be agreed that the science would be far more satisfactory, if we could base all its parts on rigorous reasoning, on first-rate evidence, on simple and palpable ideas, like those of the elements of Geometry.'[334]

Mourey had realized that the issues relating to negative and imaginary quantities arose from deficiencies in algebra—deficiencies in its definitions and fundamental principles. It was the persistent appearance of negatives and imaginaries that had revealed to Mourey that the operations of algebra were capable of being more comprehensive. John Warren had observed the same. He wrote of imaginaries:

One thing was evident respecting them; that they were quantities capable of undergoing algebraic operations analogous to the operations performed on what are called

[332]'Il suit de là qu'il est impossible d'exprimer la différence par le moyen du signe $-$, lorsque l'un des termes est inconnu ou arbitraire. [...] Il suit de là que le signe $-$, considéré comme exprimant la soustraction, ne peut pas être admis en Algèbre. L'Algèbre, étant censée ne s'occuper que de quantités inconnues ou arbitraires, ne peut point admettre de soustraction.' [145, pp1–2]

[333]'Il faut donc trouver le moyen d'exprimer la différence de deux quantités, sans recourir à la soustraction; autrement l'Algèbre resterait imparfaite.' [145, p2]

[334]'On doit convenir que la science serait beaucoup plus satisfaisante, si l'on pouvait en baser toutes les parties sur des raisonnements rigoureux, sur une évidence du premier ordre, sur des idées simples, palpables, comme celles des éléments de Géométrie.' [145, pvii]

> possible quantities, and of producing correct results: thus it was manifest, that the operations of algebra were more comprehensive than the definitions and fundamental principles; that is, that they extended to a class of quantities, viz. those commonly called impossible roots, to which the definitions and fundamental principles were inapplicable. It seemed probable, therefore, that there was a deficiency in the definitions and fundamental principles of algebra; and that other definitions and fundamental principles might be discovered of a more comprehensive nature, which would extend to every class of quantities to which the operations of algebra were applicable; that is, both to possible and impossible quantities, as they are called.[335]

There is a striking similarity between Warren and Mourey's approaches to their researches: both intended to establish a new set of definitions and fundamental principles in algebra that were applicable to all classes of quantities; both focused on algebraic operations and both used geometry as a way in.

Mourey claimed not only to have succeeded in establishing a basis for algebra that was similar to geometry but also to have discovered—unexpectedly, at the same time—a new system of geometry:

> Not only have I reached this goal [to establish a basis for algebra], but I have encountered at the same time another result, which is perhaps no less valuable; with a new system of Algebra, which I was looking for, I found a new system of Geometry, which I was not expecting. They are not, however, two sciences; it is one and the same science, one and the same theory, which has two sides, one algebraic and the other geometric. It is an Algebra which emanates from Geometry; it is a Geometry which is generalized and made algebraic.[336]

Mourey developed this dual system of algebra and geometry under a unifying concept of directed lines.

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[335] [142, p241]

[336] 'Non-seulement j'ai atteint ce but, mais j'ai rencontré, en même temps, un autre résultat qui n'est peut-être pas moins précieux; avec un nouveau système d'Algèbre, que je cherchais, j'ai trouvé un nouveau système de Géométrie, auquel je ne m'attendais pas. Ce ne sont cependant pas deux sciences; ce n'est qu'une seule science, une seule théorie, laquelle a deux faces, l'une algébrique, et l'autre géométrique. C'est une Algèbre émanée de la Géométrie; c'est une Géométrie généralisée et rendue algébrique.' [145, ppvii–viii]

Mourey anticipated fierce opposition to his new radical system; and yet, he was confident that one day it would come to be recognized as the one true theory. He writes:

> The system which I am putting forward is altogether new; it will meet with many powerful enemies: ideas which have prevailed for centuries, inveterate habits, species of vested interests ... Without any doubt it will eventually prevail, as everything that is good and true must do; but will its author enjoy this triumph?[337]

In the commentary which follows, I will highlight evidence of the far-sightedness of Mourey's work, in view of the way mathematics has since developed.

5.3.2 Definitions and fundamental principles

Recall that Mourey hoped to discover some means of expressing the difference between two quantities without resorting to subtraction. For real quantities there was, of course, already a standard approach: real quantities could be represented as points on the real line (Figure 5.7, page 131) with addition interpreted as a journey in a positive direction and subtraction as a journey in the opposite direction. But for non-real quantities, a more general theory of directed lines was required. Mourey promised a theory in which all algebraic quantities whatever could be represented geometrically by directed lines in the plane; lines which would be determined by the two parameters of length and direction.

Mourey defines a *directed line* or *path* as a line which leads in a particular direction. It is, in modern terminology, a vector.[338] To denote the path which leads from an *origin* A to a *terminus* B he writes AB and, similarly, BA for the path which leads from B to A.[339] He notes, in the first instance, that AB and BA represent the

---

[337]'Le système que je propose est tout neuf; il rencontrera des ennemis nombreux et puissants: des idées qui règnent depuis des siècles, des habitudes invétérées, des espèces de droits acquis. ... Sans doute il finira tôt ou tard par triompher, comme doit le faire tout ce qui est bon et vrai; mais est-il réservé à son auteur de jouir de ce triomphe?' [145, px]

[338][See *ligne directive*.]

[339][See *origine*, *terme*.]

same non-directed line but different paths: to be equal as non-directed lines, two lines need only have the same length; to be equal as paths, however, 'it is necessary and sufficient that they be concurrent and have the same length'.[340] With direction taken into account, AB and BA are termed *inverses* and $BA = -AB$.[341]

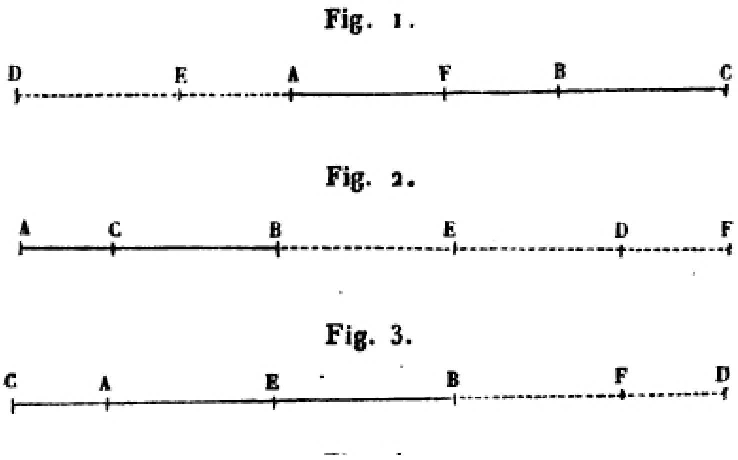

Figure 5.7: Journeys along the real line.[342]

Having defined his set of directed lines, Mourey looks to establish the rules for their combination. He defines addition first, according to the following 'fundamental principle'[343]

$$AB + BC = AC$$

That is: 'The *sum* of two consecutive paths is the single path which leads from the origin of the first to the terminus of the second'.[344] See $\triangle ABC$, for example, in Figure 5.8 (page 132). The principle extends, of course, to sums of more than two

[340] 'Donc, pour que deux chemins soient égaux, il faut et il suffit qu'ils soient concurrents et qu'ils aient même longueur.' [145, p8]

[341] [See (*chemins*) *inverses*.]

[342] [145, p2]

[343] 'L'équation $AB + BC = AC$ sera notre principe fondamental.' [145, p5]

[344] 'La *somme* de deux chemins de suite est le chemin simple qui conduit de l'origine de premier au terme du second'. [145, pp5–6]

directed lines, which Mourey demonstrates by successive addition of paths.[345]

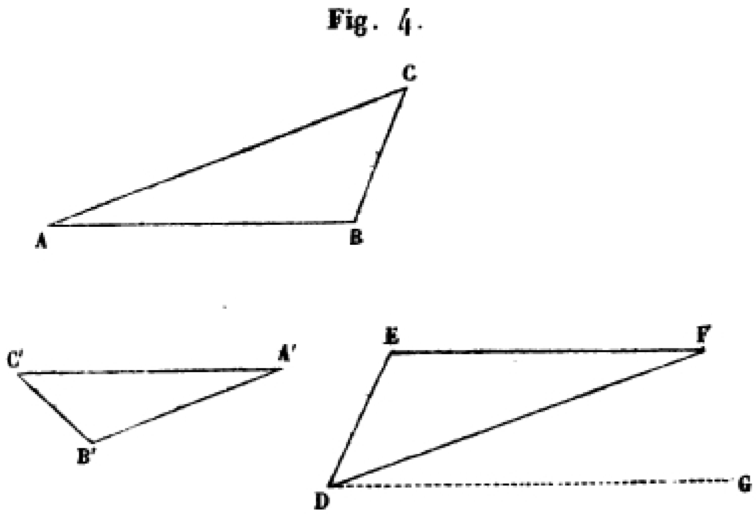

Figure 5.8: Addition of directed lines.[346]

Next, Mourey considers addition for paths that do not follow on from one another—that are not *de suite* (consecutive)—such as the paths AB and DE in Figure 5.8.[347] His approach is to trace from the point B, a path BC that is parallel to, and of the same length as, DE (translation of the vector DE in other words). Thus $AB + DE = AC$.

Mourey describes addition as a 'fundamental principle' because he believed that it provided the only means by which we might substitute for subtraction in algebra. Consider the addition of two inverse paths AB and $-AB$: if the sign $-$ expressing subtraction is interpreted as "+ inverse", then subtraction can be performed by changing the sign of the quantity to be subtracted and proceeding as in addition. Mourey writes:

> we have proved that the sign $-$, considered to express subtraction, cannot be admitted in Algebra; it is therefore appropriate to give it some other interpretation, in order

[345]See [145, pp6–7] for details.

[346] [145, p2]

[347][See (*chemins*) *de suite*.]

> to use it. The above equation [$AB + \text{inverse } AB = AB - AB$] provides us with the means; we have only to admit that this sign $-$ will signify $+$ *inverse*. Or else, we can admit that $-$ will simply mean *inverse*, and we will write $AB + -AB = 0$. We will use either one or the other of these two interpretations; but, in either case, the term will be the same, *plus inverse*.[348]

I would suggest that Mourey has re-defined subtraction by introducing the additive inverse into his system. He also has $- - AB = -BA = AB$, or $-(-x) = e.x$ in modern notation, which is a property deducible from group axioms.

Continuing with the concepts of direction and opposition, Mourey explains how they relate to abstract numbers. A *directed number* is an abstract number, considered as a quantity measured relative to an abstract unit; and for this abstract unit, Mourey takes a path of arbitrary length and direction. Thus, the positive integers $1, 2, 3, \ldots$ are defined as numbers with the same direction as the unit 1; the negative integers $-1, -2, -3, \ldots$ as numbers with a direction opposite to 1. This relationship between the integers and the unit suggests to Mourey new names for positives and negatives: *commétriques* for positives and *antimétriques* for negatives.[349]

Mourey recognizes an advantage of defining the integers in this way—that these definitions serve to dispel some of the myths surrounding negative quantities. He writes:

> Several consequences follow from these definitions which are important to emphasize in teaching, in order to keep the pupils from the false ideas naturally implied by the words *positive* and *negative*, as well as the sign $-$ when referred to by the word

[348] 'Mais nous avons démontré que le signe $-$, considéré comme exprimant la soustraction, ne peut pas être admis en Algèbre; il est donc convenable de lui donner quelque autre interprétation, afin de l'utiliser. L'équation ci-dessus [$AB + \text{inverse } AB = AB - AB$] nous en fournit le moyen; nous n'avons qu'à admettre que ce signe $-$ signifiera $+$ *inverse*. Ou bien encore, nous pouvons admettre que $-$ signifiera simplement *inverse*, et nous écrirons $AB + -AB = 0$. Nous emploierons tantôt l'une et tantôt l'autre de ces deux interprétations; mais, dans l'un et l'autre cas, l'énoncé sera le même, *plus inverse*.' [145, p10]

[349] [See *unité relative*, *nombre relatif*, *concret*, *abstrait*, *nombre directif*, (*nombre*) *positif*, (*nombre*) *négatif*.]

minus. [...] The same directed quantity will be either positive or negative as we please, depending on whether we would like to take the unit in one direction or the other. [...] Every negative quantity is as large as its positive inverse. [...] The negative quantities are just as real and palpable as the positive ones. [...] The negative quantities are not at all the result of impossible subtractions.[350]

Staying with the integers, Mourey considers how they are related to one another by the signs $<$ and $>$. While he recognizes that the same relation exists between, for instance, the numbers 1 and 0, and 0 and -1

$$+1 = 0 + \text{a positive} \quad 0 = -1 + \text{a positive}$$

he yet concludes—at odds with some contemporary algebraists—that

$$-1 > 0, -2 > -1, -3 > -2, \text{ etc.},$$

He is, therefore, ultimately unsuccessful in introducing an *order relation* onto the integers. But perhaps Mourey's intention was not to introduce an order relation onto the integers: perhaps he was looking to introduce an order relation onto the complex numbers; remember, Mourey sees the integers as a special case of the complex numbers. An ordering of the complex numbers is impossible of course. The very best we can do in this regard is to order them by their magnitudes. Doing this, we do have, in agreement with Mourey, $-1 > 0, -2 > -1, -3 > -2$, etc.

In order to depart from the real line, Mourey introduces the notion of the *directed angle*. Suppose that the directed line AB (Figure 5.9, page 135) is made to rotate about its origin A through an angle r to become the directed line AC. This geometrical operation is expressed algebraically by

$$AB_r = AC$$

[350]'Des ces définitions résultent plusieurs conséquences qu'il est important de signaler dans l'enseignement, pour préserver les élèves des fausses idées que présentent naturellement les mots *positif* et *négatif*, ainsi que le signe −, lorsqu'on l'énonce par le mot *moins*. [...] La même quantité directive sera positive ou négative à notre gré, selon qu'il nous plaira de prendre l'unité dans un sens ou dans l'autre. [...] Toute quantité négative est aussi grande que son inverse positive. [...] Les quantités négatives sont tout aussi réelles et aussi palpables que les positives. [...] Les quantités négatives ne sont point les résultats de soustractions impossibles.' [145, p13]

where the angle r is termed the *verseur* of AB.

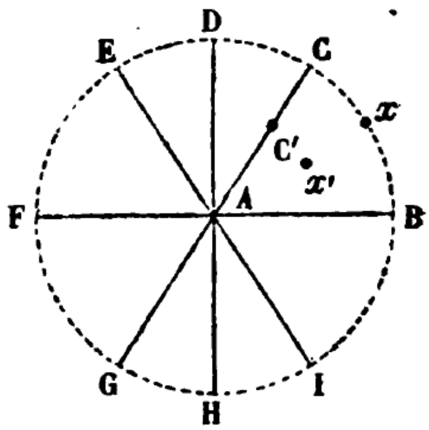

Figure 5.9: The directed angle.[351]

Now, the angle leading from AB to AC is necessarily a directed angle, as it must lead in a particular direction: either clockwise (measured by the major arc $BHFDC$) or anti-clockwise (measured by the minor arc BxC). Mourey adopts the convention that angles described by an anti-clockwise rotation are positive.

Mourey indicates angles in a peculiar fashion. He denotes the included angle between AB and AC by ABC: the first letter symbolizes the vertex, the second identifies the leading side (the side from which the angle originates which Mourey terms the *côté dirigeant*) and the last identifies the facing side (the side towards which the angle approaches which Mourey terms the *côté dirigé*).[352] If no convention is established as to the sense in which the angle turns, then additional letters are placed between the letters representing the leading and facing side. For example, the directed angle leading from AB to AC via x (Figure 5.9) is expressed by $ABxC$.

For the unit of the directed angle, Mourey chooses an anti-clockwise rotation of 90° which he denotes by a subscript 1. Thus

$$AB_{90^\circ} = AB_1 = AD$$

[351] [145, p19]

[352][See *côté dirigeant*, *côté dirigé*.]

and similarly

$$AB_{180^\circ} = AB_2 = AF, \quad AB_{-90^\circ} = AB_{-1} = AH, \quad AB_{60^\circ} = AB_{\frac{2}{3}} = AC, \quad \text{etc.}$$

Frequently, the path AB is given unit length and is taken along the positive real axis so that

$$AB = 1, \quad AD = 1_1, \quad AF = 1_2 = -1, \quad AH = 1_3 = -1_1, \quad AC = 1_{\frac{2}{3}}, \quad \text{etc.}$$

This is the realization of a statement made by Mourey in his preface: in his system, 'the algebraic (or even arithmetic) expression of a path will determine its direction, in relation to another path taken as a term of comparison, or, if we like, as unity'.[353,354]

Now, even if we were to make AB perform any number of full revolutions in addition to r, it would invariably arrive at AC. Mourey's understanding of the phenomenon is this:

> Thus, the directed angle is likely to grow to infinity; but any directed angle which is not smaller than 4 right angles, may be replaced by the excess over the largest multiple of 4 right angles which it contains.
>
> We can therefore say that two angles are equal *as directed angles*, if their difference is exactly 4^q or a multiple of 4^q.[355]

Note, the notation 4^q is a little misleading: it denotes four right angles rather than powers of 4. Mourey is not explicit about this and does not define q.

Mourey terms this species of equality, which he represents by the sign $\doteq$, as *the*

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[353]'L'éxpression algébrique (ou même arithmétique) d'un chemin en déterminera la direction, relativement à un autre chemin pris pour terme de comparaison, ou, si l'on veut, pour unité.' [145, pviii]

[354][See *verser*, *verseur*, *version*, *rapport directeur*, *angle directif*.]

[355]'Ainsi, l'angle directif est susceptible de croître à l'infini; mais tout angle directeur qui n'est pas plus petit que 4 angles droits, peut être remplacé par son excédant sur le plus grand multiple de 4 angles droits qu'il contienne. On peut donc dire que deux angles sont égaux *en tant qu'angles directeurs*, si leur différence est exactement 4^q ou un multiple de 4^q.' [145, p24]

special equality of directed angles. It is expressed in his notation, in terms of units of right angles, as

$$r \doteq r+4 \doteq r+8 \doteq r+12 \doteq \cdots \doteq r+4n \tag{5.1}$$

where n is an integer.[356] Mourey also has, for angle measured contrary to convention,

$$-r \doteq 4-r$$

Those with a background in modern mathematics might notice two things: (i) Mourey is identifying as equal, angles that are congruent modulo 2π and is, therefore, describing an equivalence relation and (ii) his equation (5.1) appears remarkably similar in form to the following equation

$$[r] = [r+4] = [r+8] = [r+12] = \ldots = [r+4n]$$

which expresses that all numbers congruent modulo 2π to r belong to the equivalence class $[r]$, the representatives of which may be indifferently $0, 4, 8, 12, \ldots, 4n$. The concept of the congruence of integers with respect to a modulus, and the associated notation, was introduced by Johann Carl Friedrich Gauss (1777–1855) in *Disquisitiones Arithmeticae* in 1801.

At the same time as rotating a directed line a_r through a given angle s, we might wish to stretch its length by a given multiple b. To this end, we multiply the directed line a_r by b_s according to Mourey's definition of multiplication

$$a_r \times b_s = (a_r \times b)_s = [(ab)_r]_s = ab_{r+s}$$

Mourey defines the directed number b_s as an *operator*, which multiplies the length of a_r by b and rotates it through an angle s. The concept of a geometrical operator was known to Hamilton in the form of a quaternion from 1844.[357]

In several places Mourey explains the significance of multiplication. (i) 'Every

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[356] [See *égalité spéciale des angles directifs.*]

[357] For verification of this, see the correspondence between Hamilton and De Morgan dated 14 July 1854 (page 183 of this thesis).

directed number is derived from the directed unit by multiplication, division and *version*'.[358] *Version* is the operation of rotation implicit in multiplication. (ii) '[The definition of multiplication] is very useful, because it follows from it [...] that all Algebraic formulae express real quantities, and that they apply very favourably to Geometry (and therefore to Mechanics)'.[359] This is not Mourey's only reference to mechanics. In the preface he writes: 'In its application to Mechanics, the advantages are not least; it even seems that the system in question is made specially for it.'[360]

For the operations of addition and multiplication, Mourey is aware that certain properties hold. For addition: commutativity and associativity. For multiplication: commutativity, associativity and distributivity over addition. Mourey proves the commutative property for addition and multiplication, and the distributive laws; but he omits the details of a proof of associativity for both operations.[361] Mourey's proof of commutativity for multiplication is entirely algebraic in nature. He has

$$a_r \times b_s = b_s \times a_r$$

Since, the first $= ab_{r+s}$ and the second $= ba_{s+r}$:

now, $ab = ba$, and $r + s = s + r$; therefore, etc.

His proof of the distributive laws is based on vector methods.

[358]'Tout nombre directif se forme de l'unité directive, par multiplication, division et *version*'. [145, p30]

[359]'On doit admettre cette convention si elle est utile; or, elle est très-utile, car il en résulte, comme on le verra, que toutes les formules de l'Algèbre expriment des quantités réelles, et qu'elles s'appliquent très-avantageusement à la Géométrie (et par conséquent à la Mécanique).' [145, p31]

[360]'Dans l'application à la Mécanique, on ne rencontrera pas de moindres avantages; il semble même que ce soit spécialement pour cette partie que soit fait le système dont il s'agit.' [145, pix]

[361]Mourey's proof of commutativity for addition relies heavily on the equality of triangles: see [145, pp8–9].

To prove the distributive laws, Mourey first constructs the similar triangles ABC and ADE (Figure 5.10). He has $AD = AB \times p$, $AE = AC \times p$ and $DE = BC \times p$, where p is positive.

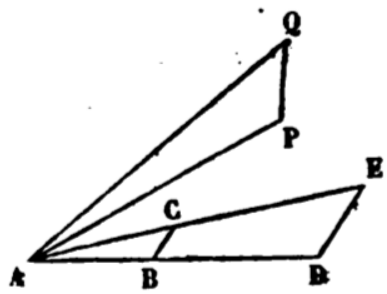

Figure 5.10: Mourey's proof of the distributive laws using similar triangles.[362]

But as vector sums,

$$AC = AB + BC \quad AE = AD + DE$$

therefore

$$(AB + BC) \times p = AB \times p + BC \times p$$

Next, Mourey rotates the triangle ADE through an angle r, about the vertex A, so that it comes to rest at APQ. By construction Mourey has $ADP = r$ and $AP = AD_r$. But AE has also moved through an angle r to become AQ, therefore, $AEQ = r$ and $AQ = AE_r$. Mourey wants to prove that $PQ = DE_r$.

Since PQ and DE are clearly the same length, all Mourey needs to prove is that $PQ \because DE = r = ADP$, where $PQ \because DE$ denotes the angle which takes DE to PQ. This is new notation introduced by Mourey to denote the angle between two directed lines that do not share the same origin.[363]

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[362] [145, p32]

[363][See *digène*.] See also [145, pp26–27].

Now, $\angle PQA = DEA$, therefore,

$$PA \because PQ = DA \because DE$$

$\angle PQA$ being the ratio of PA to PQ and likewise $\angle DEA$ being the ratio of DA to DE.[364] Switching the position of the means, Mourey has

$$PA \because DA = PQ \because DE$$

but

$$PA \because DA = AP \because AD = ADP = r$$

therefore

$$PQ \because DE = r$$

therefore

$$PQ = DE_r$$

as required.

Now

$$AQ = AP + PQ = AD_r + DE_r$$

but $AQ = AE_r = (AD + DE)_r$ therefore

$$(AD + DE)_r = AD_r + DE_r$$

From commutativity and associativity of multiplication, it follows that

$$(AB + BC) \times p_r = AB \times p_r + BC \times p_r$$

To simplify, Mourey lets $AB = c$, $BC = d$ and $p_r = q$, so that

$$(c + d) \times q = c \times q + d \times q$$

which is the right distributive law. From this, using the commutative property of multiplication, Mourey deduces the left distributive law

$$q \times (c + d) = q \times c + q \times d$$

[364][See *équi-digène.*] See also [145, pp29–30].

Then, letting $q = a + b$, Mourey derives an equation which shows how to multiply out brackets and which, he remarks, can be extended to the multiplication of polynomials

$$(a + b) \times (c + d) = a \times c + b \times c + a \times d + b \times d$$

Note, that what Mourey says in relation to the notation $\because$ suggests that he is close to the notion of the scalar product $u \cdot v = uv \cos\theta$. From an algebraic equation involving $\because$ he deduces the relative position of two paths:

> The equation $a \because b = 0$ shows us that a and b are concurrent.
> The equation $a \because b = 2$ shows us that a and b are opposite.
> The equation $a \because b = 0$ or 2 shows us a and b are parallel.
> The equation $a \because b = 1$ or 3 shows us a and b are perpendicular.

The difference is that Mourey is not taking magnitudes into account.

From the rule for multiplication Mourey deduces the rule for division

$$b_s = \left(\frac{d}{a}\right)_{z-r}$$

and the rule for taking powers

$$(a_r^m)^n = a_{r \times n}^{m \times n}$$

Note, a_r^m signifies $(a^m)_r$ and not $(a_r)^m$.[365] And from this principle Mourey deduces the nth roots of unity

$$1, \quad 1_{\frac{4}{n}}, \quad 1_{\frac{8}{n}}, \quad 1_{\frac{12}{n}}, \quad \ldots \quad 1_{\frac{4(n-1)}{n}}$$

or more generally

$$1_{\frac{\sigma}{n}}$$

where σ represents the set of multiples of 4 right angles $\{0, 4, 8, 12, \ldots\}$. The set notation is mine. Mourey notes that this is a periodic sequence of n values and

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[365] [See a_r^m.]

reasons, on the basis of this observation, that there are exactly n distinct nth roots of unity. He accounts for the plurality of the roots as follows.

Setting $x = 1_z$ the equation $x^n = 1$ becomes

$$(1_z)^n = 1_{z\times n} = 1_0$$

hence

$$z \times n = 0$$

or more generally

$$z \times n \doteq 0$$

This last equation has n roots

$$z = \frac{\sigma}{n}$$

where σ again represents the set of multiples of 4 right angles $\{0, 4, 8, 12, \ldots\}$. Therefore, the roots of $x^n = 1$ are the set of solutions

$$x = \left\{1_{\frac{\sigma}{n}}\right\}$$

Similarly, he gives the nth roots of $x^n = 1_r$ as

$$x = \sqrt[n]{1_{r+\sigma}} = \left\{1_{r+\sigma}\right\}$$

Again the set notation is mine. Note, Mourey has $\sqrt[n]{1_{r+\sigma}}$ rather than $\sqrt[n]{1_r}$ as the algebraic formula which produces the nth roots of 1_r. This is because he determines that $\sqrt[n]{1_r}$ expresses uniquely the root $1_{\frac{r}{n}}$, which in turn has a single value; and therefore, in effect, n different algebraic formulae are required in order to produce the n nth roots of 1_r.

Mourey also establishes a special convention for determining the values of the formula $\sqrt[n]{-1}$. Outside of the radical sign in his system $-1 = 1_2 = 1_6 = 1_{10} = \cdots$. If this were allowed to be the case under the radical sign, then the formula $\sqrt[n]{-1}$ would represent multiple values $1_{\frac{2}{n}}, 1_{\frac{6}{n}}, 1_{\frac{10}{n}}$, etc.; in general $1_{\frac{2+\sigma}{n}}$. Believing this to be not very useful, Mourey restricts the formula $\sqrt[n]{-1}$ to having a single value $1_{\frac{2}{n}}$. Thus, under the radical sign -1 denotes 1_2 and not $1_{2\pm\sigma}$, $-a$ denotes a_2 and not $a_{2\pm\sigma}$, and $-a_r$ denotes a_{2+r} and not $a_{2+r\pm\sigma}$. A consequence of admitting this convention is that in Mourey's system certain equalities from ordinary algebra fail to hold. For instance, in ordinary algebra $\sqrt[3]{-1} = -1$, $\sqrt[5]{-1} = -1$, $\sqrt[7]{-1} = -1$, etc.; however, in Mourey's system $\sqrt[3]{-1} = 1_{\frac{2}{3}}$, $\sqrt[5]{-1} = 1_{\frac{2}{5}}$, $\sqrt[7]{-1} = 1_{\frac{2}{7}}$, etc., and $1_{\frac{2}{3}} \neq 1_{\frac{2}{5}} \neq 1_{\frac{2}{7}} \neq -1$.

In order to accommodate his peculiar conventions for taking nth roots, Mourey introduces new notation to express the equality of two quantities under the radical sign. This species of equality, expressed by the sign $\overset{:}{=}$, is termed *super-equality*.[366] For two directed lines to be equal under the radical sign, they must have the same length and direction, and make exactly the same angle with the positive real axis (it is not good enough that their angles are congruent modulo 2π).[367] Mourey's example is 1_4 and 1_0. Although these two directed lines have the same length and direction, and their angles are congruent modulo 2π, they are not equal under the radical sign since $\sqrt{1_4} = 1_{\frac{4}{2}} = 1_2 = -1 \neq \sqrt{1_0} = 1_0$.

Continuing with the difficulties associated with the calculus of radicals, Mourey considers what happens when polynomials, rather than monomials, appear under the radical sign. Mourey runs into difficulty when it is discovered that equality is not preserved. His example is

$$(a + \sqrt{a^2 - c^2}) \times (a - \sqrt{a^2 - c^2}) \overset{:}{=} a^2 + a\sqrt{a^2 - c^2} - a\sqrt{a^2 - c^2} - (a^2 - c^2)$$

which he reduces to

$$\overset{:}{=} a^2 - (a^2 - c^2) \overset{:}{=} a^2 - a^2 - -c^2 \overset{:}{=} c_4^2$$

Now, taking the square root of both sides gives

$$\sqrt{a + \sqrt{a^2 - c^2}} \times \sqrt{a - \sqrt{a^2 - c^2}} = \sqrt{c_4^2} = c_2 = -c$$

If we let $a = c$ we have

$$\sqrt{c + \sqrt{c^2 - c^2}} \times \sqrt{c - \sqrt{c^2 - c^2}} = \sqrt{c} \times \sqrt{c} = +c$$

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[366][See (*signe*) *super-égal*, (*chemins*) *super-égaux*.]

[367][See *prime-directeur*.]

however, if we let $a = 0$ we have

$$\sqrt{\sqrt{-c^2}} \times \sqrt{-\sqrt{-c^2}} = \sqrt{c_1} \times \sqrt{-c_1} = \sqrt{c_1} \times \sqrt{c_3} = (\sqrt{c})_{\frac{1}{2}} \times (\sqrt{c})_{\frac{3}{2}} = c_{\frac{4}{2}} = c_2 = -c$$

Mourey concludes that there is no general method for multiplying polynomials under a radical sign. Let A and B be any two polynomials and let $A \times B = P$. We should not conclude, he says, that $\sqrt[n]{A} \times \sqrt[n]{B} = \sqrt[n]{P}$ but only that $\sqrt[n]{A} \times \sqrt[n]{B} = \sqrt[n]{P}_{\frac{\sigma}{n}}$; that is, that the product is any one of the roots of the equation $x^n = P$.

These few examples evidence some very serious consequences of Mourey's peculiar conventions. Although Mourey was aware of the problems in his system, and knew that they arose from the conventions he had admitted for taking nth roots, it seems that he was willing to accept that they might surface occasionally.

Mourey concludes his introductory section by showing how logarithms can make the process of multiplying directed lines easier. His example is $x = 7_{\frac{1}{2}} \times -15 \times 10 \times 1_{\frac{2}{3}}$. He calculates the logarithm of the length of each path, sums the result and then finds the antilogarithm using tables. This gives him the length of x. The direction of x, or its *prime-directeur*, is found by adding the *prime-directeurs* of all the factors of x.

5.3.3 Applications, including a proof of the fundamental theorem of algebra

Having established the preliminaries, Mourey attempts to show how his theory of directed lines in the plane relates to trigonometry. With reference to Figure 5.11 (page 145), he establishes an association between trigonometric lines and paths. His approach is to express trigonometric lines in terms of their position with respect to the real axis.

First, he establishes AB (Figure 5.11) as a line drawn in a positive direction

$$AB = posit. = 1$$

Next, he identifies the angles

$$\angle ABC = C \quad \angle ABC' = C' \quad \angle ABCC'C'' = C'' \quad \angle ABCC'C''C''' = C'''$$

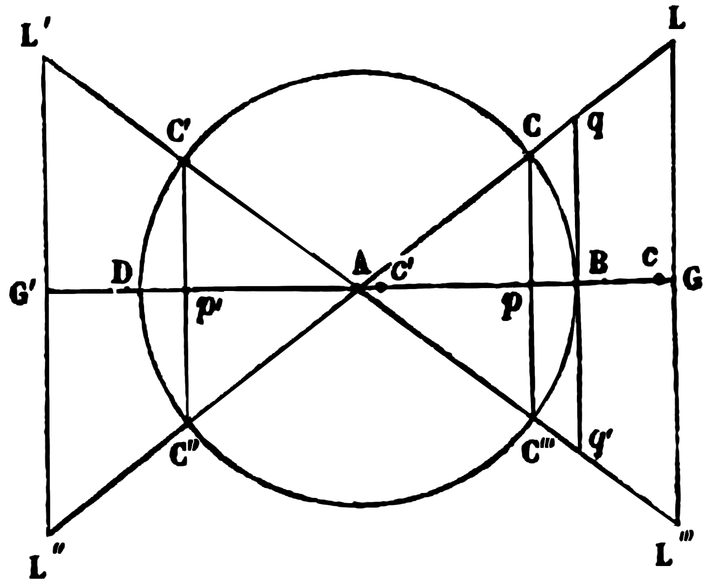

Figure 5.11: Relating trigonometric lines and paths.[368]

Thus he has for sine, for example,

$$\begin{aligned}
\sin C = \sin C' = pC = p'C' = pc_1 = (pos.)_1 = pos. \times \sqrt{-1} \\
\sin C'' = \sin C''' = p'C'' = p'C''' = pc_3 = (pos.)_3 = pos. \times -\sqrt{-1} \\
\text{or } = (pc')_1 = (neg.)_1 = neg. \times \sqrt{-1}
\end{aligned}$$

and, therefore, in general

$$\sin x = (posit.)_1 \text{ or } (neg.)_1 = (paral.)_1$$

that is, the sine of an angle corresponds with a path that is perpendicular to the real axis.

[368] [145, p58]

Similarly

$$\begin{aligned}
\cos x &= posit. \text{ or } negat. = parallel \\
\tan x &= (posit.)_1 \text{ or } (neg.)_1 = (paral.)_1 \\
\sec x &= (posit.)_x \text{ or } (neg.)_x = (paral.)_x \\
\sin \text{ vers } x &= posit.
\end{aligned}$$

Note, sin vers x is the versed sine of x, that is, the sine of x rotated through an angle of $90°$. It is related to $\sin x$ by the formula vers $x = 1 - \cos x$, which Mourey deduces from Figure 5.11.[369] The versed sine featured historically in the mathematics of engineering and surveying.

Mourey also deduces from Figure 5.11, the formula for $\sec x$ in terms of $\tan x$

$$(\sec x)_{=x} = 1 + (\tan x)_{=1}$$

and Euler's identity

$$1_x = (\cos x)_{=} + (\sin x)_{=1} \tag{5.2}$$

In the new notation $a_{=}$ denotes a path of length a which is parallel to unity and $a_{=1}$ denotes a path of length a which is perpendicular to unity.[370]

Mourey also establishes—from Figure 5.11, by means of similar triangles—that $\cos x : 1 :: \sin x : \tan x :: 1_x : \sec x$, from which the trigonometric formulae for $\tan x$ and $\sec x$ follow. Equating the first two ratios, we have

$$\tan x = \frac{\sin x}{\cos x}$$

Equating the first and third, we have

$$\sec x = \frac{1_x}{\cos x}$$

[369] See [145, p60] for details.

[370] [See $a_{=}$, $a_{=}1$, *mi-déverseur*.]

He then goes on to show how we can derive the compound angle formulae for $\cos(x+z)$ and $\sin(x+z)$ using the familiar method of equating two expressions for 1_{x+z}.[371]

Application of trigonometry to the calculus of equations

Next, Mourey outlines a method whereby we can use Euler's identity to write the nth roots of the equation $x^n = 1$ in the form $a + b\sqrt{-1}$. Recall, the roots of $x^n = 1$ are $\left\{1_{\frac{\sigma}{n}}\right\}$. Substituting $\frac{\sigma}{n}$ for x in (5.2) gives

$$1_{\frac{\sigma}{n}} = \cos\frac{\sigma}{n} + (\sin\frac{\sigma}{n})_{=1} = \cos\frac{\sigma}{n} + (\sin\frac{\sigma}{n})_{=}\sqrt{-1}$$

where $\sigma = 0, 4, 8, 12, \ldots$

From Euler's identity, Mourey establishes the polar form of a directed line

$$a_r = a \times 1_r = a\cos r + a\sin r$$

Note, he frequently drops the subscript notation.

Next, Mourey considers three problems: in the first two he explains how to approach the geometrical operation of addition algebraically; in the third he works through a method for solving the cubic.

In order to add two paths p_r and q_s we need to solve the following equation for the unknowns x and u

$$p_r + q_s = x_u$$

which becomes, when p_r and q_s are in polar form,

$$x_u = (p\cos r + q\cos s) + (p\sin r + q\sin s)$$

Again, Mourey drops the subscript notation. Identifying the real and imaginary parts, so that if $x_u = \pm a \pm b_1$, we have

$$\pm a = p\cos r + q\cos s$$

$$\pm b = p\sin r + q\sin s$$

---

[371]See [145, pp62–64] for details.

From here we can find $\tan u$ using the formula Mourey deduces from Figure 5.11

$$\frac{\pm b_1}{\pm a} = \frac{\sin u}{\cos u} = \tan u$$

and from there, find u using tables. We can then find x using the following formula, which Mourey deduces from Figure 5.11 by considering similar triangles,

$$x = \frac{\pm a}{\cos u}$$

Alternatively, Mourey suggests we might use the following formulae

$$x = \sqrt{a^2 + b^2}$$

$$(\sin u)_= = \frac{\pm b}{x}$$

Mourey's method for solving the cubic is as follows. He begins with the reduced general cubic equation

$$x^3 + px + q = 0 \tag{5.3}$$

and proceeds initially according to the method of Tartaglia given in Girolamo Cardano's (1501–1576) *Ars Magna* (1545).[372]

He substitutes $y + z$ for x into (5.3) and lets

$$y \times z = \frac{-p}{3} \tag{5.4}$$

which gives

$$y^3 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} = A \tag{5.5}$$

$$z^3 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} = B \tag{5.6}$$

from which it follows that

$$y = (\sqrt[3]{A})_{\frac{\sigma'}{3}} \qquad z = (\sqrt[3]{B})_{\frac{\sigma''}{3}}$$

[372]The issue of priority in the solution of the cubic is discussed in [158].

since any root of $x^n = a$ is $x = 1_{\frac{\sigma}{n}} \times \sqrt[n]{a} = (\sqrt[n]{a})_{\frac{\sigma}{n}}$.

Next, he substitutes for y and z into (5.4), which gives

$$(\sqrt[3]{A}\sqrt[3]{B})_{\frac{\sigma'+\sigma''}{3}} = \frac{1}{3}p_2 \tag{5.7}$$

Now, the signs of A, B, p are unknown. If A, B, p are positive then the values of $\sqrt[n]{A}$, $\sqrt[n]{B}$, $\sqrt[n]{p}$ will be the positive numbers for which A, B, p are the nth powers. If A, B, p are negative, Mourey says make $A = A_{+\alpha}$, $B = B_{+\beta}$ and $p = p_{+\pi}$. The notation A_+ signifies a directed line of the same length as A but always taken in a positive direction. For example, $-2 = 2_{+2}$.[373] Proceeding under the assumption that A, B, p might be negative, Mourey has

$$\sqrt[n]{A} = (\sqrt[n]{A_+})_{\frac{\alpha}{n}} \qquad \sqrt[n]{B} = (\sqrt[n]{B_+})_{\frac{\beta}{n}} \qquad \sqrt[n]{p} = (\sqrt[n]{p_+})_{\frac{\pi}{n}}$$

so that

$$y = (\sqrt[3]{A_+})_{\frac{\alpha+\sigma'}{3}} \qquad z = (\sqrt[3]{B_+})_{\frac{\beta+\sigma''}{3}}$$

Equation (5.7) then becomes

$$(\sqrt[3]{A_+}\sqrt[3]{B_+})_{\frac{\alpha+\beta+\sigma'+\sigma''}{3}} = \frac{1}{3}p_{+\pi+2}$$

from which it follows that two conditions must hold:

1. Respecting lengths: $\sqrt[3]{A_+}\sqrt[3]{B_+} = \frac{1}{3}p_+$
2. Respecting angles: $\frac{\alpha+\sigma'}{3} + \frac{\beta+\sigma''}{3} \doteq \pi + 2$

To verify that condition (1) is satisfied, Mourey suggests we multiply together the expressions for A and B given in equations (5.5) and (5.6).

Condition (2):

Next, Mourey deduces the values of α, β, π for which the second condition is satisfied. He takes σ' as arbitrary and writes σ'' in terms of σ', so that we have

$$x = y + z = (\sqrt[3]{A_+})_{\frac{\alpha+\sigma'}{3}} + (\sqrt[3]{B_+})_{\frac{3\pi-\alpha+6-\sigma'}{3}}$$

---

[373][See AB_+, *déverseur*.]

Substituting in $\sigma' = 0, 4, 8, 12, \ldots$ gives

$$x = (\sqrt[3]{A_+})_{\frac{\alpha}{3}} + (\sqrt[3]{B_+})_{\frac{3\pi-\alpha+6}{3}}$$
$$x = (\sqrt[3]{A_+})_{\frac{\alpha+4}{3}} + (\sqrt[3]{B_+})_{\frac{3\pi-\alpha+2}{3}}$$
$$x = (\sqrt[3]{A_+})_{\frac{\alpha+8}{3}} + (\sqrt[3]{B_+})_{\frac{3\pi-\alpha-2}{3}}$$

Mourey explains that there are only three distinct x values because $\frac{\sigma'}{3}$ has only three distinct values.

Mourey considers a number of particular cases, in which various combinations of the signs of p and q appear. In each, he deduces the values of α, β and π. The most interesting case is the irreducible case, where p is negative, q is positive or negative, and $\left|\frac{p^3}{27}\right| > \frac{q^2}{4}$.[374]

The irreducible case

Recall

$$y^3 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} = A \quad z^3 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} = B$$

To simplify, Mourey writes

$$-\frac{q}{2} = \pm m \quad \frac{q^2}{4} + \frac{p^3}{27} = -n = n_2$$

where m and n are positive paths, so that we have

$$A_{+\alpha} = \pm m + (\sqrt{n})_1 \quad B_{+\beta} = \pm m - (\sqrt{n})_1$$

Next, he identifies $\pm m$ and $(\sqrt{n})_1$ in Figure 5.11 (page 145)

$$\pm m = AG(\text{or } AG')$$
$$(\sqrt{n})_1 = GL(\text{or } G'L')$$
$$\text{where } A_{+\alpha} = AL(\text{or } AL')$$
$$\text{and } \angle\alpha = ABC(\text{or } ABCC')$$

[374]Typographical error: Mourey has $\frac{p^3}{27} < \frac{q^2}{4}$ which is incorrect. See [145, p72].

from which it follows that

$$-(\sqrt{n})_1 = GL'''(\text{or } G'L'')$$
$$B_{+\beta} = AL'''(AL'')$$
$$\text{where } \beta = ABCC'C''C'''(ABCC'C'')$$

therefore

$$B_+ = A_+ \quad \beta = 4 - \alpha$$

and since p is negative, $\pi = 2$ and

$$p = p_{+2}$$

therefore

$$x = (\sqrt[3]{A_+})_{\frac{\alpha}{3}} + (\sqrt[3]{A_+})_{-\frac{\alpha}{3}}$$
$$x = (\sqrt[3]{A_+})_{\frac{\alpha+4}{3}} + (\sqrt[3]{A_+})_{-\frac{\alpha+4}{3}}$$
$$x = (\sqrt[3]{A_+})_{\frac{\alpha+8}{3}} + (\sqrt[3]{A_+})_{-\frac{\alpha+8}{3}}$$

Since these expressions are all of the form $a_r + a_{-r}$ and since $a_r + a_{-r} = 2a\cos r$, Mourey deduces

$$x = 2\sqrt[3]{A_+}\cos\frac{\alpha}{3}$$
$$x = 2\sqrt[3]{A_+}\cos\frac{\alpha+4}{3}$$
$$x = 2\sqrt[3]{A_+}\cos\frac{\alpha+8}{3}$$

These x values are all clearly real, which Mourey duly notes. Now, the values of A_+ and α can be deduced from

$$A_{+\alpha} = \pm m + (\sqrt{n})_1$$

using the formulae

$$\tan\alpha = \frac{\sqrt{n}}{\pm m} = \frac{\sqrt{\frac{-q^2}{4} - \frac{p^3}{27}}}{-\frac{q}{2}}$$

$$A_+ = \frac{\pm m}{\cos\alpha} = \frac{-\frac{q}{2}}{\cos\alpha}$$

or, alternatively, using[375]

$$A_+^2 = m^2 + n = +\frac{q^2}{4} + \Big(-\frac{q^2}{4} - \frac{p^3}{27}\Big) = \frac{-p^3}{27}$$

from which it follows that

$$A_+ = \sqrt{\frac{-p^3}{27}} = \sqrt{\frac{p_+^3}{27}}$$

Finally, to find α substitute for A_+ in the following equation

$$\cos\alpha = -\frac{\frac{q}{2}}{A_+}$$

Mourey remarks that this method for solving the cubic can be applied to equations with imaginary or complex coefficients and that it might be extended to the solution of quartic equations.

Mourey's proof of the fundamental theorem of algebra

A modern statement of the theorem is, as we find in [159, p52]:

Fundamental theorem of algebra If $p(z)$ is a polynomial in z with coefficients in $\mathbb{C}$ then there is a number $c \in \mathbb{C}$ with $p(c) = 0$.

Mourey's statement of the theorem is: *Every equation has at least one root.*[376]

On his proof, Mourey writes in the preface: 'The proof, which is elementary and which seems to me very rigorous, is so general that it even encompasses the case

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[375]Typographical error: Mourey has $A_+^2 = m^2 + n^2$ which is incorrect. The rest of the equation, however, is correct. See [145, p74].

[376]'*toute équation a au moins une racine*'. [145, pviii]

where the coefficients are what we call imaginary.'[377] His proof is as follows.[378] It begins with a re-statement of the problem in terms of geometry.

The problem stated geometrically Given n fixed points situated in the plane $A, B, C, D, E, \ldots$ (Figure 5.12) find a point P also in the plane, such that the product of the paths $AP, BP, CP, DP, EP, \ldots$ is equal to a given path g.

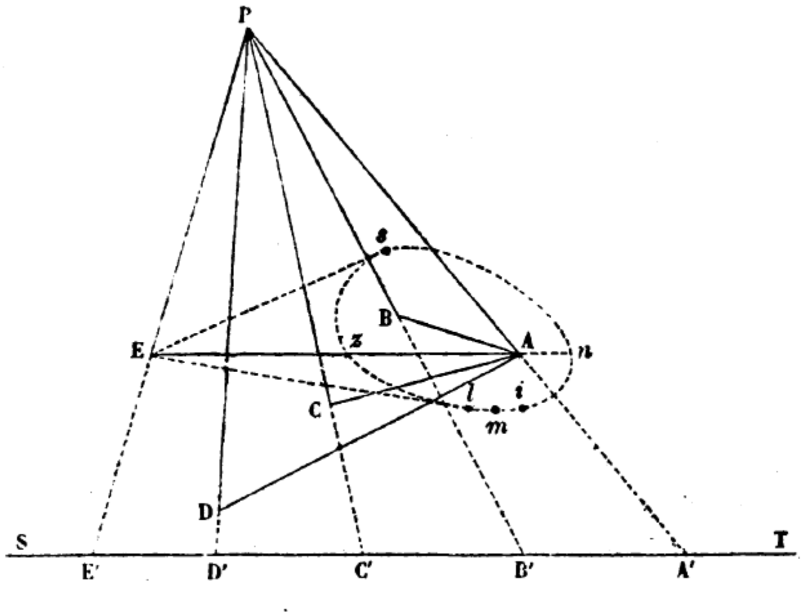

Figure 5.12: Mourey's geometrical interpretation of the F.T.A.[379]

Mourey writes

$$AP = x_{+u} \quad BP = x'_{+u'} \quad CP = x''_{+u''} \quad \text{etc., and} \quad g = g_{+r}$$

where x_+ denotes a positive path of the same length as x. With notation in place, he

[377]'La démonstration, qui est élémentaire, et qui me paraît très-rigoureuse, est tellement générale, qu'elle embrasse même le cas où les coefficients sont ce qu'on appelle imaginaries.' [145, ppviii-ix]

[378]A detailed summary of Mourey's proof of the F.T.A. also appears in [160, pp58–62].

[379] [145, p76]

is able to reduce the geometrical problem to the following system of two equations

$$x_+ \times x'_+ \times x''_+ \times \cdots \;=\; g_+ \qquad \text{(i)}$$

$$u + u' + u'' + \cdots \;\doteq\; r \qquad \text{(ii)}$$

The first equation says simply that the product of the lengths of AP, BP, CP, etc., must equal the length of g. The second requires that the sum of the *prime-directeurs* of AP, BP, CP, etc., must equal (modulo 2π) the *prime-directeur* of g. The *prime-directeur* of a directed line is the angle the line makes with the positive real axis.

Writing each of the paths BP, CP, etc., as a vector sum involving $AP = x$, Mourey has

$$BP = BA + AP = b + x$$

$$CP = CA + AP = c + x$$

$$DP = DA + AP = d + x$$

$$\vdots$$

etc.

which enables him to combine equations (i) and (ii) to form an algebraic equation of degree n in one unknown, x

$$x \times (b + x) \times (c + x) \times (d + x) \times \cdots = g$$

or, multiplied out,

$$x^n + (b+c+d+\cdots)x^{n-1} + (bc+cd+bd+\cdots)x^{n-2} + \cdots + (bcd\cdots)x - g = 0 \quad \text{(A)}$$

The solution of the geometrical problem is equivalent to the solution of this algebraic equation of degree n.

To prove the fundamental theorem Mourey has to prove:

First — '*every problem of this kind can be solved at least in one way*'[380]

[380]'*Tout problème de cette nature peut être résolu, au moins d'une manière*'. [145, p78]

Second — '*every equation with only one unknown quantity is the translation of a problem of this kind*'[381]

It is a proof in two parts. Following Mourey, we will consider his proof of each statement in turn.

Every problem of this kind can be solved at least in one way

In order to prove this first part of the proof, Mourey needs to show that it is always possible to satisfy the conditions expressed in equations (i) and (ii). He addresses condition (i) first, which concerns the product of the lengths.

Condition (i): $x_+ \times x'_+ \times x''_+ \times \cdots = g_+$
This condition is satisfied easily by fixing u (the direction of AP) in the direction of an arbitrary path AN. In modern vector notation $\overrightarrow{AN} = k\overrightarrow{AP}$. See Figure 5.13.

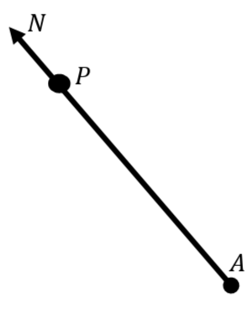

Figure 5.13: AP taken along AN.[382]

As P may be situated anywhere on AN, it is afforded continuous movement along the line. When P is at A, the product of the lengths is zero. When P is infinitely far from A, the product of the lengths is infinitely large. In between,

[381]'*Toute équation à une seule inconnue est la traduction d'un problème de cette nature*'. [145, p84]

[382]Figures 5.13–5.15 do not appear in Mourey.

the value of the product increases continuously: as the product—being a product of continuous functions of P—is itself a continuous function of P. Therefore, (by implicit use of the intermediate value theorem) there is some intermediate stage at which the product of the lengths $x_+ \times x'_+ \times x''_+ \times \cdots$ is equal to g_+. Note, Mourey proves neither the continuity of sums and products of continuous functions, nor that the paths AP, BP, etc., are continuous functions of P.

Now, since u is arbitrary, AN may be taken as any one of the infinitely many radii emerging from A (Figure 5.14). Along each radius there is a position for P such that condition (i) is satisfied. Joining all such points P on the radii, Mourey forms a curve δ (represented by *nszmn* in Figures 5.12 and 5.14) which surrounds A. Thus, all points on δ satisfy condition (i).

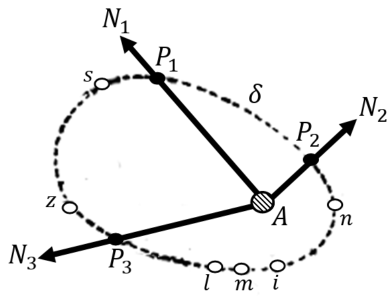

Figure 5.14: The curve δ.

Condition (ii): $u + u' + u'' + \cdots \doteq r$

Mourey wants to show that there is at least one point on the curve δ which satisfies condition (ii) which relates to the sum of the *prime-directeurs*. Again, his argument involves the non-algebraic concept of continuity.

He intends to move P in a continuous manner, in a positive sense, around the curve δ and show that during its progress, the sum of the *prime-directeurs* $u + u' + u'' + \cdots$ increases continuously from a value less than r to a value greater than r. From this, he will conclude—again by implicit use of the intermediate value theorem—that at some intermediate stage the sum $u + u' + u'' + \cdots$ equals r.

Mourey considers the following three cases, which relate to the position of A (the origin of the path AP) in relation to the curve δ. In each case, Mourey is looking for the change in the value of the *prime-directeur* of AP after a full revolution of P around δ. This is the contribution of the path AP to the sum of the *prime-directeurs*.

Case 1: A lies inside δ

Suppose that the position of A is fixed inside δ. The positions of P as it moves around δ are recorded in Table 5.1. When P returns to m (posn. 5), having travelled via n, s and z, the *prime-directeur* of AP will have described 4 positive right angles.

Position no.	*Position of P*
1	m
2	n
3	s
4	z
5	m

Table 5.1: Positions of P on δ for A inside δ.[383]

Case 2: A lies outside δ

Suppose that the position of A is fixed outside δ. (If referring to Figure 5.12 on page 153, take E as the point A.) The positions of P on δ are recorded in Table 5.2 (page 158) along with some indication of the value of the *prime-directeur* of AP (or EP) at that position. It is evident that after one full revolution, the *prime-directeur* has gained a balance of positive and negative contributions, which amounts to zero

[383] Tables 5.1–5.4 do not appear in Mourey.

overall.

Position no.	*Position of P*	*Prime-directeur*
1	m	negative
2	n	zero
3	s	positive
4	z	zero
5	m	negative

Table 5.2: Positions of P on δ for A outside δ.

Case 3: A is on δ

Suppose that the position of A is fixed at m on δ and the distance between l and i is infinitely small. The positions of P as it moves around δ are recorded in Table 5.3 (page 159).

As P moves from i to l (posn. $2 \to 6$), AP describes 2 positive right angles, so that the *prime-directeur* of AP increases from v say, to $v + 2$. But as P returns to i via m (posn. $6 \to 7$), a sudden jump occurs in the value of the *prime-directeur*, from $v + 2$ to v (or $v + 4$, as P has made a full revolution around δ). See Figure 5.15 (page 159). A gradual transition cannot be made from l to i without causing a jump in the value of the *prime-directeur* of AP, which in turn would affect a jump in the sum of the *prime-directeurs*.

Fortunately, this situation cannot arise, as Mourey explains. Suppose that one of the given points, C for instance, lies on δ. When P, in the course of its journey around δ, sits at C, the length of CP will be zero. Thus, condition (i) is satisfied for the trivial case, $g = 0$. Now, since CP has no length its direction u'' is undetermined, therefore, the sum of the *prime-directeurs* is also undetermined. Hence, case 3 can be discarded.

Position no.	*Position of P*
1	m
2	i
$\vdots$	$\vdots$
6	l
7	i

Table 5.3: Positions of P on δ for A on δ.

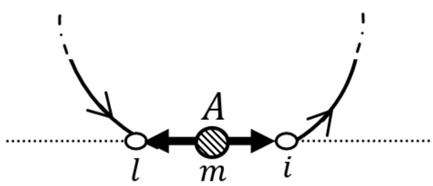

Figure 5.15: Discontinuity at m.

It still remains to prove that the sum of the *prime-directeurs* passes from a negative value to an infinitely large positive value. Mourey considers the case where A *is alone inside* δ. Table 5.4 (page 160) summarizes the contributions of AP to the sum of the *prime-directeurs* for each of the three cases discussed above, for A alone in δ.

Case	*Contribution to $u + u' + u'' + \cdots$*	
	After 1 revolution of δ	*After n revolutions of δ*
1	4 right angles	$4n$ right angles
2	0	0
3	———*discarded*———	

Table 5.4: Contributions of AP to the sum of the *prime-directeurs*, for A alone in δ.

Mourey's approach to ensuring that the sum of *prime-directeurs* $u+u'+u''+\cdots$ is negative, and therefore less than r, is to measure the *prime-directeurs* from left to right (contrary to convention) so that each *prime-directeur* is negative and

$$u + u' + u'' + \cdots = -s$$

After n revolutions

$$u + u' + u'' + \cdots = -s + 4n \to \infty \quad (n \to \infty)$$

(My notation.) Thus, the sum of the *prime-directeurs* will pass from $-s < r$ to infinity with n and, therefore, will be equal to r at some intermediate stage (again by implicit use of the intermediate value theorem). It follows that there is at least one point Q on δ such that if P is at Q condition (ii) is satisfied; and since Q lies on δ condition (i) is also satisfied. Hence, the geometrical problem has at least one solution and the proposition is proved.

Moreover, Mourey argues that since the procedure of constructing a curve δ can be repeated around each of the n fixed points A, B, C, etc., there are n solutions in total and equation (A) has n roots.

To prove this second statement, Mourey sets out to prove the equivalent statement: any equation of degree n in 1 unknown, x

$$x^n + px^{n-1} + qx^{n-2} + \cdots + sx + t = 0 \tag{B}$$

can be transformed into equation (A). Recall equation (A)

$$x^n + (b+c+d+\cdots)x^{n-1} + (bc+cd+bd+\cdots)x^{n-2} + \cdots + (bcd\cdots)x - g = 0 \tag{A}$$

Now, to transform (B) into (A) we need to find $b, c, d, \ldots$ etc., such that they satisfy the following system of $n-1$ equations

$$\begin{aligned} b + c + d + \cdots &= p \\ bc + cd + bd + \cdots &= q \\ \vdots \qquad & \quad \vdots \\ bcd\cdots &= s \end{aligned}$$

and set $t = -g$.

Mourey forms the following equation of degree $n-1$ in one unknown, z

$$(z-b) \times (z-c) \times (z-d) \times \cdots = z^{n-1} - pz^{n-2} + qz^{n-3} - \cdots \pm s = 0 \tag{C}$$

He reasons that if this equation has $n-1$ roots these roots will be $b, c, d, \ldots$ etc., the $n-1$ solutions of the system of $n-1$ equations; and having solved the system of equations, equation (B) can be transformed into equation (A) which completes the proof.

Mourey believes that he has proved something further: that any equation of degree n has n roots. He has shown that if an equation (C) of degree $n-1$ has

$n-1$ roots, then an equation (A) of degree n has n roots; and by recalling that any equation of degree 1 has 1 root, he proves—by implicit use of mathematical induction—that any equation of degree n has n roots for all values of n. Of course, if he proves this, then he also proves the lesser statement that any equation of degree n has at least one root. He explains that he did not set out to prove that any equation has as many roots as dimensions but that it came out naturally in his proof of the existence of at least one root.

Remark about Mourey's proof: lack of rigour in analysis

It is disappointing, perhaps, that Mourey is not as rigorous in analysis as he is in algebra; that his proof on the existence of roots relies on insufficiently precise notions of continuity. Perhaps our disappointment is borne out of our own unrealistic expectations.

The problem common to early proofs of the fundamental theorem is explained by Stillwell in his *Elements of Algebra*: 'Early attempts to prove the theorem were incomplete, mainly because they failed to reckon with the existential part of the proof—usually an application of the extreme value theorem or the intermediate value theorem.'[384] He cites among such early proofs, those by Laplace (1795) and Gauss (1816), which required the intermediate value theorem, and those of d'Alembert (1746) and Argand (1806), which required the extreme value theorem. It was close to half a century after Mourey (in 1874) when Karl Weierstrass (1815–1897) was able to provide rigorous proofs of the intermediate and extreme value theorems—once a definition of real numbers was secured—thus enabling complete and rigorous proofs of the fundamental theorem.[385] The definition of real numbers came by virtue of Richard Dedekind (1831–1916), in terms of Dedekind cuts, and Georg Cantor (1845–1918), in terms of Cauchy sequences.

[384] [159, p55]

[385][Ibid.]

Application of directed algebra to plane curves

In this section Mourey takes a kinematic approach to constructing curves. Imagine a curve drawn in the plane. From an arbitrary fixed point A in the plane (Figure 5.16) draw the path AQ, such that the point Q sits somewhere on the curve. Q is allowed to move but it must remain on the curve. As Q moves, the distance between A and Q will vary according to the parameter r which is the *prime directeur* of AQ. The curve (or, equivalently, the locus of Q) can be expressed in polar co-ordinates or the rectilinear equivalent. The simplest curve, the circle, is expressed in polar co-ordinates as $AQ = a_r$.

Now, AQ may be split up into a succession of paths (Figure 5.16). As the sum of two paths

$$AQ = AP + PQ$$

or

$$q_{+r} = x_{+s} + y_{+t}$$

To determine the nature of the given curve, Mourey says that we must establish the relationship between x and s, t and s, and y and t.

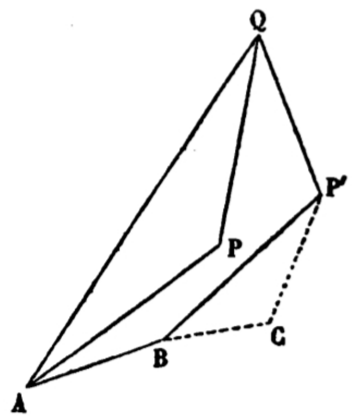

Figure 5.16: Application to plane curves, polar co-ordinates.[386]

[386] [145, p86]

In terms of rectilinear co-ordinates (Figure 5.17) Mourey has

$$AQ = AP + PQ$$

or

$$q = x + y$$

The procedure here is to fix the line $AP = x$ and find an expression for y in terms of x. Mourey considers a linear relationship between x and y and one expressed by the complete quadratic equation

$$y^2 + Ay + Bxy + Cx^2 + Dx + E = 0$$

which he simplifies, with a change of co-ordinates, to give

$$y^2 = a^2x^2 + b^2x + c^2$$

He considers three cases: for $a = 0$, $b = 0$, $c = 0$. Now, when the y's are mutually parallel the equation of second degree expresses a conic section; so for each of the three cases Mourey determines the conditions for parallel y's and identifies the conic section described by the locus of Q.

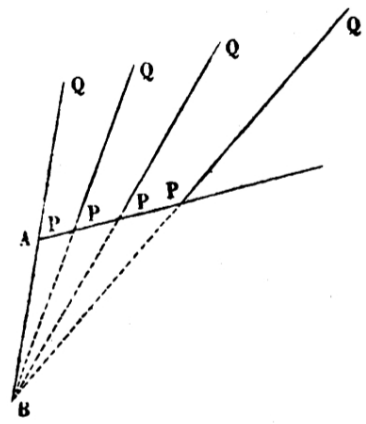

Figure 5.17: Application to plane curves, rectilinear co-ordinates.[387]

[387] [145, p89]

Of course, there is nothing new in a kinematic approach to generating curves. We need only think of the geometrical work of Descartes (1596–1650), l'Hôpital (1661–1704) and de Witt (1625–1672) two centuries prior.

Supplementary section

In his final section Mourey works through some additional examples to show how his theory might be applied to a variety of problems involving quantities of a variety of species. In the main body of the work he has already shown: how to track the gains and losses of a gentleman's fortune, by means of a path with a moving terminal point, and how to calculate durations of time, by representing indivisible instants of time as points on the real line. The problems covered in this supplementary section are of a similar nature.

5.3.4 Characteristic features of Mourey's mathematics

If called to identify two characteristic features of Mourey's mathematics, I might advance: (i) the interaction of algebra and geometry, and (ii) the modernity of his approach and his mathematical concepts.

Evidence of the interaction between algebra and geometry is everywhere in Mourey's work. We recall: he devises an algebraic expression for the geometrical operation of rotation, and algebraic definitions of addition and multiplication; he proves the distributive laws using vector methods; he translates the geometrical problem of addition into the solution of an algebraic equation, which is solved by trigonometric formulae deduced from a geometric figure; from algebraic equations he deduces the relative positions of two paths, etc. We encounter further examples still, in the applications of his theory.

Mourey's approach is a modern-algebra approach: he defines his set of directed lines and then his operations; he identifies the properties of these operations and then proves that they hold. Some of the concepts introduced or developed by him are sophisticated, innovative and far-reaching: the additive inverse, the equivalence relation, the geometric operator, etc.

Sometimes he comes across as pedantic, often his mathematics is too involved and not everything he does is rigorous enough by modern standards. However, Mourey believed earnestly that he was working things out properly for the first time. If we are mindful of this, then his shortcomings are easier to understand.

5.4 Notable references to Mourey

This section surveys the most notable references to Mourey. From the nineteenth century, I have collated all the references to Mourey that I have been able to find. From the twentieth and twenty-first centuries, I have only included those references which venture beyond a brief reference to Mourey in terms of the Argand diagram. Of particular significance are the references made by Warren in England, Lefébure de Fourcy and Liouville in Paris, and Hamilton and De Morgan in Ireland and England.

The information is also presented graphically in a timeline in Figure 5.18 (page 179). The timeline runs up until the end of the nineteenth century. It gives some indication of geography and of the noticeable gaps between references so that we might better understand the dissemination of knowledge of Mourey's contribution.

The Revd. John Warren, a Fellow of Jesus College, Cambridge, appears to have been the first to notice Mourey's contribution. The first individuals in Paris to refer to Mourey were associated with the École Polytechnique. They were: Louis-Etienne Lefébure de Fourcy (1787–1869), Examiner for Admissions, and Joseph Liouville (1809–1882), Professor of Analysis and Mechanics.

5.4.1 Louis-Etienne Lefébure de Fourcy (1787–1869): a biographical sketch

Lefébure studied at the École Polytechnique between 1803 and 1805. In 1807 he took up a position at the École as assistant *répétiteur* in analysis, progressing to *répétiteur* in analysis in 1813. Later, he taught applied analysis and descriptive geometry there. Between 1826 and 1861 he functioned as Examiner for Admissions. In December 1828 he was one of three examiners who refused Evariste Galois (1811–1832) entry into the École. Lefébure was also employed as a mathematics teacher at Louis-le-Grand, where he taught Victor Hugo, and as a professor of mathematics at the Collège Royal de Saint-Louis, where he taught Joseph Liouville. In 1825 he became assistant to Lacroix at the Faculté des Sciences, succeeding him in the Chair of Differential and Integral Calculus in 1843.[388]

5.4.2 Joseph Liouville (1809–1882): a biographical sketch

Joseph Liouville is remembered at the founder and editor of the *Journal de mathématiques pures et appliquées*. He established the journal in 1836 and remained as editor until 1874. The journal—which now serves as an important record of mathematical activity during the period—became known as the *Journal de Liouville*.

Liouville is also remembered for his connection with Evariste Galois. Liouville encountered Galois' mathematical writings in 1843. Galois had died eleven years prior, at the age of twenty-one; in a dual, at the hands of an 'unknown adversary'.[389] Liouville published some of Galois' work in the *Journal de mathématiques* in 1846. This brought the significance of Galois' work on the solvability of algebraic equations to the public's attention. Liouville thereby 'participated, indirectly, in the elaboration of modern algebra and of group theory'.[390]

Several parallels can be drawn between Mourey and Galois, with respect to: (i) chronology, (ii) geographical proximity and (iii) Liouville's publicity of their little-known work. In addition, it is not unreasonable to consider both as innovators in mathematics but to a much lesser extent in Mourey's case, of course.

Liouville was educated at the Collège Royal de Saint-Louis, where he was taught by Lefébure; the École Polytechnique (1825–1827) and the École des Ponts et Chaussées (1827–1830). In 1831 he took up a position as *répétiteur* in analysis and mechanics at the École Polytechnique. In 1838 he succeeded Mathieu as Professor in Analysis and Mechanics. He remained in this post until 1851. During this

[388]Sources of biographical information on Lefébure de Fourcy: [161] and [162].

[389] [163]

[390] [164]

period he also taught at a number of other institutions in the city: at the École Centrale des Arts et Manufactures, where he taught mechanics and mathematics (1833–1838), and at the Collège de France, where he substituted for Biot (between 1837 and 1843). In 1851 he took up Libri's chair at the Collège de France and in 1857 he succeeded Sturm as Professor of Mechanics at the Faculté des Sciences. He was elected to the Astronomy Section of the Académie des Sciences (1839) and to the Bureau des Longitudes (1840). His mathematical interests were truly diverse. Encompassing both pure and applied mathematics, they included: mathematical physics, mechanics, mathematical analysis, algebra, number theory and geometry.[391]

5.4.3 Dissemination of knowledge of Mourey's contribution

This is a survey of the reception of Mourey's work, which includes details of the author, publication and reference to Mourey.

1828 C.-V. Mourey [145]

The first edition of Mourey's work is published in Paris by Bachelier. On the title page Mourey's address is given as 'Paris [...] rue des Quatre-Vents, no. 8'.

1829 John Warren [166] [142, pp251–254] [143, pp339–340]

In December 1828 Warren communicates Mourey's proof of the F.T.A., with a correction, to the Cambridge Philosophical Society. His correction takes into account the possibility that two or more points might be enclosed by a single curve δ and, instead of proving the existence of n roots by geometry, he prefers to use the fact that the existence of n roots follows by polynomial induction once the existence of one root is proved.[392]

In 1829 Warren publishes two follow-up papers to his 1828 treatise. In the first

[391] Uncited sources of biographical information on Liouville: [165].

[392] If a polynomial $p(z)$ of degree n has a root c, then we can factorize $p(z)$ into $(z-c)q(z)$, where $q(z)$ is a polynomial of degree $n-1$, and then repeat the process with $q(z)$, and so on.

he brings to light the contributions of Buée and Mourey: he remarks on the similarity between his and Mourey's approaches to the subject and describes Mourey's proof of the F.T.A. as 'remarkably clear and satisfactory, and an example of the advantages which mathematicians may derive from a knowledge of the true theory of the quantities improperly called impossible or imaginary'.[393] Warren's second paper is inspired by Mourey. In his 1828 work [145, pp94–95] Mourey had alluded to the fact that—in a larger unpublished manuscript—he had represented certain forms geometrically as directed lines in the plane, namely $a^{\sqrt{-1}}$, $a_{\sqrt{-1}}$, $\sin\sqrt{-1}$, etc. Warren realized that if this was the case, then Mourey had discovered a geometrical representation of all algebraic quantities. As circumstances had prevented Mourey from publishing his manuscript in full, Warren was provided with an opportunity to further develop his own researches.

1834 Moritz Wilhelm Drobisch [167, pxvi]

Drobisch (1802–1896)—Professor of Mathematics at the University of Leipzig—publishes a textbook on the analytic and geometric properties of higher numerical equations. In the preface he cites Mourey (he writes 'Mouray') within a list of mathematicians who gave meaning to the expression $a + b\sqrt{-1}$.[394]

1835 Louis-Etienne Lefébure de Fourcy [169, pii]

Lefébure recalls Mourey's work in the preface to the 2nd edition of *Leçons d'algèbre*, in the context of the difficulties associated with the calculus of imaginaries. He reports Mourey's aim as being to free analysis from imaginary quantities entirely and remarks on Mourey's success in introducing into the calculus the two species of quantities referred to in analytical geometry as "polar co-ordinates", as a means of representing imaginaries.

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393 [142, p254]

394 Knowledge of Drobisch was acquired through [168]. I am grateful to Stefanie Eminger for translating Drobisch's preface into English.

1839 Joseph Liouville [170]

Liouville publishes a paper on Mourey in his *Journal de mathématiques.* He writes that mathematicians will have been reminded of Mourey's work by Lefébure's inclusion of a proof (by Liouville and Sturm) of Cauchy's theorem in the new edition of *Leçons de géométrie analytique* (1840). He explains that another proof of the theorem by Sturm [171] is based on the same lemma used by Mourey in his proof of the F.T.A. The lemma relates to the gains and losses, after a complete revolution, of the *prime-directeurs* of radii drawn from any point within, without or on, a closed circuit drawn in the plane. He goes on to provide a brief exposition of Mourey's proof of the F.T.A. which, for the benefit of the reader, he writes in language more familiar to algebraists: he believes that Mourey's original terms and notation are unworthy of replacing the established terminology and notation. His only other criticism of Mourey is in regard to the incompleteness of his proof: Mourey ought to have proved that the points P form a true curve, which would require him to prove that the radius vector AP varies continuously with its *prime-directeur* (i.e. the angle between the radius vector and the positive real axis). In addition, Liouville provides an alternative proof of the F.T.A. It is, however, nothing other than Gauss' third proof which was published in 1816.[395] Liouville does make reference to Gauss but by no means does he make priority clear.[396]

1840 Joseph Liouville [173]

Liouville publishes an addition to his 1839 paper, in which he attempts to deal with the technical issues which relate to the curve δ in Mourey's proof.[397]

1840 Louis-Etienne Lefébure de Fourcy [174, ppv–vi]

Lefébure refers to Mourey in the preface to the 4th edition of *Leçons de géométrie analytique.* He is writing to inform the reader about an addition made to the sec-

[395] For verification see [172, p99–102].

[396] A summary of Liouville's 1839 paper is given in [160, pp63–65].

[397] A summary of Liouville's 1840 paper is given in [160, pp65–66].

tion on analytical geometry in two dimensions: it is to include, for the first time, a proof of Cauchy's theorem on the number of complex roots of an algebraic equation situated inside a given contour. The proof is by Sturm and Liouville and it was published in the *Journal de mathématiques* in August 1836. On Mourey, Lefébure writes: 'Without the ingenious idea of providing a geometrical representation to the imaginary quantities, M. Cauchy's theorem could not exist; and with regard to this, it is my duty to recall here a most curious little work, published by M. Mourey, in 1828'.[398] The description of the work which follows is essentially the same as that given in *Leçons d'algèbre* (1835), except that here Lefébure notes that Mourey proves that an equation of degree m has m roots.

1845 Louis-Etienne Lefébure de Fourcy [175, pp216–219] [176, pp214–219]

Lefébure considers Mourey's mathematics in-depth for the first time; providing an exposition of the basics of his methods in the 5th edition of *Leçons d'algèbre*. The section on Mourey follows on from one detailing how to manipulate expressions of the form $a + b\sqrt{-1}$; taking powers and nth roots, etc. Lefébure intends to provide the reader with a brief summary of the way Mourey proposed to free analysis from imaginary quantities; sufficient for the reader to grasp the meaning of $\sqrt{-A^2}$ in the new algebra. In Mourey's system, he remarks, the expression $\sqrt{-A^2}$ represents perpendicularity rather than impossibility. It is significant—with algebraic reform in mind—that Lefébure credits Mourey with having reconstructed algebra in its entirety; by establishing the rules of algebraic calculus, then moving on to equations.

Lefébure extends his exposition of Mourey's methods in the 6th edition of *Leçons d'algèbre* (1850). Emphasis in this edition is on the agreement between the results of the new algebra and those of ordinary algebra—made possible by Mourey's notion of

[398]'Sans l'idée ingénieuse de donner une représentation géométrique aux quantités imaginaires, le théorème de M. Cauchy ne pourrait point exister; et la justice exige que je rappelle ici à ce sujet un petit ouvrage fort curieux, publié par M. Mourey, en 1828'. [174, ppv–vi]

verseurs. Lefébure hopes that his exposition of Mourey's new doctrine is sufficient to show that imaginaries would have presented few difficulties to mathematicians had they had, from the outset, as clear and precise a meaning as Mourey has given them.

It is interesting to query why a period of seventeen years elapsed before Lefébure wrote at length about Mourey. Perhaps the 1845 publication was the first opportunity that arose to include a longer discussion on Mourey. Perhaps he chose to wait until elements of the new doctrine became popular. I suggest the possibility that there may be a link with the position he assumed in 1843 as successor to Sylvestre François Lacroix (1765–1843) at the Faculté des Sciences.

When Lefébure became an Examiner for Admissions at the École Polytechnique in 1826, he was not able to continue teaching there because the roles of examiner and teacher were considered incompatible.[399] As a result of this, the new position brought with it the luxury of much free time, which Lefébure spent in writing textbooks for students of mathematics. His books proved remarkably popular: in effect, they provided the only means by which entrant students to the École could familiarize themselves with their examiner's mathematical interests and so they constituted invaluable revision material. They were, however, severely criticized for their simplicity and for the author's lack of originality and innovation.[400] In 1843 Lefébure succeeded Lacroix in the Chair of Differential and Integral Calculus at the Faculté des Sciences. I imagine that the role might have demanded improvements in the standard of Lefébure's publications: including an exposition of Mourey's methods would have been an effective way to add substance.

1845 Ambroise Faure [177, preface]

Faure (1795–1871)—Professor of Mathematics and Physics at the Collège de Gap and at the École Normale de Gap—publishes a book on the theory and interpretation

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[399] [161]

[400] 'Mais, écrira Larousse, ils sont "totalement dépourvus d'idées originales et de goût pour les innovations, même les plus légitimes".' [162, p2]

of imaginary quantities. In the preface he cites Mourey, along with Buée, Argand and Français, as one who attempted to find meaning in the abstract symbols of imaginaries. He states, for the sake of priority, that in 1828—before Mourey's book was published—his manuscript was seen by Joseph Fourier (1768–1830), Secretary of the Académie des Sciences.[401]

1846 Charles W. Hackley [178, pp244–246]

Hackley (1809–1861)—Professor of Mathematics and Astronomy at Columbia College, New York—publishes the textbook, *A Treatise on Algebra*. It is based on the finest English, French and German sources and contains 'all that is important in the higher parts of Algebra'.[402] It includes a translation of the section on Mourey from Lefébure's *Leçons d'algèbre* (1845).

1846 *Nouvelles annales de mathématiques* [147]

An appeal is placed for biographical information on Mourey.

1852–1862 William Rowan Hamilton and **Augustus De Morgan [179]**

Hamilton and De Morgan discuss Mourey at intervals in correspondence during this period. Hamilton's copy of Mourey is borrowed from De Morgan.

1858 Augustus De Morgan [180]

De Morgan publishes a paper on the existence of roots in which he provides a summary of Mourey's proof, with improved notation, and an algebraic substitute for a geometric step in the proof.

[401]Knowledge of Faure's reference to Mourey was acquired through [160, p67f(no.17)]. The same source [160, pp63–66] also provided additional insights into Léfebure (1835, 1840) and Liouville (1839, 1840).

[402] [178, piii]

1861 C.-V. Mourey [145]

Mourey (1828) is re-printed in Paris. The author's address is not given in this second edition. There are no other revisions.

1868 Abel Transon [181, pp193f, 199, 202]

Transon (1805–1876)—Examiner for Admissions at the École Polytechnique—publishes a paper on the application of directed algebra to geometry in the *Nouvelles annales de mathématiques*. Therein, he makes a number of brief references to Mourey. He recognizes in Faure's proof that every equation has at least one root—given in his 1845 work—the same issues that Liouville had called attention to regarding Mourey's proof. He addresses his argument for the existence of imaginaries to 'cet ami de l'Évidence', imitating Mourey.[403]

1870 Michel Chasles [182, p62]

Chasles' report on the progress of geometry is published. Therein, he refers to Mourey and Warren as re-inventors of the doctrine of Argand. He writes that Mourey has generalized the ideas of Argand and Buée.[404]

1886 P. G. Tait [146, p447]

Tait's article, 'Quaternions' is published in the 9th edition of the *Encyclopaedia Britannica*. Therein, he refers to Mourey and Warren as independent re-inventors of the new doctrine of imaginaries.

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[403] [181, p199]

[404] Michel Chasles (1793–1880): student at the École Polytechnique (1812–1815); Professor of Geodesy, Mechanics and Astronomy at the École Polytechnique (1841–1851) and Chair of Higher Geometry at the Sorbonne (1846–1880). A member of the Académie des Sciences (elected 1851); a Fellow of the Royal Society of London (elected in 1854, received the Copley Medal in 1865) and the first foreign member of the London Mathematical Society (elected 1867). His 1837 work, *Aperçu historique* ('Historical View of the Origin and Development of Methods in Geometry') established him both as a mathematician and an historian of mathematics. [183]

1887 Gino Loria [184]

Loria publishes a paper on the F.T.A. in *Acta Mathematica*, in which he remarks on the similarity between Mourey's proof of the F.T.A. and one published recently by a Norwegian mathematician named Holst.[405] He argues that the proofs are very similar in substance.[406]

1904 Elie Cartan [186, pp337–338, 341f(no.58)]

In Cartan's article on complex numbers—published in the *Encyclopédie des sciences mathématiques pures et appliquées*—he refers to Mourey as one among a number in the early nineteenth century whose work justified and legitimized the calculus of imaginaries. He makes an important distinction between the geometrical *representation* of complex numbers—where we associate $a + b\sqrt{-1}$ with a point (a, b) on the plane—and the geometrical *theory* of complex numbers—in which complex numbers are defined by vectors which are subject to defined operations. He sees Mourey's work as a rigorous exposition of the vector theory and notes that Mourey is aware of the need to define operations. Making the comparison with Argand and Français, Cartan notes that in Argand's work the distinction between representation and theory is not clear and that Français makes no reference to the properties of the operations such as commutativity, etc.[407]

1924 Julian Coolidge [188, pp26–27]

In his book, *The Geometry of the Complex Domain*, Coolidge includes Mourey in his account of the history of the representation of the binary domain. He remarks that Mourey 'writes with a notable exuberance [. . .] but [. . .] is by no means lacking in penetration and mathematical insight'.[408] He notes that Mourey has: (i)

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[405]For Holst's paper see *Acta*, volume VIII.

[406]Gino Loria (1862–1954): Italian mathematician and historian of mathematics; Professor of Higher Geometry at Genoa University (1886–1935). [185]

[407]Knowledge of Cartan's reference to Mourey came from [187, p724].

[408] [188, p26]

an awareness of the conditions for the equality of directed and non-directed lines; (ii) the notion of the directed angle and associated notation; (iii) the notion of a geometrical operator in the rule for multiplication; (iv) the nth roots of unity deduced from the rule for taking powers and (v) a proof of the F.T.A. He also remarks that nowhere does Mourey stress that any directed line may be represented by a linear combination of 1 and 1_1. I believe this is implicit in Mourey's remark that all directed numbers can be formed from the unit line by multiplication, division and *version*: recall that Mourey has for the unit line $1_r = \cos r + \sqrt{-1}\sin r$.

1929 G. Windred [189, pp538–539]

Windred publishes an article, 'History of the Theory of Imaginary and Complex Quantities' in *The Mathematical Gazette*. Therein, he describes Mourey's work as one of two 'notable contributions' to appear in 1828.[409] The other is Warren's. Like Coolidge, Windred notes that Mourey has the notion of a geometrical operator. In addition, he suggests that Mourey was perhaps the first to see the need for stating the conditions of equality for vectors and remarks that while Mourey's notation is no longer used, some of the problems he treated remain of 'distinct mathematical value'.[410]

1997 Gert Schubring [168, pp12–13]

Schubring contributes a chapter to *Le Nombre une hydre à n visages*. Therein, he refers to Mourey from the perspective of the history of the vector concept. Much of what he writes about Mourey I do not agree with, in particular: that Mourey persisted with the definition of subtraction in arithmetic as he moved into algebra and refused to accept a broader meaning of subtraction in algebra; and that, like Buée, he had success in geometrically (but not algebraically) bringing together the notions of length and direction. Still, two remarks of his are of interest: first, that he was unable to obtain any biographical information about Mourey and second,

409 [189, p538]

410 [189, p539]

his suggestion that Hermann Grassmann (1809–1877)—who worked on developing a general calculus of vectors from 1832—may have been influenced by Mourey's 1828 work, as it was cited in the work of Moritz Wilhelm Drobisch (1802–1896) who was a popular German mathematician of the period.

1997 Christian Gilain [160, pp58–66, 67f(no.17)]

The work done by Gilain in his chapter in *Le Nombre une hydre à n visages* constitutes the most in-depth study of Mourey's mathematics since the nineteenth century (apart from my own). He describes Mourey's proof of the F.T.A. in detail, following Mourey's text closely, and makes a brief comparison with Argand's proof: both have the same level of generality but Mourey's proof is longer and less straight-forward. He also makes some comments on the lack of rigour in Mourey's proof. A survey of the reception of Mourey's work in the nineteenth century is also provided. It covers: Warren, Lefébure, Liouville and Faure. A summary of Liouville's papers on Mourey also appears. Gilain remarks that Liouville made fewer changes to Mourey's proof than we would expect; notably, he retains the geometrical aspects of Mourey's proof. Gilain suggests that Sturm's proof of Cauchy's theorem may have been inspired by Mourey's work: Sturm may have acquired knowledge of Mourey through Lefébure's reference to Mourey in the preface to *Leçons d'algèbre* (1835).

2005 Gert Schubring [190, pp569–570]

In *Conflicts Between Generalization, Rigor, and Intuition*, Schubring gives his interpretation of Mourey:

> Mourey's publication confirms that the epistemological orientation was to reject any generalizability of algebra, and in claiming geometry to be the authoritative instance of mathematics that guaranteed sense and meaning. Mourey admitted subtraction only in arithmetic, provided that the subtrahend was smaller than the minuend. [...] Mourey was so radical in this that he excluded subtraction from the operations of algebra altogether.[411]

[411] [190, p569]

My interpretation of Mourey is quite different: I see him as a progressive, rather than as an opponent of change.

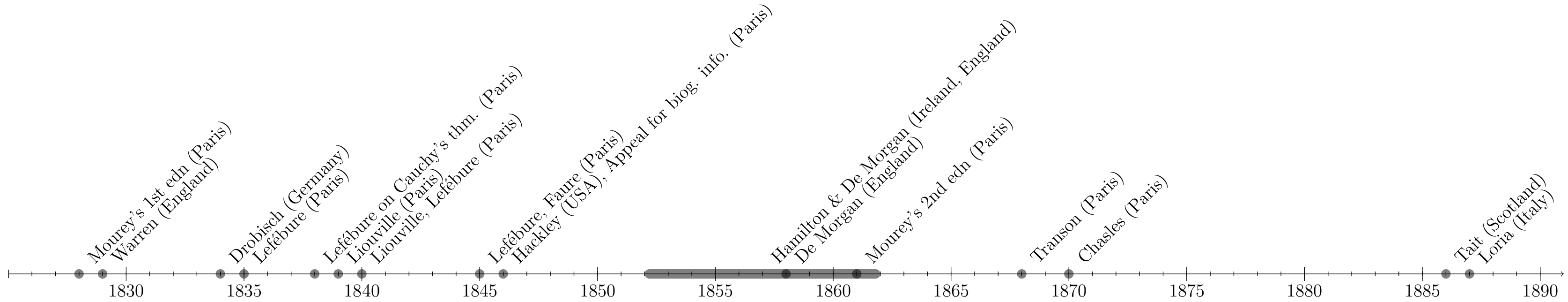

Figure 5.18: Timeline to show dissemination of knowledge of Mourey's contribution, with some indication of geography.

5.4.4 Hamilton and De Morgan's correspondence on Mourey, 1852–1862

At intervals during the ten-year period between 1852 and 1862, Sir William Rowan Hamilton and Augustus De Morgan (1806–1871) corresponded on C.-V. Mourey's 1828 work. The correspondence—which was published, in part, in Graves' *Life of Hamilton* (1889)—provides valuable information on: (i) the availability of Mourey's book and the distribution of copies; (ii) Hamilton's initial interest in Mourey and (iii) Hamilton and De Morgan's impression of Mourey's mathematics and their assessment of the evidence which might have suggested that Mourey had anticipated Hamilton in the discovery of quaternions.

The correspondence between the two men was initiated by De Morgan in 1841. In his first letter to Hamilton [179, p245] De Morgan reminded Hamilton of their first encounter, in London twelve years prior, and expressed an interest in Hamilton's theory of triplets which had featured in Hamilton's 'Essay' [191] in the *Transactions of the Royal Irish Academy*.

At the time of the correspondence, Hamilton was Andrews Professor of Astronomy at Trinity College, Dublin and De Morgan was Professor of Mathematics at University College, London.

The correspondence between Hamilton and De Morgan relating to Mourey: abridged and with editorial additions to establish context

Hamilton to De Morgan — 8 January 1852.

Hamilton is still working towards publication of his *Lectures on Quaternions*. In this correspondence he confesses to De Morgan his fear of having been 'too diffuse' and lays out before him plans to remedy the fault: he proposes to produce 'a copious table of contents' and 'a concise and readable preface' which might include some historical references.[412] In closing, he asks of De Morgan: 'Can you assist me to pro-

[412] H to DeM (8 Jan. 1852) in [179, p314].

cure Mourey, Paris, 1828?'[413] Clearly, Hamilton intends to include Mourey in his sketch of the historical developments which preceded his discovery of quaternions.[414]

De Morgan to Hamilton — 10 January 1852.

De Morgan writes to Hamilton to inform him that he is able to furnish him with a copy of Mourey. He writes:

> I can lend you Mourey; as you ought to see, with this letter. You will take care of it I know; and you may return it at leisure.
>
> Mourey would have been very remarkable if Warren had not appeared in the same year. [...]
>
> By and bye, when the French—tardily—begin to cultivate algebra as a science, they will declare that Mourey did it all. So I would not on any account lose Mourey.[415]

Hamilton to De Morgan — 13 January 1852.

In reply to De Morgan's satirical remarks about French priority, Hamilton refers to Servois' correspondence in Gergonne's *Annales* in which Servois had suggested the trinomial form $(p\cos\alpha + q\cos\beta + r\cos\gamma) \times (p'\cos\alpha + q'\cos\beta + r'\cos\gamma) = (\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 1$ but was unable to advance any further, not knowing what form p, q, r, p', q', r' would take.[416] Hamilton writes:

> Thanks for your promise to lend me Mourey, of which I shall take every care when it arrives; and the post to this place appears to be safe, though slow. I heard of it a good while ago, and shall be very glad to see it, though I fancy the book to be little else than double algebra.[417] If the French want an anticipation, though not a very

[413]H to DeM (8 Jan. 1852) in [179, p315].

[414]See [141, pp(31–32f)] for Hamilton's reference to Mourey in *Lectures on Quaternions.*

[415]DeM to H (10 Jan. 1852) in [179, p316].

[416]See [192, p235] for Servois' suggestion of the trinomial form.

[417]In using the term 'double algebra' to describe Mourey's mathematics, Hamilton is crediting Mourey with having constructed an algebraic system, which is something quite apart from ordinary arithmetic.

> *complementary* one! of the quaternions [see the word "absurdes"], I can point out what might with some plausibility be claimed by them as such.[418]

Hamilton then reproduces Servois' trinomial form and writes: 'You see that *I* solve his problem by $p = i$, $q = j$, $r = k$, $p' = -i$, $q' = -j$, $r' = -k$.'[419]

14 January 1852.

Continuing with the same letter, Hamilton acknowledges receipt of Mourey. He has already begun a careful consideration of its contents. He writes:

> P.S.—The Mourey has arrived, and shall be taken every care of. *In a sense*, I have already read it through, but must re-consider the proof of the existence of a root. I see that *you or* I—but I hope it will be you—must write, some time or other, a history of $\sqrt{-1}$.[420]

De Morgan to Hamilton — 15 January 1852.

De Morgan makes some remarks on Servois' trinomial form. He closes the correspondence—in reply to Hamilton's comment on 13 January—with confirmation that 'Mourey is *nothing* else but a *part* of double algebra'.[421]

Hamilton to De Morgan — 2 June 1852.

Hamilton updates De Morgan on his latest mathematical investigations and offers to lend to him Cauchy's 1825 memoir on definite integrals with imaginary limits, which relates to his (Hamilton's) current investigations. He suggests that he might return De Morgan's copy of Mourey at the same time as sending the Cauchy memoir; but, for the time-being, he wishes to hold on to them both.[422]

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[418]H to DeM (13 Jan. 1852) in [179, p316].

[419]H to DeM (13 Jan. 1852) in [179, p317].

[420]H to DeM (14 Jan. 1852) in [179, p317].

[421]DeM to H (15 Jan. 1852) in [179, p318].

[422]H to DeM (2 June 1852) in [179, p371].

Hamilton to De Morgan — 14 July 1854.

In this correspondence, Hamilton relates to De Morgan the progress he has made to date with Mourey. He has produced copious notes which constitute an abstract of the book, so that he might return Mourey but continue to discuss the work in correspondence with De Morgan. Hamilton is interested in Mourey's definitions and fundamental principles, not his applications of the theory. He is keen to discuss Mourey's concept of *version* and to address the issue of priority over geometrical operators. He writes:

> Having so long procrastinated about returning your Mourey, I wished to retain some notes of the work, which might assist my own memory, and enable me perhaps to write a little to you about it, when it shall be in your hands again, as I really hope it soon will be, for I have lately made nearly as full an abstract of the book, in one of my own manuscript volumes, as I wish to have at hand. In fact between extracts (copied in the French, for practice), abridgements, and comments of my own, I have already filled fifteen large pages of such a volume, and got as far as his solution of a cubic, so that not much more remains. I should like to consider, if I can spare some quiet hours for it, his proof (or alleged proof) that every algebraic equation has a root; but I care more for his conceptions, definitions, and notations, which I think that I now perfectly understand, than for such applications of his theory. What he says about version, versors, &c., *might* no doubt have set me on the track of the quaternions, if I had seen his book as early as I did Warren's; but I had not only formed my own general views, but had published the *name* and *sign* (i.e. *my* sign U) of the *versor of a quaternion*, at least as early as July, 1846, in the *Philosophical Magazine*, if not elsewhere before that date; and at a time when I had (certainly) not seen, and (I think) not heard of, the work by Mourey. The *conception* of the quaternion, as a *geometrical operator*, which at once turns (or *verts*) and stretches (or *tends*) a line, was familiar to me at least as early as 1844 [...] Warren, I think, was shy of putting *rotation* prominently forward, if at all (but I must look into his book again); with Mourey it is quite a *key*, for the *plane*, as with me for *space* ...[423]

[423]H to DeM (14 July 1854) in [179, pp488–489].

Hamilton's notebook MS.1492/95

The manuscript volume which Hamilton refers to in the above letter is extant in the Manuscripts Library of Trinity College, Dublin. Within MS.1492/95, a 13" ×8" leather-bound ledger with 287 pages, there are thirty-four pages on Mourey, written predominantly in 1854.[424]

Hamilton worked his way through De Morgan's copy of Mourey (1828) during the summer of 1854; copying extracts from the book in French and writing in English [in square brackets] when he came across some aspect of Mourey's work which he felt was worthy of remark. To highlight important sections, Hamilton drew a single or double vertical line in the margin of the notebook. Interspersed within the mathematics is the correspondence between Hamilton and De Morgan on Mourey copied into the notebook, usually by Hamilton from memory. Hamilton included the correspondence in the notebook so as to keep all his material on the same subject together, as was his practice.

Hamilton seems to have devoted more attention to the consideration of Mourey's definitions and principles, than to Mourey's applications of his theory. However, there are two exceptions: (i) inspired by Mourey's application of his theory to the solution of the cubic, Hamilton devised an alternative method using quaternions; and (ii) he discussed at length the merits and faults of Mourey's proof of the F.T.A. and was inspired to write his own paper on the subject, which is transcribed in Appendix F.[425] Hamilton never published the paper. The following letter, from Hamilton to De Morgan (17 July 1854), is a good example of the correspondence which passed between them concerning Mourey's proof.[426]

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[424]See MS.1492/95, pp. 241–274.

[425]For Hamilton's solution of the cubic using quaternions, see MS.1492/95, pp. 273–274.

[426]H to DeM (17 July 1854), MS.1492/95, pp. 259–261. This letter was not published in Graves' *Life of Hamilton*. I am grateful to the Keeper of the Manuscripts at Trinity College, Dublin for giving their permission to publish it here.

Obs July 17th 1854

My Dear De Morgan

I have pretty well satisfied my curiosity about Mourey's work, & a very few additional memoranda will suffice for any purposes of mine. It (the book) has only been taken up by me at some rare moments, but it appears to me to deserve to be better known. I think that Mourey really does prove I^{st}, that every eq^{n} of the form (A), page 104,

$$\begin{array}{r|r|l}
x^n + b & x^{n-1} + bc & x^{n-2} + ... + (bcd...)x - g = 0; \\
+c & +cd & \\
+d & +bd & \\
+... & +... &
\end{array} \qquad \text{(A)}$$

has at least one root, x, namely a certain line AP, if $b, c, d, \ldots$ & g denote given lines: though I don't admit the validity of his proposed proof, pages 113, 114, that *this* eq^n (A) has as many as n roots (comp. p.266.) However it was quite unnecessary that he sh^{d} attempt to do so, at that stage. He ought to have been content with proving, as I think he has fairly done, that "toute problème *de cette nature*", namely wh. conducts to an eq^{n} of the form (A), "peut être résolu, *au moins d'une manière*" (p.104); or in other words that an eq^{n} *of this form* (A) has always at least *one* root x. Let that one root be x_1; we may, as in all elementary books on Algebra, divide by $x - x_1$, & so depress the degree by a unit in the exponent. Suppose it be otherwise & previously known, $\mathrm{II}^{\mathrm{nd}}$, that an eq^{n} of this *lower* degree $n-1$, say his eq^{n} of p.115,

$$z^{n-1} - pz^{n-2} + qz^{n-3} - \cdots \pm s = 0, \qquad \text{(C)}$$

has always $n-1$ roots, $b, c, d, \ldots$ so that up to some *given* value of $n-1$, we are sure of the *existence* of $b, c, d, \ldots$ in the $n-1$ equations

$$\begin{aligned}
b + c + d + \cdots &= p \\
bc + bd + cd + \cdots &= q, \\
bcd \cdots &= s;
\end{aligned}$$

then the eq$^{\rm n}$ of the next higher degree, p.114,

$$x^n + px^{n-1} + qx^{n-2} + \cdots + sx + t = 0 \tag{B}$$

whatever may be the values of its coeff$^{\rm s}$ $p, q, \ldots, s, t$, takes the *form* (A), where $g = -t$, & $\therefore$ by his geom$^{\rm l}$ proof (which I have just now turned roughly into a more algebraic form), it has at least one root x_1, as above. Applying then *again* the II$^{\rm nd}$ premise, or admitting that there are always $n-1$ roots of an eq$^{\rm n}$ (C) of the $(n-1)^{\rm th}$ degree, we see that there are n roots in all, of the *eq^n* (A) of the $n^{\rm th}$ degree; & $\therefore$ of every eq$^{\rm n}$ of that degree.

Thus (tho' I know that you don't want these illustrations), the eq$^{\rm n}$ (C) is linear & has *one* root, $b = p$, when $n = 4$; $\therefore$ the gen$^{\rm l}$ quadratic eq$^{\rm n}$ $x^2 + px + q = 0$, (B), is reducible to the *form* (A), $x(x + b) - g = 0$; $\therefore$ it has *one* root x_1; $\therefore$ it can be depressed to the first degree, & $\therefore$ it has one other root x_2; the *gen^l quad^c* eq$^{\rm n}$ has $\therefore$ 2 roots, x_1, x_2. Again, this being admitted, the gen$^{\rm l}$ cubic $x^3 + px^2 + qx + r = 0$ may be put under the form $x(x + b)(x + c) - q = 0$; it has $\therefore$ by Mourey's principle, *one* root, & $\therefore$ may be depressed by div$^{\rm n}$ to a quad$^{\rm c}$ wh. will supply two other roots.

Thus all depends on (A) having *one* root; & Mourey has only (as I think) spoiled the statement of his argument, by seeking to show, in the latter part of p.113 & in the earlier part of p.114, by *geom^l* considerations, that (A) itself has n roots. Nor does he (as I conceive) succeed in showing this. He shows indeed that in discussing the eq$^{\rm n}$ $AP \cdot BP \cdot CP \cdot DP \cdot EP = g$, we may *select any one*, say A, of the given points A, B, C, D, E; describe a certain closed curve δ about it; & determine on that curve at least one position P, wh. satisfies the cond$^{\rm ns}$ of the question: and then commence anew with any other of the given points, as B, & find a point P' on the new curve, wh. likewise satisfies the conditions proposed. But he gives no reason for P & P' being *different.* The n curves δ might be conceived to intersect in one common point P. Therefore altho' "le problème peut être resolu de n manières" [c'est-a-dire par n *méthodes*, ou avec n origines differents, employés comme autant de points auxiliares] it is not yet proved, in this way, that the unknown line AP admits of n distinct *values*, or that the eq$^{\rm n}$ (A) has n roots. But one suffices. Yours very sincerely, W$^{\rm m}$. R$^{\rm n}$. Hamilton.

Correspondence in *Life of Hamilton* cont.

Hamilton to De Morgan — 23 November 1857.

In a letter to Hamilton on 20 November, De Morgan had claimed to have discovered a proof that every equation has a root; a proof which he described as 'so elementary that it must be *the* proof in future'.[427] Hamilton writes in reply:

> I am very glad to hear that you think you have discovered *the* proof of the general existence of a root of an algebraic equation. Notwithstanding all that has been done by others, it was (for perfect repose to the mind) a thing still to be sought for. Perhaps Cauchy's proof came nearest to satisfying me, of any which I have tried to examine. But even Mourey's method, with some correction of details, was not (I thought) a total failure. You understand, therefore, that I am not asserting (or admitting) the insufficiency of all known proofs, when I say that I look forward with much interest to the communication of some *new* and *simple* proof by you.[428]

De Morgan to Hamilton— 27 November 1857.

Thus reminded of Mourey's proof, De Morgan is encouraged to revisit it and makes the comparison with his own. He writes: 'Your mention of Mourey's proof—which I had forgotten the existence of—made me look at it again, and I find no affinity with mine, which is of the Cauchy family, and would have struck Cauchy, if he had lived, I suppose.'[429]

The same date, later.

De Morgan writes:

> I have been looking at Mourey's proof to-day, *i.e.* of the existence of the root of an equation. It is perfectly sound, and is, I feel sure, *the* double algebra proof. But it wants a little *intelligiblization* to suit it to modern notions. I never did more than

[427] DeM to H (20 Nov. 1857) in [179, p530].

[428] H to DeM (23 Nov. 1857) in [Ibid.].

[429] DeM to H (27 Nov. 1857) in [179, p532].

glance at it; but I now see I wanted, what I have been doing lately, to get it *glanced* into me. I shall get it into my paper on the subject.[430]

De Morgan's paper on the existence of roots

Just over a week after this correspondence (on 7 December 1857) De Morgan presented his paper on the existence of roots to the Cambridge Philosophical Society. From an examination of Sturm's proof of Cauchy's theorem on imaginary roots [171], De Morgan had been led to a demonstration of the existence of roots which he believed to be 'the natural prefix to Sturm's demonstration'.[431] He found, in addition, an extension of Cauchy's theorem. Both are incorporated in his paper, which also includes a consideration of the proofs by Argand and Mourey on the existence of roots.

De Morgan had issue with Mourey's notation and with his reliance on geometry in his proof of the F.T.A. Unimpressed by Mourey's original notation, he attempted to introduce improved notation in his summary of Mourey's proof. He also felt that it was necessary to supply an algebraical substitute for the geometric step in Mourey's proof; the step relating to the change in the *prime-directeurs* of the radii.[432] De Morgan's view with regard to mathematical proofs was that one should to be able to fill any gaps which appear in the proofs using algebra; otherwise, the proof is not algebraic, nor does it have the potential to be so. He believed that a proof ought to be rigorous and based on accepted principles that are closely related to the theorem in question. Mourey's proof of the F.T.A. has gaps: it is intuitive, rather than rigorous, on issues of continuity. Had Mourey demonstrated far more rigour in analysis, then perhaps De Morgan would have been convinced that the geometrical arguments in the proof were worthy of being retained and might, in

[430][Ibid.]

[431] [180, p261]

[432]For details of De Morgan's substitution see [180, pp264–265].

fact, do just as well as algebra. It has already been remarked (page 162) that it is unreasonable to expect this.

Peacock's criticism of Mourey

De Morgan's views on Mourey were clearly influenced by the Revd. George Peacock (1791–1858) who had taught De Morgan at Trinity College, Cambridge. In his paper on the existence of roots, De Morgan cites Peacock's criticism of Mourey from his 1834 report (quoted below). Note that Mourey's name is misspelt and the year of publication is also incorrect. Peacock writes:

> It is not very difficult to establish this fundamental proposition [the existence of roots of equations] by reasonings derived from the geometrical representation of impossible quantities. This was done, though imperfectly, by M. Argand [...] and has been since reconsidered by M. Murey [*sic*], in a very fanciful work upon the geometrical interpretation of imaginary quantities, which was published in 1827 [*sic*]. It seems to me, however, to be a violation of propriety to make such interpretations which are conventional merely, and not necessary, the foundation of a most important symbolical truth, which should be considered as a necessary result of the first principles of algebra, and which ought to admit of demonstration by the aid of those principles alone.[433]

To think that the F.T.A. can be proved by algebraic principles alone is unrealistic. We now understand that the theorem is really about complex numbers and so we expect proofs to involve concepts from analysis where complex numbers are defined.

It is not surprising that Peacock is critical of Mourey. Like Mourey, Peacock was concerned with the fundamental principles of algebra and was determined to reform algebra in order to overcome the problems associated with negative and complex numbers; however, his understanding of how generalization ought be introduced into algebra—through his Principle of the Permanence of Equivalent Forms—was rather more limited than Mourey's.[434]

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[433] [193, p305]

[434] Peacock's Principle of Equivalent Forms allowed the operations of arithmetical algebra to be adopted into symbolic algebra. See [194] for a paper on Peacock's development of symbolic

Correspondence in *Life of Hamilton* cont.

Hamilton to De Morgan — 4 January 1858.

In a letter to Hamilton on 1 January, De Morgan wrote of his recent interest in Argand and remarked on the similarity between Argand and Cauchy's proof of the existence of a root. In reply, Hamilton writes:

> Your remark about Argand's investigation respecting roots of equations, which I never had time to read, reminded me that I had said something hastily, which might seem depreciating as regarded Mourey's analogous investigation. I thought the latter, which I *did* read (in 1854), quite satisfactory, so far as concerned the existence of *one* root; and that he ought to have stopped there, and referred the *rest* to *algebra*.[435] But he went on to make the superfluous effort, in which I thought he failed, to prove *all the n roots by geometry*.[436]

Hamilton to De Morgan — 25 March 1862.

Hamilton has returned Mourey to De Morgan, after borrowing it for a second time; though, apparently, he has not returned all of it! He writes: 'I trust that you have received your "Mourey", minus a paper *wing*, which may yet be found'.[437]

De Morgan to Hamilton — 1 April 1862.

De Morgan acknowledges receipt of Mourey, which affects the end of their correspondence relating to him. He writes: 'I have received the Mourey. Never mind the cover. *Uno avulso non deficit alter.*[438] I am thoroughly embusinessed for a few

algebra, with respect to his *Principles of Algebra* (1830) in particular. The paper also contains a short discussion on Hamilton's initial criticism of Peacock's symbolic algebra and his invention of quaternions as an exercise of algebraic freedom.

[435]By 'referred the *rest* to *algebra*' Hamilton meant that once the existence of one root is proved, the existence of all the other roots follows by polynomial induction.

[436]H to DeM (4 Jan. 1858) in [179, p541].

[437]H to DeM (25 March 1862) in [179, p577].

[438]This is a quotation from the 6th book of Virgil's *Aeneid*. It translates as: 'When one is removed, there is no shortage of another.' I am grateful to my brother, Benjamin for the reference and

days—after which at you again.'[439]

Two years later Hamilton returned to the subject of Mourey, in correspondence with his personal friend, the mathematician, Dr. Andrew Hart (1811–1890). At the time of the correspondence, Dr. Hart was a Senior Fellow of Trinity College, Dublin.

Hamilton to Hart — 30 June 1864.

It appears that Hamilton has acquired a copy of the second edition of Mourey. Once again he is concerned over priority. He writes:

> In driving home yesterday I looked into the Second Edition (Paris, 1861) of Mourey's very ingenious little work. It was lucky that I could at once supply Salmon with a reference to the page of my Preface to the *Lectures*, in which I had cited the First Edition (Paris, 1828). But it is *foolish* to consider any such work as an *anticipation* of the quaternions. This brilliant and patriotic notion occurred lately to a French correspondent of our friend Salmon, who was so good as to send me the letter to read. The relation is rather of *contrast* than of *resemblance*, as in this very note to you I partly show.[440] Systems which interpret +1 differently cannot have much in common. I forget in what *year* it was that I first heard of Mourey; nor is it of the slightest importance. As long ago as 1829 my attention was called by John T. Graves to the work of Mr. Warren, published in Cambridge, in 1828, *On the Square Roots of Negative Quantities.* The systems of Warren and Mourey, *both* published in 1828, are substantially the same; though the Frenchman was livelier and smarter. What was best in both had been anticipated in France by Argand.[441]

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translation.

[439]DeM to H (1 April 1862) in [179, p579].

[440]See MS.1492/195, pp. 57–59, in the Manuscripts Library of Trinity College, Dublin, for Hamilton's comparison of the systems: Graves chose not to include this section of the correspondence in *Life of Hamilton.*

[441]H to Hart (30 June 1864) in [179, pp189–190].

Salmon's French correspondent

Hamilton became acquainted with the Revd. George Salmon (1819–1904) in 1841, when Salmon joined the staff in the mathematics department at Trinity College, Dublin.[442] From the letters which passed between Hamilton and Salmon, we know that the 'French correspondent', referred to above, was a M. Kuntz who was a student at the Lycée Charlemagne in Paris. He had written to Salmon on 8 June 1864 with the intention of bringing Mourey's work to Salmon's attention. He had read Salmon's textbook, *A Treatise on the Analytic Geometry of Three Dimensions* (1862), in which Salmon had included a section in the appendix on Hamilton's quaternions. Kuntz believed that while Hamilton might be credited with having invented the terminology associated with quaternions—scalars, vectors and tensors—it was Mourey who deserved priority as the first to have the concept of directed lines in the plane.

Salmon sent Kuntz's letter on to Hamilton; telling Hamilton, in an accompanying letter, that he had never heard of Mourey before. Kuntz's letter and the subsequent related correspondence between Hamilton and Salmon were copied into another of Hamilton's notebooks, which again is preserved in the Manuscripts Library of Trinity College, Dublin.[443]

On 11 June Hamilton replied to Salmon:

> Mourey however was by no means the *first* to treat of *directed lines*, [at least within the plane] [...] Argand (1806–1813) was decidedly prior to him. [Your correspondent may take comfort from the thought that Argand also was a Frenchman.][444] [...] The *thing* (or *thought*), which I express by the *word* "*vector*", was *never* claimed by me

[442] Salmon was appointed to a divinity lectureship at Trinity College, Dublin in 1845 and as Donegall Lecturer in Mathematics in 1848. [195]

[443] See MS.1492/195, pp. 1–9 in the Manuscripts Library of Trinity College, Dublin. Kuntz's letter was copied into the notebook by Hamilton's wife, Helen. Again, I am grateful to the Keeper of the Manuscripts at Trinity College, Dublin for giving their permission to publish this correspondence.

[444] We now know that Argand, though resident in Paris, was of Swiss nationality.

as my own: nor can I, on the other hand, admit that Mourey, or his predecessor Argand, had *in any degree anticipated the quaternions.*

Vectors *might* have been familiarly known for two or three thousand years, instead of merely fifty or sixty, and the world been as far off as ever from quaternions.[445]

Hamilton returned Kuntz's letter to Salmon on 15 June. Initially, Hamilton had intended to respond publicly to the young Frenchman in the *Philosophical Magazine*. However, Salmon reassured him that such action was not necessary and explained that he had already dealt with the matter by writing to Kuntz personally:

My Dear Sir W^{m},

I do not think it is necessary for you to take any public notice of my French Correspondent's letter, who for all I know may be a young man of no mark.

If his criticisms had appeared anywhere in public, the case would be different.

I wrote telling him he ought not to judge of your system by the meagre sketch which was appended to my book on surfaces, but that he ought to consult your own Lectures. I told him of the reference you had given to Mourey's work: & I availed myself of your hint & in telling [him] that Mourey had been anticipated made the communication more pleasant by speaking of his "Countryman" Argand.

Finally I added that nothing that had been done by Mourey, Argand or anyone else for the Geometry of the Plane could be considered as an anticipation of Quaternions as I thought my correspondent himself would fully recognize if he made himself better acquainted with the latter system.

I remain
very sincerely yours
Geo. Salmon.[446]

Hamilton thanked Salmon for his conduct but explained that he was not content to let the matter rest and would return to the issue in future correspondence with Salmon.

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[445]H to S (11 June 1864), MS.1492/195, pp. 3–4.

[446]S to H (16 June 1864), MS.1492/195, pp. 8–9. It appears that Hamilton never published on the matter in *Phil. Mag.*

De Morgan and the history of mathematics

De Morgan had a keen interest in the history of mathematics. Cajori writes: 'Few contemporaries were as profoundly read in the history of mathematics as was De Morgan.'[447] Smith writes: 'He devoted considerable attention to the history of mathematics, but his articles are not only eccentric but unreliable.'[448]

De Morgan's interest in the history of mathematics led to a number of publications, in: Smith's *Dictionary of Greek and Roman Biography* (1862–1864); the *Penny Cyclopaedia* (1833–1843), published by the Society for the Diffusion of Useful Knowledge; the *Companion to the Almanac* (various years) and his *Arithmetical Books* (1847). This last has been described as 'probably the first significant work of scientific bibliography'.[449]

De Morgan's collection of scientific books—which comprised around three thousand titles—was bought after his death by Lord Overstone who then presented the collection to the library of University College, London.[450] This is the current location of De Morgan's 1828 edition of Mourey. The University of London's Senate House Library also has a copy of the first edition of Mourey: this particular copy once belonged to John T. Graves. It is peculiar that Hamilton had asked De Morgan to help him acquire a copy of Mourey when his friend, John Graves owned a copy. The British Library also has a copy of the first edition of Mourey.

5.5 Benefits of the algebraic perspective

In studying Mourey from an algebraic perspective we have derived a number of advantages. The true focus of Mourey's researches has come to light and we have seen more readily: (i) evidence of a logical, rigorous and even axiomatic approach to his

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[447] [196, p316]

[448] [197, p462]

[449] [198]

[450] According to De Morgan's entry in [13].

development of the new algebra and (ii) implicit evidence of what Wussing terms 'group-theoretic thought'.[451] We also find ourselves in a better position to appreciate the relevance of Mourey's work to today's mathematicians and historians of mathematics: Mourey's mathematics is about the fundamental principles of algebra which is the foundation of our reasoning in the subject; it is also a fresh example of the interaction of number theory, algebraic equations and geometry, which have been described by Wussing as the 'three historical roots of abstract group theory'.[452]

I am not suggesting that Mourey ought to occupy a place in the history of the development of algebra or group theory. I believe his work to be, in some respects, an isolated incident that had little or no impact on contemporary developments: it attracted no real publicity; notably, it failed to achieve the level of publicity which Mourey himself predicted, in the form of opposition to his radical new system.

What I do hope for is that the reader may now appreciate the following. (i) Mourey was concerned with establishing a rigorous foundation for algebra. (ii) He understood that deficiencies in contemporary algebra were inhibiting progress. (iii) He had a clear sense that arithmetical algebra had to be left behind and that rigour, generalization and a higher level of abstraction had to be embraced. He understood that this was key in reviving algebra and driving it forward. The fact that Mourey was an innovator, ahead of his time, is plain in his mathematics. With the benefit of hindsight, historians of mathematics can now recognize the quality and far-sightedness of some of Mourey's mathematical ideas.

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[451] [199, p17]

[452] [199, p16]

Overall final remarks

To conclude this thesis, I will suggest some of the ways in which it might contribute to the history of mathematics and, more generally, to the history of science.

— I have highlighted the existence of primary source material relating to Tait, facilitating fellow researchers in their efforts.

— Through original research, I have supplemented the existing literature on Tait, providing new insights into: Tait's family history; his unique contribution to the ever-relevant science-versus-religion debate; his involvement in the fields of probability and statistics; and, with respect to the history of the Argand diagram, Bishop Terrot's work on the geometrical representation of complex numbers.

— In addition, I have given an identity to the elusive Frenchman, C.-V. Mourey who, for the past 186 years, has remained an unknown to historians of mathematics. I have re-examined his 1828 work and have brought to light an unpublished paper by Sir W. R. Hamilton on the F.T.A. which was inspired by Mourey.

In covering a variety of aspects of Tait's life and work, it is hoped that this thesis has given an indication of the vast scope of his expertise and interests, and has appealed to the varied interests of a diverse readership. The intention was to stimulate interest in Tait, who has been unfortunately overlooked in recent times.

Seldom in a generation does it fall to the lot of the scientific worker to become a pre-eminent leader in his subject; and not often in the course of the centuries is such a leader also pre-eminent as a clear and original writer, as an unusually effective lecturer, and as an exerter of powerful personal influence. Each of these qualifications by itself would be sufficient to make its possessor a man of distinction. He who possesses them all is truly great as a worker and teacher, great also as a man when his personal aims and influences are all in the direction which he deems to be highest and best. Other men gather the long record of his work which he leaves behind, and it is passed on to help the work of other ages. His works do not cease with his life, they follow him. His influence continues to spread too, if in diminished stream, unavoidably and unconsciously, by means of those who have come under the compulsion of his personality. He exerts compulsion, not on his pupils merely, but on his compeers also; and through this his influence redoubles itself. From the record of his life and work, men who never knew him may come under the magic spell and be enrolled in the list of his disciples. Such a man was Peter Guthrie Tait, Natural Philosopher.[453]

[453] [200, p1] This quotation is taken from an obituary tribute to Tait written by Professor William Peddie (1861–1946): assistant to Tait (1883) and lecturer in natural philosophy (1892–1907) at Edinburgh, later Professor of Physics at University College, Dundee (1907–1942) and President of the Edinburgh Mathematical Society (1896–1897). Peddie gives a good account of Tait's contributions to mathematics and physics, with particular emphasis on Tait's experimental work in his Edinburgh laboratories and his influence as a teacher. I found the seven-page booklet amongst the Papers of the R.S.E., in a folder containing documents relating to the Tait Memorial Fund: perhaps the booklet was circulated to gather support for the Tait Memorial Movement. A copy of the booklet is also preserved in Tait's scrapbook.

Bibliographical essay

In addition to the three primary unpublished sources on which this thesis is based—Tait's scrapbook, the Tait–Maxwell school-book and Tait's pocket notebook—I have made use of a variety of archival material and a number of secondary sources.

For information on Tait's family history, I consulted the digitised family history records for Scotland, which are available at The ScotlandsPeople Centre, Edinburgh. This supplemented information available from *ancestry.co.uk* and from a genealogical table for the Ronaldson family (MS.36998/38, Special Collections, the University of St Andrews) which was presented to the University of St Andrews by Tait's granddaughter, Miss. Margaret Tait on 26 April 1974. I was also privileged enough to have personal communication with Susan Rutherford (Tait), paternal great-granddaughter of P.G.T.

At the Edinburgh Academy archive, I found a number of documents relating to Tait's schooldays. The following proved especially useful. *The Edinburgh Academy List, 1824–1974* is a record of the pupils and staff at the Academy during the period between 1824 and 1974. It was published by the Edinburgh Academical Club in 1974 for the 150th anniversary of the Academy. The Academy's *Prize List* is a published list of pupils receiving prizes at the Academy's Public Exhibition Day—the school's annual prize-giving event in July, which was first held in 1825. The pupils' prize-winning poetry is also published alongside the list of prize winners. The *Annual Report* of the Rector and Directors contains information on the general running of the school; finances, staff appointments, etc. *The Edinburgh Academy Chronicle* is the school's newsletter. It was launched in 1893, with a number of editions published per year. It features: important school events (including guest lectures, sports matches, etc.), news on former pupils, literary articles written by current pupils, etc.

At the National Library of Scotland in Edinburgh, I consulted the Papers of the R.S.E. (Acc. 10000), giving special consideration to: the minute books, letter books, bound volumes of visitors to the R.S.E. and two folders of documents relating to the Tait Memorial Fund (Acc.10000/166,167). Acc.10000/166 contains documents relating to the setting up of the Tait Memorial Committee, decisions about a fitting memorial to Tait and the raising of funds. Acc.10000/167 contains correspondence between the R.S.E. and Tait's son, William Archer Porter Tait, and between the R.S.E. and the University of Edinburgh. The correspondence relates to the investment and distribution of donations.

At the Manuscript Library of Trinity College, Dublin, I consulted Hamilton's notebooks. Amongst the library's collection of two-hundred Hamilton notebooks, MS.1492/95 and MS.1492/195 were of particular interest in relation to C.-V. Mourey. MS.1492/95 is a 13"×8" ledger with 287 pages. It contains: a copy of Buée's 1806 paper, copied into the notebook by Hamilton's sister, Eliza, together with some remarks by Hamilton; a consideration of the work of Carnot and Grassmann; and thirty-four pages relating to Mourey (pages 241–274). MS.1492/195 is another ledger notebook. It contains: correspondence between Hamilton and the Revd. George Salmon, Dr. Andrew Hart and others, including Tait. On the inside cover of the notebook (page iii) Hamilton recorded Tait's address at Greenhill Gardens, Edinburgh. The correspondence with Salmon and Hart is principally on Mourey and concerns priority in the discovery of quaternions.

In search of biographical information on Mourey, I went to the Archives de Paris, at 18 boulevard Sérurier 75019 Paris. I consulted the ledger volumes that contain: information on property owners in Paris (their name, occupation etc.); property valuations and a record of when properties changed hands and for what reason (e.g. through sale or succession). The addresses I had for Mourey were in volume nos. DQ18.206, DG18.323 and DQ18.364. I also consulted microfilmed copies of the reconstituted civil records for the period between the C16th and 1859.

The principal secondary source relating to P. G. Tait is the scientific biography written by C. G. Knott (1856–1922), published in 1911: Cargill Gilston Knott, *Life and Scientific Work of Peter Guthrie Tait* (Cambridge : at the University Press, 1911). Knott was a former student of Tait's at Edinburgh and his research assistant in the natural philosophy department between 1876 and 1883. His biography of Tait was described by Sir Edmund Taylor Whittaker (1873–1956), Royal Astronomer of

Ireland, as "'one of the best scientific biographies ever written"'.[454]

Knott's biography supplemented the two volumes of Tait's scientific papers which were published in 1898 and 1900 by Cambridge University Press: Peter Guthrie Tait, *Scientific Papers*, 2 vols. (Cambridge : at the University Press, 1898 and 1900).

A bibliography of Tait's published works is given in Knott's biography and in a paper compiled by Chris Pritchard for the Tait Centenary meeting at the R.S.E. in 2001:'Provisional Bibliography of Peter Guthrie Tait', ed. Chris Pritchard on behalf of the Royal Society of Edinburgh, a paper presented at *Peter Guthrie Tait* (*1831–1901*): *Centenary Meeting*, July 2001; The Royal Society of Edinburgh. <http://www.maths.ed.ac.uk/∼ aar/knots/taitbib.htm>. Chris Pritchard has also published a number of papers on Tait's involvement in knot theory, golf science and the promotion of quaternions. For details consult the References section at the end of this thesis.

454 [33, p49]

Appendices

A FAMILY TREE REPORT FOR THE TAIT FAMILY

Information on the Tait family has come from a variety of sources: personal communication with Susan Rutherford, paternal great-granddaughter of P.G.T., searches at The ScotlandsPeople Centre in Edinburgh; information available through *ancestry.co.uk* and that contained in a genealogical table for the Ronaldson family (MS.36998/38, Special Collections, the University of St Andrews) which was presented to the University of St Andrews by Tait's granddaughter, Miss. Margaret Tait on 26 April 1974. Only sources other than these have been referenced in the report. Readers may find it helpful to consult the genealogical tables in Figures A.1 and A.2 (pages 218–219).

Table A.1: Family group report for the Tait family.

Peter Guthrie Tait	
Father	John Tait (1785–1837) Private Secretary to the 5th Duke of Buccleuch, Walter Francis Montagu Douglas Scott (1806–1884). Son of Patrick Tait (c.1746–1819), a shoemaker who died in Selkirk, and Margaret Guthrie (b. c.1747), daughter of Robert Guthrie of Selkirk. Married Mary Ronaldson on 27 June 1829 in Dalkeith, Midlothian.
Mother	Mary Ronaldson (1795–1846) Daughter of John Ronaldson (1756–1820) (a tenant farmer in Sauchland, Crichton, Midlothian) and Ann(e) Turnbull (d.1837) who were both buried in Borthwick cemetery, Midlothian.
Birth	28 April 1831. Dalkeith, Midlothian.
Baptism	10 June 1831. Dalkeith, Midlothian. Witnesses: William Lamb (Selkirk); John Ronaldson (maternal uncle; writer, Edinburgh).
Marriage	13 October 1857 to Margaret Archer Porter. Shankill, Antrim, Ireland.

Death	4 July 1901. Age 70. At Challenger Lodge, Wardie, Leith. Usual residence: 38 George Square, Edinburgh. Occupation: Emeritus Professor, Edinburgh University. Cause of death: arteriosclerosis (stiffening of the arteries); weak heart for 7 months. Death certified by: Gibson. Death certificate signed by: J. G. Tait (son), resident at 38 George Square, Edinburgh. The funeral: 6 July 1901 at St John's Episcopal Church, Edinburgh. Place of interment: St John's Church Yard, to the east of the church.[455] Challenger Lodge was the residence of Sir John Murray (1841–1914), Tait's friend and former pupil, who had invited Tait to stay at the residence in order to recuperate, following a period of ill health which began when his son Freddie was killed in the South African War in February 1900.[456]
Will	Written 15 January 1875. Updated 23 March 1896. Trustees: Margaret Archer Tait (Porter); Alexander Crum Brown [Brother-in-law, Doctor of Medicine, Prof. Chemistry and Chemical Pharmacy]; and William Ramsay Kermack [Writer to the Signet, Edinburgh]. Additional new trustees: Tait's sons John Guthrie [Prof. in the Central College, Bangalore, India], William [civil engineer, Edinburgh], Freddie [Lieutenant 2nd Battalion Black Watch, Royal Highlanders] and Alexander Guthrie [residing at 38 George Square]; and Harry Cheyne [Writer to the Signet, Edinburgh]. The beneficiaries. To his wife: £500 for providing family mournings and household and other expenses; rents, dividend, interests, whole free annual income and profits of remainder of estate and effects. If she enters into a second marriage, provision immediately ceases and she will no longer be a trustee. If she dies or marries again, monies will be divided equally between the children: boys to receive at age 21; girls age 21 or on marriage, whichever is first.

[455] [18, p41]

[456] [18, p40]

Assets	On 28 August 1901, assets included: cash £39.7; house furniture and effects (38 George Square Edinburgh) £175.18.6; current accounts in the National Bank of Scotland, £376.5.7, at Mackenzie and Kermack £215.1.1; savings in the National Bank of Scotland £1436.18.6; stocks and shares, William Younger and Co. Ltd. £465.12.6, City Property Investment Trust Corporation Ltd. £979.19, John Fraser and Sons Ltd. £205; bonds in the Royal and Ancient Golf Club St Andrews £25.10.2; rents, for stables in Meadow Lane, Edinburgh £910 for the half year to Martemmas 1901; fenduty payable at Whitsunday yearly, Duke of Buccleuch for subjects of Dalkeith annual sum of £11; ground annuals payable at Whitsunday for subjects at Taitshill, Selkirk £306 and payable at Martemmas for subjects at Taitshill, Selkirk £266; sum in bond and disposition in security for £2500 granted (May 1895) by son William Archer Porter Tait to P.G.T. reduced to £400; life assurance policies, The Scottish Provident Institution £5990.10, Scotland Annuable Life Assurance Society £3652.18.6; pension from the University of Edinburgh £953.19.2 per annum; fee due for examining a D.Sc. thesis for the University of Edinburgh £3.3; sum due from son John Guthrie Tait for advance on premium of insurance of his life £54.13.4.; interest in estate of his uncle, the late John Ronaldson of Somerset Cottage, Edinburgh £1935.16.3, and of his aunt, the late Margaret Ronaldson £774.18.9. Estate in Scotland: £24795.18.1. Estate in England (£143.7.4): balance owing to the publishers Macmillan and Co. £18.7.4; with Macmillan, interest in copyright for *Heat*, *The Unseen Universe* and *Quaternions* with Kelland £25; with A & C Black, interest in copyright for *Light*, *Properties of Matter*, *Dynamics* and *Newton's Laws of Motion* £100. Testate. Total assets in the U.K.: £24,939.5.5. Duty paid: 4.5% £1,268.8.11.
Residences—indicated by the 10 year census.	
1841	**23 Warriston Crescent, Edinburgh.** P.G.T. age 10, living with mother Mary (45) [independent means]; sisters Anne (8) and Mary (5); and 1 servant.
1851	**Peterhouse College, Cambridge.** P.G.T. age 19 [undergraduate pensioner], living with (amongst others): William A. Porter (26); William J. Steele (19); Frederick Fuller (31); and Edward J. Routh (20). See Figures A.3 and A.4 (pages 220–221). Tait collaborated on a book with William Steele (1831–1855), *Dynamics of a Particle* (1856); married William Porter's sister, Margaret (1857); and competed against Fuller and Routh for the Chair of Natural Philosophy at the University of Edinburgh (1860).
1861	**13 Buccleuch Place, Edinburgh.** P.G.T. age 29 [Professor of Natural Philosophy], living with: wife Margaret (21); child Edith (1); and 2 servants.
Sometime between 1861 & 1871	**6 Greenhill Gardens, Edinburgh.**[457] In William Thomson's obituary tribute to Tait, Thomson recalls that Tait produced a list, etched in charcoal, on the bare plaster of his study wall at this residence. Tait had ordered the current scientific figures by merit: Hamilton, Faraday, Andrews, Stokes and Joule ranked first in the column. Clerk Maxwell was as yet too young to appear.[458]

[457] [201, p193]

[458] [28, p364–365]

1871	**17 Drummond Place, Edinburgh.** P.G.T. age 39 [Professor of Natural Philosophy, University of Edinburgh], living with: wife Margaret (31); children Edith (11) [scholar], John (9) [scholar], Mary (6) [scholar], William (5) [scholar], Freddie (1); and 4 servants.
1881	**38 George Square, Edinburgh.** P.G.T. age 49 [M.A. Professor of Natural Philosophy], living with: wife Margaret (41); children John (19) [student at Cambridge], William (15) [scholar], Freddie (11) [scholar], Alex (8) [scholar]; and 3 servants.
1891	**38 George Square, Edinburgh.** P.G.T. age 59, living with: wife Margaret (51); children Edith (31), Alex (18) [Student of Arts], William (25) [Civil Engineer]; and 2 servants.
1901	**38 George Square, Edinburgh**. P.G.T. age 69, living with: wife Margaret (61); children Edith (41), John (39) [Professor of English], Alex (28) [Merchant's Clerk]; and 3 servants.
Education.	
Early education at Dalkeith Grammar School and Circus Place School, Edinburgh.[459]	
1841–1847	**The Edinburgh Academy.** In Dr. Cumming's class. [4, p120] Associations: schoolboy friendship with fellow student, James Clerk Maxwell (1831–1879).
1847—1848	**The University of Edinburgh.** Studied mathematics under Philip Kelland (1808–1879) and natural philosophy under James David Forbes (1809–1868).
1848—1852	**The University of Cambridge.** Peterhouse College: admitted 21 June 1848 (pensioner); matriculated in the Michaelmas term of 1848. Coached by private mathematics tutor, William Hopkins (1793–1866). B.A. (1852); Senior Wrangler (2nd Scot on record) and 1st Smith's Prizeman; elected a Fellow of Peterhouse (1853); M.A. (1855).[460] See Figure A.5 (page 222) for the Tripos Examination results. Associations: William Steele; and the Porter brothers, William Archer Porter (1824–1890) and James Porter (1827–1900). See Figure A.6 (page 223) for a photograph of Tait and Steele as graduates in 1852.
Occupations: Professor of Mathematics; Professor of Natural Philosophy.	
1854–1860	**The Queen's College, Belfast.** Professor of Mathematics: elected 14 September 1854; also held voluntary classes to supplement honours lectures in natural philosophy. Associations: Thomas Andrews (1813–1885), James Thomson (1822–1892) (William Thomson's brother), Charles Wyville Thomson (1830–1882) and James McCosh. Andrews introduced Tait to experimental work and to Sir William Rowan Hamilton (1805–1865).[461]

[459] [18, p3]

[460] [13]

[461] [18, p12]

1860–1901	**The University of Edinburgh.** Chair of Natural Philosophy: succeeded Forbes. Associations: William Robertson Smith (1846–1894), Tait's lab assistant (1868–1870); and Balfour Stewart (1828–1887), Forbes' former student and lab assistant.
Memberships, fellowships, prizes, honours.	
Royal Society of Edinburgh. Elected Ordinary Fellow (07/01/1861); proposed by Philip Kelland. Served as: Councillor (1861–1864); Secretary to the Ordinary Meetings (1864–1879) and General Secretary (1879–1901). Awarded the Society's Keith Prize (1867–1869, 1871–1873) and Gunning Victoria Jubilee Prize (1887–1890).[462] Associations: William Thomson (1824–1907).	
The Royal Society of London. Awarded their Royal Medal (1886); however, Tait chose never to become a Fellow of the Society.[463]	
Honorary member of The Literary and Philosophical Society of Manchester (1868). Honorary degrees: The Catholic University of Ireland (Sc.D., 1875); The University of Glasgow (LL.D., 1901) and The University of Edinburgh (LL.D., 1901). See Figure A.7 (page 224) for Tait's notification of his honorary degree from Edinburgh. Honorary fellowships: Societas Regia Hauniensis, Copenhagen (1876); The Edinburgh Mathematical Society (1883); The University of Cambridge, Peterhouse (1885); Société Bavate de Philosophie Experimentale, Rotterdam (1890); Societas Regia Scientiarum, Upsala (1894) and The Royal Irish Academy (1900).[464]	
John Ronaldson (maternal uncle)	
Father	Only son of John Ronaldson (1756–1820), a tenant farmer from Sauchland, Crichton, Midlothian.
Mother	Ann(e) Turnbull (d.1837).
Birth	17 October 1812. Sauchland, Crichton, Midlothian. Baptised 1 December 1812, Sauchland, Crichton, Midlothian.
Marriage	Single.
Death	26 December 1864. Age 52. Somerset Cottage, Raeburn Place, Edinburgh. Cause: Disease of the aorta. Death certificate signed by: P.G.T. (nephew, living at 6 Greenhill Gardens). Occupation: writer. Single. Place of interment: Dean cemetery, Edinburgh.

462 [29]

463 [18, p49]

464 [18, p47]

Will	Written 22 July 1862. Trustees: P.G.T. [Master of Arts, Professor of Natural Philosophy, University of Edinburgh] and William Ramsay Kermack [Writer to the Signet]. Beneficiaries: entire estate to sister, Margaret Ronaldson, residing at Somerset Cottage; upon her death, the residue of the estate to go to P.G.T. (nephew), Anne Margaret Tait and Mary Tait (nieces, both resident at Somerset Cottage).
Assets	On 2 February 1865, assets: cash in the house £10, household furniture and silver plate £263.8, current account at the National Bank of Scotland £89.8.7, shares in the British Railway Company £807.2 and £1008.17.6, promissory note by John Macnab of the Oriental Bank Corporation £507.6.11, balance on current account Messers Mackenzie and Kermack £45.14.7, stocks in the National Bank of Scotland including dividends £2088, stocks in the Edinburgh and Glasgow railway £338.6. Total assets £5965.5.
Residences—indicated by the 10 year census.	
1841	**Claremont Street, Midlothian.** J.R. age 25 [writer], living with: sister Margaret Ronaldson (25); Ann Ronaldson (20); and 1 servant.
1851	**Somerset Cottage, Raeburn Place, Edinburgh.** J.R. age 37 [Clerk in the National Bank of Scotland], living with: sister Margaret Ronaldson (39) [Annuitant]; nieces Anne M. Tait (17), Mary Tait (15) [scholar]; and 2 servants.
1861	**Somerset Cottage, Raeburn Place, Edinburgh.** J.R. age 48 [Banks Clerk], living with: sister Margaret Ronaldson (50), nieces Anne M. Tait (27), Mary Tait (25); 1 servant; and a visitor Peter Melrose [a shoe maker].
Occupations: writer; clerk in the National Bank of Scotland.	
Influence on P.G.T.	
Following the death of both their parents (John in 1837 and Mary in 1846), P.G.T. and his two sisters went to live with Mary's bachelor brother, John Ronaldson, and her maiden sister, Margaret Ronaldson, at Somerset Cottage. Although John was a banker by profession, he had a keen interest in scientific investigation and spent much time with P.G.T., enjoying scientific pursuit which must have influenced P.G.T.'s enthusiasm for science.[465]	
ANNE MARGARET TAIT (SISTER)	
Father & mother	Same as *P.G.T.*

465 [18, p3]

Birth	14 April 1833. Dalkeith, Midlothian. Baptised 16 June 1833, Dalkeith, Midlothian, in the presence of the congregation.
Marriage	Single.
Death	26 June 1915. Age 83. 15 Fettes Row, Edinburgh, Midlothian. Cause: Pneumonia 6 days. Death certificate signed by Edith Tait (niece, residing at 19 Abercromby Place, Edinburgh). Single. Place of interment: Dean cemetery, Edinburgh.
Mary Tait (sister)	
Father & mother	Same as *P.G.T.*
Birth	4 August 1835. Dalkeith, Midlothian. Baptised 15 September 1835, Dalkeith, Midlothian. Witnessed by John Patterson, agent of the Leith Bank, Dalkeith, and John Ronaldson, writer, Edinburgh.
Marriage	Single.
Death	26 April 1913. Age 77. Somerset Cottage, Raeburn Place, Edinburgh. Cause: Carcinoma (malignant tumour) of the intestine. Death certificate signed by Mary Cathcart (niece, residing at 44 Melville Street, Edinburgh). Single. Fundholder. Place of interment: Dean cemetery, Edinburgh.
Mary was a pupil of the Scottish landscape and marine painter, Samuel Bough (1822–1878) at the Royal Scottish Academy.	
Residences—indicated by the 10 year census.	
Tait's maternal aunt and uncle, Margaret Ronaldson (1809–1892) and John Ronaldson (1812–1864), lived in Somerset Cottage with Tait's sisters, Anne Margaret Tait (1833–1915) and Mary Tait (1835–1913). It appears that John, Margaret, Anne and Mary were all unmarried, had no children of their own and remained resident at Somerset Cottage until their deaths. All four were buried in Dean cemetery in Edinburgh. Somerset cottage was built in 1832. It was one of three detached villas; two of which were replaced by tenements and no longer exist. The property was bought by the Edinburgh Academy, probably soon after Anne and Mary died. During the 1920s, it was converted into two villas, upper and lower, to provide accommodation for Academy staff. The Academy sold the property in the 1960s. It became a guest house and then the Raeburn House Hotel. Following recent redevelopment, the grade B listed Somerset Cottage is now operating as 'The Raeburn', a boutique hotel with bar and restaurant. The property is situated on the south east corner of the Edinburgh Academy sports fields. It is a fifteen minute walk from Somerset Cottage to the Edinburgh Academy.	
Margaret Archer Tait (Porter) (Spouse)	
Father	James Porter (c.1788–1851) Presbyterian minister in Drumlee, Castlewellan, Co. Down, Ireland. Died in a fall from his horse.

Mother	Eliza Archer (d. c.1877)
Margaret was a sister to the Porter brothers, William Archer and James, who Tait had known at Cambridge. Tait grew close to the family during his time in Belfast, from 1854. Margaret was one of twelve children.	
Birth	1 May 1839. Co. Down, Ireland. (Date from her grave stone in St John's Church Yard, Edinburgh.)
Marriage	13 October 1857 to Peter Guthrie Tait. Shankill, Antrim, Ireland.
Death	27 October 1926. Age 87. The Rectory, Colinton, Midlothian. Cause: Cardiac failure due to old age. Death certificate signed: H. Reid of the Rectory, Colinton (son-in-law, resident at The Rectory, Colinton, Midlothian). Place of interment: Tait family grave, at St John's, Edinburgh.
Residences—indicated by the 10 year census.	
1861–1901	See *P.G.T.*
1911	**38 George Square, Edinburgh.** M.A.P.T. age 71 [widow], living with: son William (45) [Civil Engineer, Employer]; daughter-in-law Anne (34) [John Guthrie's wife] and Anne's children Margaret (3), Patrick (1); and 4 servants.
ALEXANDER CRUM BROWN (BROTHER-IN-LAW)	
Born 26 March 1838 in Edinburgh. Son of the Revd. John Brown (1784–1858) who was biblical scholar and minister of the United Presbyterian Church of Scotland, Broughton Place, Edinburgh. Alexander's mother, Margaret Fisher Crum (1799–1841) was John's second wife. Alexander had a step-brother and step-sister from his father's first marriage. Alexander married Jane Bailie Porter (1836–1910), sister to Tait's wife and the Porter brothers, in Belfast in 1866. He died on 28 October 1922, aged 84, at 8 Belgrave Crescent, Edinburgh. He studied medicine at the University of Edinburgh: M.A. (1858); M.D. (1861) and D.Sc. (1862, London). Between 1863 and 1869, he was Lecturer in Chemistry at the Edinburgh Extramural School and from 1869 to 1908, Professor of Chemistry at the University of Edinburgh.[466] Alexander was a Fellow of the R.S.E. (07/12/1863), proposed by Sir Lyon Playfair. He served as: Councillor (1865–1868, 1869–1872, 1873–1875, 1876–1878, 1911–1913), Secretary to the Ordinary Meetings (1879–1905) and Vice-President (1905–1911). He was awarded the Society's Keith Prize (1873–1875) and the Makdougall-Brisbane Prize (1866–1868) which was a joint award.[467] He was President of the Edinburgh Medical Missionary Society between 1911 and 1916.[468]	

466 [202, p895]

467 [29]

468 [202, p895]

Children of P.G.T. and Margaret Archer Porter (6)	
Edith Tait (c.1860–1948); John Guthrie Tait (1861–1945); Mary Guthrie Tait (1864–1946); William Archer Tait (1866–1929); Frederick Guthrie Tait (1870–1900); Alexander Guthrie Tait (1873–1934).	
Edith Reid (Tait)	
Birth	c. 1860. Belfast, Ireland.
Marriage	24 June 1902. St John's Church, Edinburgh. To Harry Seymour Reid (1866–1943): born in Glasgow; son of Daniel Reid, a ship worker; later Rt. Revd. Bishop of Edinburgh (1929–1939). At time of marriage: Harry was a clerk in holy orders, residing at 5 Granville Terrace, Edinburgh; Edith was residing at 38 George Square, Edinburgh. Harry died at 3 Pilmour Place, St Andrews, aged 76. Harry Seymour Reid's portrait (from 1930) is housed in the National Portrait Gallery, London, as part of the 'Photographs of Anglican Bishops, 1860s–1940s' set of portraits.
Death	1 September 1948. Age 88. At 3 Pilmour Place, St Andrews. Cause: Cardiovascular degeneration. Death certificate signed: William Shaw Andrews (intimate friend; resident at The Eastory, St Andrews). Widow.
Residences—indicated by the 10 year census.	
1861 & 1871	See *P.G.T.*
1881	Cannot be found on the Scottish or English 1881 Censuses.
1891 & 1901	See *P.G.T.*
1911	**27 Argyle Crescent, Portobello, Edinburgh.** Edith age 51, living with husband Harry (44) [Episcopal Clergyman]; and 2 servants.
John Guthrie Tait	
Birth	24 August 1861. 6 Greenhill Gardens, Edinburgh.

Marriage	7 January 1904. Bangalore, Madras, India. To Anne Smith Cook (1876–1951): born Forfar, Arbroath; daughter of John Cook (teacher in the mathematics department at Abroath High School; Principal of Doveton College, Madras (1882–1907); Principal of the Central College Bangalore (1882–1907); F.R.S.E. (1894, proposed by P.G.T.).[469]) Anne died at 38 George Square, Edinburgh, aged 75. She is buried in Morningside cemetery in Edinburgh, while John Guthrie is buried in the Tait family grave in St John's Church Yard.
Death	4 October 1945. Age 84. 38 George Square, Edinburgh. Cause: Enlargement of the prostrate, urinary bladder fistula, suppression of urine, rheumatoid arthritis. Death certificate signed: M. Tait (daughter). Occupation: retired College Principal.
Residences—indicated by the 10 year census.	
1871 & 1881	See *P.G.T.*
1891	Cannot be found on the Scottish or English 1891 Censuses: he went to India in 1890.[470]
1901	**38 George Square, Edinburgh.** See *P.G.T.* Presumably he returned for a short period following his brother Freddie's death.
1911	Cannot be found on the Scottish or English 1911 Censuses. We can presume he is still in India as *The Edinburgh Academy Register* has his address in 1914 as 'Central Coll. House, Bangalore, S. India'.[471]
Education.	
1871–1877	**The Edinburgh Academy.** Mr. Carmichael's class.[472]
1880–1883	**The University of Cambridge.** Admitted Peterhouse [pensioner]; matriculated Michaelmas 1880; classical Tripos 1st class 1883; B.A. 1884; M.A. 1890.[473]
Sporting achievements.	

[469] [29]

[470] [4, p336]

[471] [Ibid.]

[472] [4, pp335–336]

[473] [13]

Rugby: Played for A.F.C. (The Academical Football Club at the Edinburgh Academy) and London Scottish F.C.; blues for Cambridge (1880, 1882); played for Scotland, versus Ireland (1800, 1885). Golf: Semi-final of the Golf Amateur Championship, Hoylake (1887); London Scottish Golf Gold Medal, Hope Grant Medal, Bangalore Golf Club Gold medal (twice). Winner of prizes at Bangalore Rifle meetings and S. India Rifle Association meetings.[474]	
Occupations: barrister-at-law, scholar of English Literature, College Principal.	
1884	Barrister-at-law: admitted Lincoln's Inn; called to the bar 25 April 1888.[475]
1890	Government education department, Mysore, India.[476]
1908	Principal of Mysore Government Central College, Bangalore.[477] Previously Professor of Languages there and Vice-Principal.[478]
Examiner in English, the University of Madras.[479]	
Military service.	
1914–1919	Served in the Great War: Lieutenant Colonel, Bangalore Rifle Volunteers.[480]
Memberships, fellowships, prizes, honours.	
1937	**Royal Society of Edinburgh.** Elected Fellow (01/03/1937), proposed by: Sir D'Arcy W. Thompson, William Peddie, A. Crichton Mitchell, Sir Edmund T. Whittaker.[481]
Fellow of Madras India University.[482]	
MARY GUTHRIE CATHCART (TAIT)	
Birth	22 July 1864. 6 Greenhill Gardens, Edinburgh.

[474] [4, p336]

[475] [13]

[476] [4, p336]

[477] [Ibid.]

[478] [13]

[479] [Ibid.]

[480] [Ibid.]

[481] [29]

[482] [13]

Marriage	10 September 1885. Episcopal Church, St Andrews. To Charles Walker Cathcart (1853–1932): born 42 Drummond Place, Edinburgh; surgeon; son of James Cathcart, a wine merchant in Leith. At time of marriage: Charles resident at 44 Melville Street, Edinburgh; Mary resident at 4 Alexander Place, St Andrews. Witnesses: James Porter (Mary's uncle, officiating minister), R. W. Irvine, J. G. Tait (Mary's brother). Charles Cathcart was a Senior Lecturer in Clinical Surgery (1913–1917) at the University of Edinburgh; previously Lecturer in Anatomy at Surgeons' Hall, Edinburgh (1881–1885). He was a Consultant Surgeon at the Edinburgh Royal Infirmary and Conservator of the Museum of the Royal College of Surgeons in Edinburgh (1887–1900). He was also a Lieutenant-Colonel in the Royal Army Medical Corps (Territorial Force), at the 2nd Scottish General Hospital, at Edinburgh War Hospital and at Edenhall Hostel for Limbless Sailors and Soldiers (1914–1919).[483] He wrote a number of medical books, including *A Surgical Handbook*, jointly with Professor F. M. Caird, which ran to twenty editions.[484] Charles died at 12 Randolph Crescent, Edinburgh, aged 78. His usual residence: 13 Newbattle Terrace, Edinburgh.
Death	22 November 1946. Buxton, Derbyshire. Usual residence 16a Trinity Church-Square, London. Widow. It seems that Mary moved to London following her husband's death.
Residences—indicated by the 10 year census.	
1871	See *P.G.T.*
1881	**The School House, St Andrews.** St Andrews School for Girls, St Andrews, Fife. Mary age 16 [scholar], living as a boarder.
1891	**8 Randolph Crescent, Edinburgh.** Mary age 26, living with: husband Charles (38) [surgeon]; and 2 servants.
1901	**8 Randolph Crescent, Edinburgh.** Mary age 36 [living by own means], living with: children Theodora (8) [scholar], Francis (6) [scholar], Helen (3); brother William (35) [civil engineer]; and 3 servants. Mary's husband Charles does not appear on the Scottish or English 1901 Censuses.
1911	All the Cathcarts disappear from the Scottish and English 1911 Censuses, except for Theodora: she is a student, aged 18, a visitor at St Margaret's School in Polmont, East Stirling, perhaps on teacher training. In 1913, however, the Cathcarts reappear; resident at 44 Melville Street, Edinburgh.
William Archer Porter Tait	
Birth	25 March 1866. 6 Greenhill Gardens, Edinburgh.

[483] [203]

[484] [204]

Marriage	Single.
Death	23 June 1929. Age 63. 17 Greenhill Gardens, Edinburgh. Cause: Parkinson's disease 5 years, cerebral haemorrhage. Death certificate signed: J. G. Tait (brother, residing at 38 George Square). Single. Occupation: Civil Engineer, retired. National Probate Calendar has William resident as 38 George Square, Edinburgh and 72 George Street, Edinburgh. *The Edinburgh Academy Register* has William resident (in 1914) at 38 George Square, Edinburgh.[485] Place of interment: Tait family grave, at St John's, Edinburgh.
Residences—indicated by the 10 year census.	
1871–1891	See *P.G.T.*
1901	**8, Randolph Crescent, Edinburgh.** See *Mary Guthrie Cathcart (Tait).*
1911	**38 George Square, Edinburgh.** See *Margaret Archer Tait (Porter).*
Education.	
1875–1881	**The Edinburgh Academy.** Mr. Carmichael's class.[486]
1885	**The University of Edinburgh.** BSc Engineering, D.Sc.[487]
Occupations: civil engineer.	
1887–1890	Training (Engineer): Sir J. Wolfe Barry and H. M. Brunel (son of I. K. Brunel).[488]
1891–1894	Assistant Engineer, Glasgow Central Railway.[489]
1894–??	Partner in J. & A. Leslie & Reid. Involved in the building of the Talla Reservoir, which was opened in 1899.[490]

485 [4, p360]

486 [Ibid.]

487 [29]

488 [Ibid.]

489 [Ibid.]

490 [Ibid.]

Engineer, Edinburgh and District Water Trust and several other authorities; Manager of Edinburgh Royal Infirmary.[491]	
Memberships, fellowships, prizes, honours.	
1891	**Institute of Civil Engineers.** Awarded Miller Prize (1891) and Telford Premium (1906) for contribution of professional papers. William was a member of the Inst. C.E.[492]
1898	**Royal Society of Edinburgh.** Elected Fellow (02/05/1898), proposed by: Sir W. Thomson, Charles Alan Stevenson, George Barclay, John Stugreon MacKay. Served as: Councillor (1914–1917, 1918–1921); Vice-President (1921–1924).[493]
Sporting achievements.	
Played for: London Scottish F.C.; West of Scotland F.C. and A.F.C. (The Academical Football Club at the Edinburgh Academy).[494]	
Frederick "Freddie" Guthrie Tait	
Birth	11 January 1870. 17 Drummond Place, Edinburgh.
Marriage	Single.
Death	7 February 1900. Age 30. Koodoosberg Drift. Killed in action, during reconnaissance under General Macdonald. He was shot through the heart while making an advance of 50 yards on the Boer position. His body was buried on the banks of the Riet River. P.G.T. received the news of Freddie's death by telegram on 14 February; it affected him profoundly.[495] There is a monument in memory of Freddie at the Tait family grave at St John's, Edinburgh.
Residences—indicated by the 10 year census.	
1871–1881	See *P.G.T.*
1891	**Aldershot Hampshire.** Freddie age 21, living in barracks, 2nd Battalion Infantry.

491 [4, p360]

492[Ibid.]

493 [29]

494 [4, p360]

495 [53, pp220–226]

Education.	
1879–1883	**The Edinburgh Academy.** Mr. Carmichael's class.[496]
May 1883–Dec. 1886	**Sedberg School.** A boarding school for boys in Cumbria. Sent on the advice of Bishop Sandford, Rector of St John's, Edinburgh and a family friend. Freddie excelled at mathematics and French.[497]
Military career.[498]	
Sept. 1889	Enters Sandhurst. Promoted to Corporal at Christmas 1889. Passes out (July 1890) with special honours in riding and military administration.
Oct. 1890	Gazetted to 2nd Battalion of the Leinster Regiment, the 109th foot. Spends time in Folkstone and Aldershot.
1890	Promoted to 2nd Lieutenant. Promoted to Lieutenant (1893).
June 1894	Joins 2nd Battalion of the 42nd Royal Highlanders (The Black Watch). Stationed at Edinburgh Castle for two years, until autumn 1896 when sent to Ballater (Scotland) as her Majesty's Guard. A period at York follows, until the end of 1897.
Aug. 1898	Appointed as Inspector for the Scottish District, following brief spell in Colchester, Eastern District. Resident in Glasgow from October, with weekends spent in and around Edinburgh.
18 Oct. 1899	Resignation from staff appointment sanctioned: Freddie had applied to rejoin the Black Watch in case of active service. On declaration of War in South Africa, he voyages to South Africa on 'The Orient' (24 Oct.).
11 Dec. 1899	Wounded in action: hit in thigh. Recuperated at Wynberg Hospital, then Claremont Sanatorium.
7 Feb. 1900	Killed in action: shot in the heart on first day of action following recuperation.
Sporting associations and achievements.	

[496] [4, p379]

[497] [53, p35]

[498] Source: [53, various places].

Played rugby for A.F.C. (The Academical Football Club at the Edinburgh Academy).[499] Colours for rugby football (1889–1890): Sandhurst XV, 'one of the best forwards in the team'.[500] Introduced golf to Sandhurst: 'Before Tait's time golf had been unknown at Sandhurst, so he laid out a short course for his fellow cadets, and instructed them in the ways and mysteries of the sport.'[501] Member of the Royal and Ancient Club at St Andrews (joined Spring 1890).[502] Scotland's amateur golf champion (1896, 1898); runner-up (1899).[503]	
ALEXANDER GUTHRIE TAIT	
Birth	2 February 1873. 17 Drummond Place, Edinburgh.
Marriage	Single.
Death	11 May 1934. Age 61. Highfield Dreghorn Loan Colinton, Midlothian. Usual residence: Ladebraes Villa, St Andrews. Cause: Carcinoma (tumour) of the rectum 1.5 yrs, cachexia (wasting syndrome) 6 months. Death certificate signed: J. G. Tait (brother, resident at 38 George Square). Single. Occupation: Glass merchant, retired. Place of interment: Tait family grave, at St John's, Edinburgh.
Residences—indicated by the 10 year census.	
1881–1901	See *P.G.T.*
1911	**100 Upper Parliament Street, Liverpool.** A.G.T. age 37 [glass merchant, a boarder]. *The Edinburgh Academy Register* has Alex at the same residence in 1914.[504]
Education.	
1883–1885	**The Edinburgh Academy.** Mr. Shipton's class, later Mr. McBean's class.[505]
1891	**The University of Edinburgh.** M.A.[506]

[499] [4, p379]

[500] [53, p59]

[501] [Ibid.]

[502] [53, p60]

[503] [4, p379]

[504] [4, p397]

[505] [Ibid.]

[506] [Ibid.]

Occupations: glass merchant.	
GRANDCHILDREN OF P.G.T. AND MARGARET ARCHER PORTER (7)	
By *John Guthrie* (3)	Margaret Tait (1907–1996); Patrick Tait (1909–1935); Peter Guthrie Tait (1914–1984). All three children were born in Bangalore, India. Margaret was an academic who lived in St Andrews. She died aged 88. Patrick was in the Mysore service. He had a B.A. from Cambridge. He died of tuberculosis, aged 26, at the family home, 38 George Square, Edinburgh. His father John signed the death certificate. Patrick's body was interred in the family grave at St John's, Edinburgh. Peter Guthrie was a District Commissioner in the Colonial Civil Service. He died in Selkirk, aged 70. He married Marjorie Agnes Hope Gillespie in Edinburgh on 19 February 1938. The couple had seven children.
By *Mary Guthrie* (4)	Theodora Cathcart (1892–1951); Francis John Cathcart (1894–1916); Helen M. Cathcart (1897–1992); Louisa Marion Cathcart (1902–1988). The children were born at 8 Randolph Crescent, Edinburgh. Theodora was a teacher. She never married. She died in London, aged 59. Francis was a 2nd Lieutenant in the Royal Field Artillery. He was educated at Loretto school in Edinburgh and studied engineering at the University of Edinburgh. He served at Gallipoli and at Mesopotamia during the advance to Bagdad (1916). He was killed in action on 3 June 1916, aged 21. He never married. Helen married Archibald Mcneilage, a mechanical engineer, in Edinburgh on 17 June 1924. She died in Edinburgh, aged 95. Louisa worked as a private secretary. She married Christopher F. Millett, an advertising artist from London, on 23 December 1930. Harry Seymour Reid, Bishop of Edinburgh, officiated at the marriage, which took place at Christ Church in Morningside, Edinburgh. Louisa died in Morningside, aged 85.

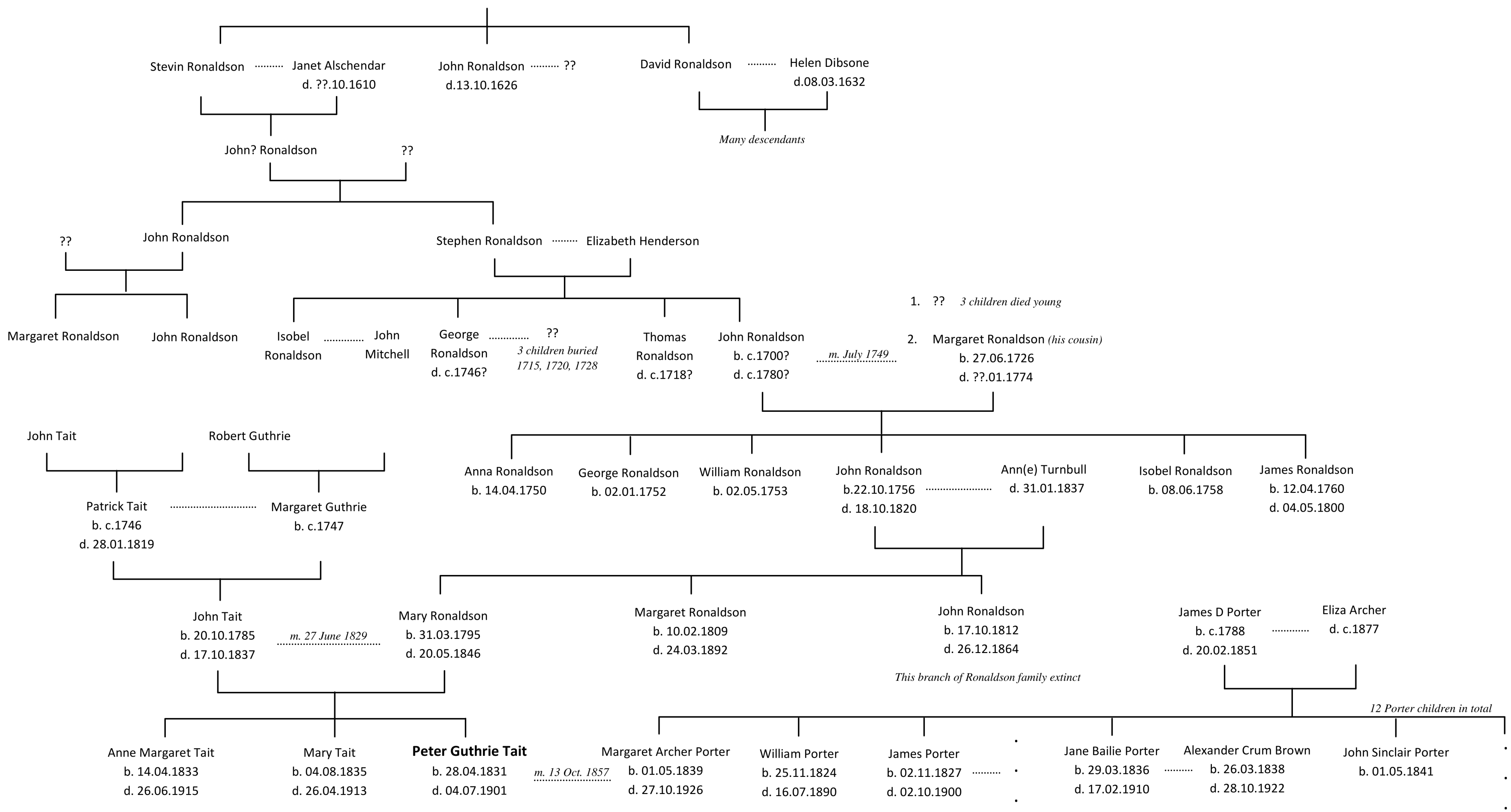


Figure A.1: Ancestors of P. G. Tait (1831–1901).

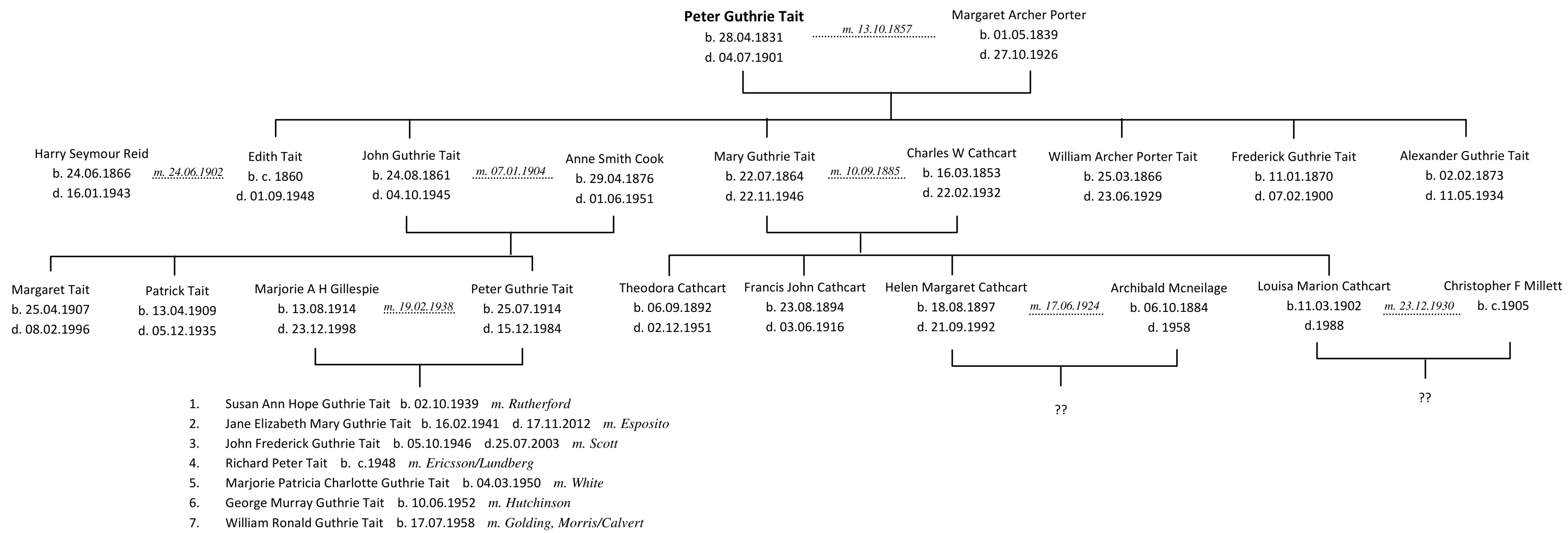
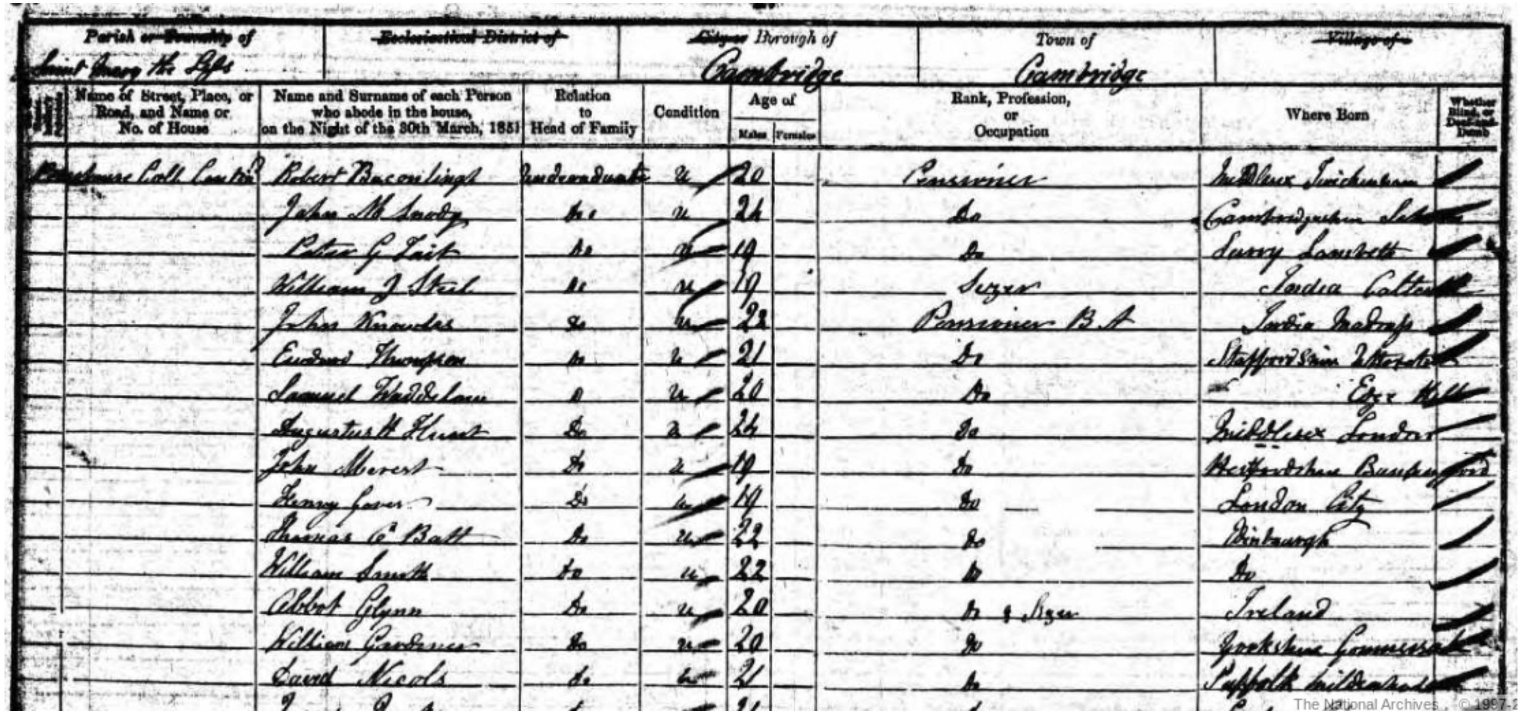
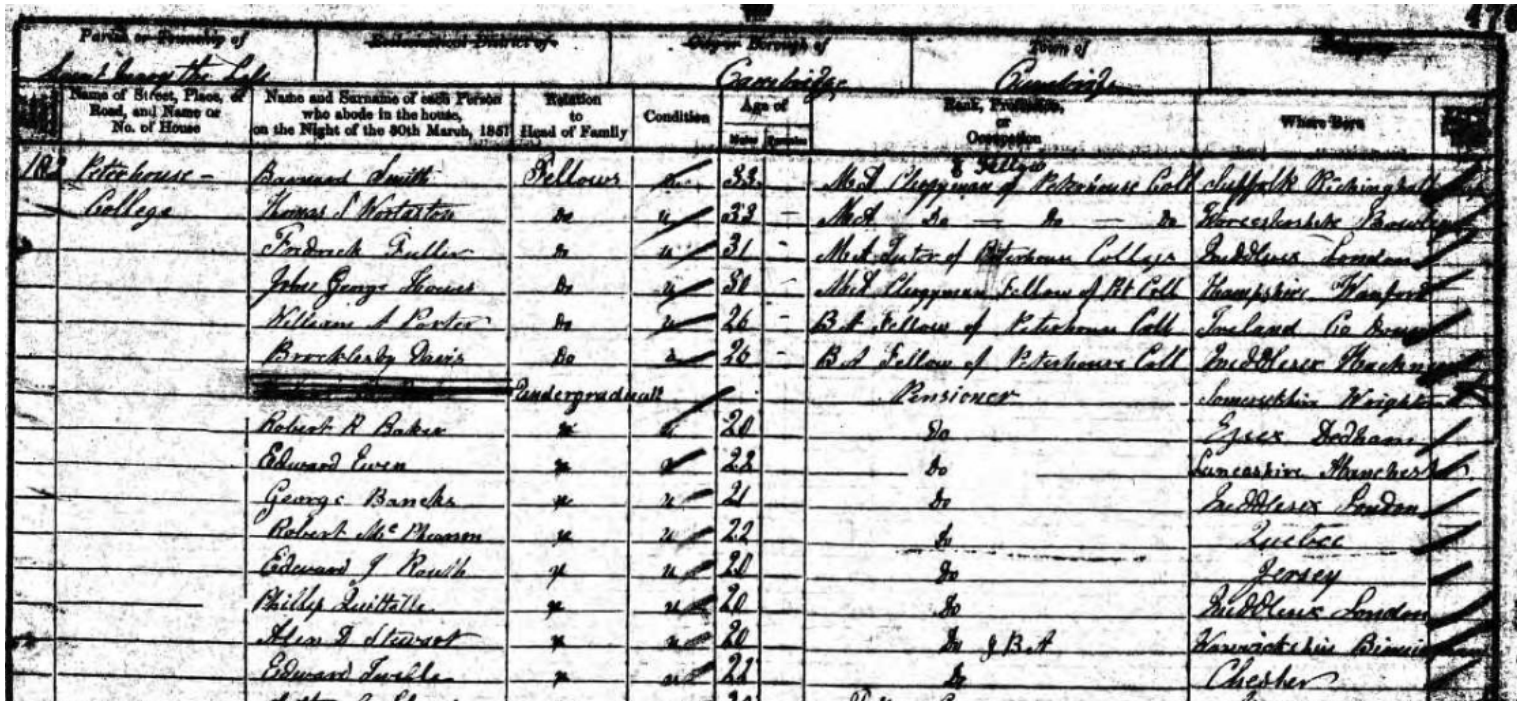


Figure A.2: Descendants of P. G. Tait (1831–1901).



Name of Street, Place, or Road, and Name or No. of House	Name and Surname of each Person who abode in the house, on the Night of the 30th March, 1851	Relation to Head of Family	Condition	Age of Males	Age of Females	Rank, Profession, or Occupation	Where Born	Whether Blind, or Deaf-and-Dumb
Peterhouse Coll Cambs	Robert Baconlingh	Undergraduate	U	20		Pensioner	Middlesex Twickenham	
	John M Saudy	Do	U	24		Do	Cambridgeshire [illegible]	
	Peter G Tait	Do	U	19		Do	Surrey Lambeth	
	William J Steel	Do	U	19		Sizar	India Calcutta	
	John Knowles	Do	U	22		Pensioner B.A	India Madras	
	Edward Thompson	Do	U	21		Do	Staffordshire [illegible]	
	Samuel Waddelow	Do	U	20		Do	Essex [illegible]	
	Augustus H Hunt	Do	U	24		Do	Middlesex London	
	John Mercer	Do	U	19		Do	Hertfordshire Buntingford	
	Henry Lowe	Do	U	19		Do	London City	
	Thomas C Ball	Do	U	22		Do	Edinburgh	
	William Smith	Do	U	22		Do	Do	
	Albert Glynn	Do	U	20		Do [illegible]	Ireland	
	William Gardener	Do	U	20		Do	Yorkshire [illegible]	
	David Nicols	Do	U	21		Do	Suffolk [illegible]	

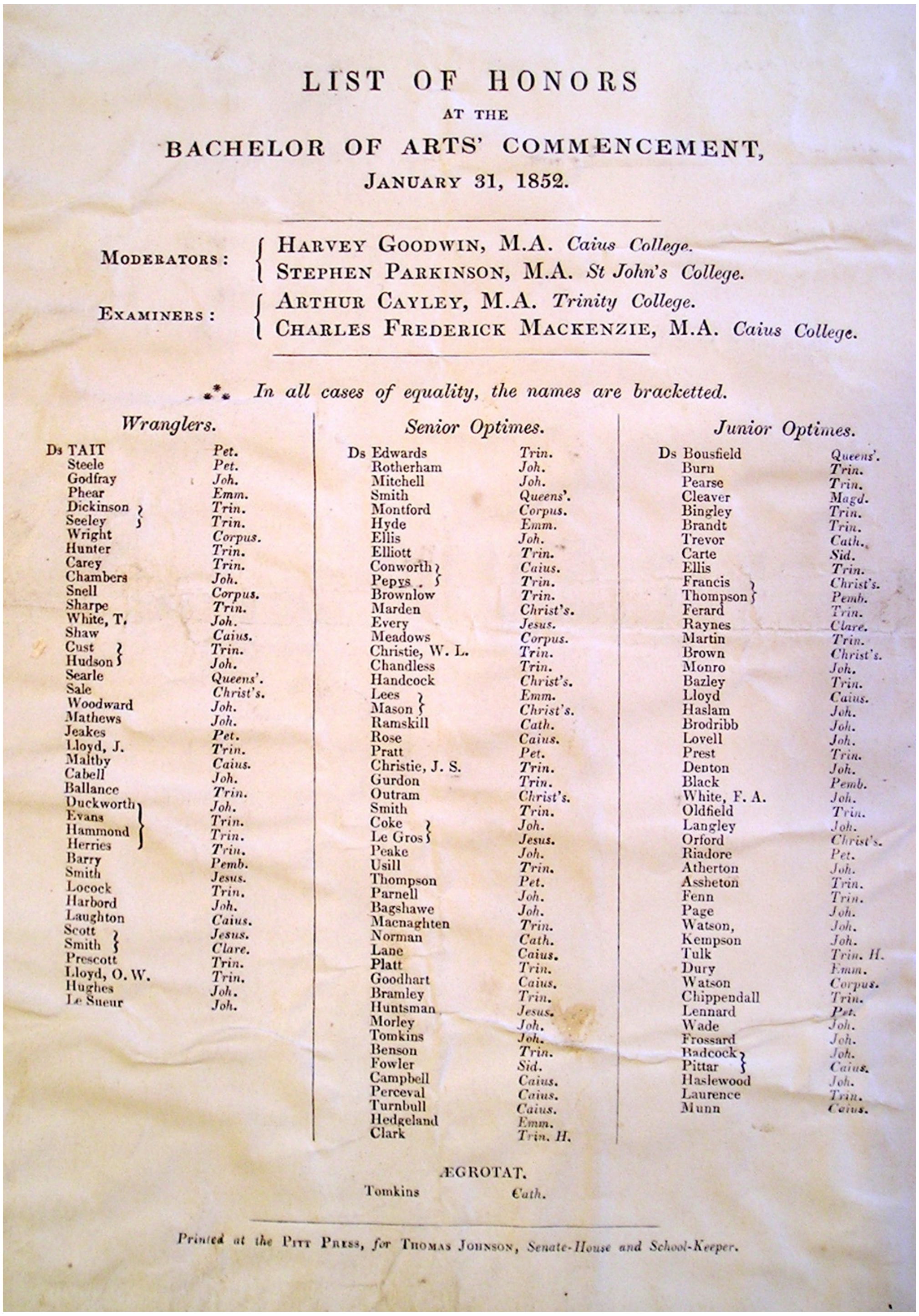
Figure A.3: Tait [no.3] and William Steele [no.4] resident at Peterhouse College, Cambridge, 1851. From the England Census of 1851. [205] Tait's birth place is given as Lambeth, Surrey, England, which is incorrect. As a pensioner, Tait would have had to pay for his own food and tuition. Tait was admitted to Peterhouse College on 21 June 1848; he matriculated in the Michaelmas term that year and in 1852 came out as Senior Wrangler and First Smith's Prizeman.



Parish or Township of Saint Mary the Less | Ecclesiastical District of | City or Borough of Cambridge | Town of Cambridge

Name of Street, Place, or Road, and Name or No. of House	Name and Surname of each Person who abode in the house, on the Night of the 30th March, 1851	Relation to Head of Family	Condition	Age of Males	Age of Females	Rank, Profession, or Occupation	Where Born
182 Peterhouse College	Bernard Smith	Fellows	U	33		M.A. Clergyman & Fellow of Peterhouse Coll	Suffolk Rickinghall
	Thomas S Worsterton	do	U	33	–	M.A. do do do	Worcestershire [illegible]
	Frederick Fuller	do	U	31	-	M.A. Tutor of Peterhouse College	Middlesex London
	John George Howes	do	U	30	–	M.A. Clergyman Fellow of Pt Coll	Hampshire [illegible]
	William A Porter	do	U	26	-	B.A. Fellow of Peterhouse Coll	Ireland Co Down
	Brocklesby Davis	do	U	26	-	B.A. Fellow of Peterhouse Coll	Middlesex [illegible]
	[illegible]	Undergraduate				Pensioner	Somersetshire Wright[illegible]
	Robert R Baker		U	20		do	Essex Dedham
	Edward Ewen		U	22		do	Lancashire Manchester
	George Banks		U	21		do	Middlesex London
	Robert McPherson		U	22		do	Quebec
	Edward J Routh		U	20		do	Jersey
	Philip Quittalle		U	20		do	Middlesex London
	Alex D Stewart		U	20		do & B.A.	Warwickshire Birmingham
	Edward Tuckle		U	21		do	Chester

Figure A.4: Others resident with Tait at Peterhouse: Frederick Fuller [no.3], William A. Porter [no.5] and Edward J. Routh [no.12]. From the England Census of 1851. [205]



LIST OF HONORS
AT THE
BACHELOR OF ARTS' COMMENCEMENT,
JANUARY 31, 1852.

MODERATORS: HARVEY GOODWIN, M.A. *Caius College.*
STEPHEN PARKINSON, M.A. *St John's College.*

EXAMINERS: ARTHUR CAYLEY, M.A. *Trinity College.*
CHARLES FREDERICK MACKENZIE, M.A. *Caius College.*

*** *In all cases of equality, the names are bracketted.*

Wranglers.

Ds TAIT	*Pet.*
Steele	*Pet.*
Godfray	*Joh.*
Phear	*Emm.*
Dickinson	*Trin.*
Seeley	*Trin.*
Wright	*Corpus.*
Hunter	*Trin.*
Carey	*Trin.*
Chambers	*Joh.*
Snell	*Corpus.*
Sharpe	*Trin.*
White, T.	*Joh.*
Shaw	*Caius.*
Cust	*Trin.*
Hudson	*Joh.*
Searle	*Queens'.*
Sale	*Christ's.*
Woodward	*Joh.*
Mathews	*Joh.*
Jeakes	*Pet.*
Lloyd, J.	*Trin.*
Maltby	*Caius.*
Cabell	*Joh.*
Ballance	*Trin.*
Duckworth	*Joh.*
Evans	*Trin.*
Hammond	*Trin.*
Herries	*Trin.*
Barry	*Pemb.*
Smith	*Jesus.*
Locock	*Trin.*
Harbord	*Joh.*
Laughton	*Caius.*
Scott	*Jesus.*
Smith	*Clare.*
Prescott	*Trin.*
Lloyd, O. W.	*Trin.*
Hughes	*Joh.*
Le Sueur	*Joh.*

Senior Optimes.

Ds Edwards	*Trin.*
Rotherham	*Joh.*
Mitchell	*Joh.*
Smith	*Queens'.*
Montford	*Corpus.*
Hyde	*Emm.*
Ellis	*Joh.*
Elliott	*Trin.*
Conworth	*Caius.*
Pepys	*Trin.*
Brownlow	*Trin.*
Marden	*Christ's.*
Every	*Jesus.*
Meadows	*Corpus.*
Christie, W. L.	*Trin.*
Chandless	*Trin.*
Handcock	*Christ's.*
Lees	*Emm.*
Mason	*Christ's.*
Ramskill	*Cath.*
Rose	*Caius.*
Pratt	*Pet.*
Christie, J. S.	*Trin.*
Gurdon	*Trin.*
Outram	*Christ's.*
Smith	*Trin.*
Coke	*Joh.*
Le Gros	*Jesus.*
Peake	*Joh.*
Usill	*Trin.*
Thompson	*Pet.*
Parnell	*Joh.*
Bagshawe	*Joh.*
Macnaghten	*Trin.*
Norman	*Cath.*
Lane	*Caius.*
Platt	*Trin.*
Goodhart	*Caius.*
Bramley	*Trin.*
Huntsman	*Jesus.*
Morley	*Joh.*
Tomkins	*Joh.*
Benson	*Trin.*
Fowler	*Sid.*
Campbell	*Caius.*
Perceval	*Caius.*
Turnbull	*Caius.*
Hedgeland	*Emm.*
Clark	*Trin. H.*

Junior Optimes.

Ds Bousfield	*Queens'.*
Burn	*Trin.*
Pearse	*Trin.*
Cleaver	*Magd.*
Bingley	*Trin.*
Brandt	*Trin.*
Trevor	*Cath.*
Carte	*Sid.*
Ellis	*Trin.*
Francis	*Christ's.*
Thompson	*Pemb.*
Ferard	*Trin.*
Raynes	*Clare.*
Martin	*Trin.*
Brown	*Christ's.*
Monro	*Joh.*
Bazley	*Trin.*
Lloyd	*Caius.*
Haslam	*Joh.*
Brodribb	*Joh.*
Lovell	*Joh.*
Prest	*Trin.*
Denton	*Joh.*
Black	*Pemb.*
White, F. A.	*Joh.*
Oldfield	*Trin.*
Langley	*Joh.*
Orford	*Christ's.*
Riadore	*Pet.*
Atherton	*Joh.*
Assheton	*Trin.*
Fenn	*Trin.*
Page	*Joh.*
Watson,	*Joh.*
Kempson	*Joh.*
Tulk	*Trin. H.*
Dury	*Emm.*
Watson	*Corpus.*
Chippendall	*Trin.*
Lennard	*Pet.*
Wade	*Joh.*
Frossard	*Joh.*
Badcock	*Joh.*
Pittar	*Caius.*
Haslewood	*Joh.*
Laurence	*Trin.*
Munn	*Caius.*

ÆGROTAT.

Tomkins *Cath.*

Printed at the PITT PRESS, *for* THOMAS JOHNSON, *Senate-House and School-Keeper.*

Figure A.5: Tripos examination results, Cambridge, 1852. Tait is Senior Wrangler. Sourced from Tait's scrapbook. Reproduced with the kind permission of the J.C.M. Foundation. Tait had been coached by the private mathematics tutor, William Hopkins (1793–1866) who is referred to by Craik [14, p107] as the "wrangler maker". Hopkins also tutored Maxwell, Thomson, Stokes and Cayley.

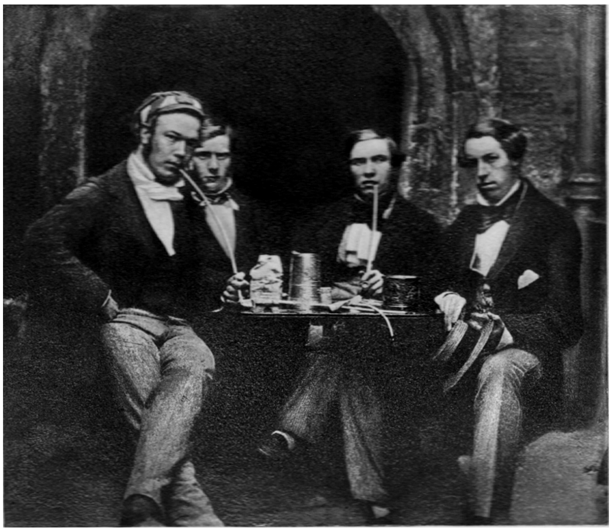


Figure A.6: Tait (first on the left) and William Steele (1831–1855) (third on left); graduates at Cambridge, 1852. [18, pfacing 11] Original from Tait's scrapbook. While at Cambridge, Tait developed a close friendship with William Steele. After graduation, both became Fellows of Peterhouse and they began collaborating on a book, *Dynamics of a Particle* which was published in 1856. Sadly, Steele died with barely a few chapters written so Tait finished the work.

University of Edinburgh.

The Senatus Academicus of this University have resolved to offer to you the Honorary Degree of

Doctor of Laws.

The honour of your presence at the Graduation Ceremonial on Saturday 27th July is requested in order that the Degree may then be conferred upon you.

Edinburgh 3rd June 1901

W. Muir Principal.

L. J. Grant. Sec. Sen. Acad.

To Professor P. G. Tait M. A. D. Sc.

Note. Should you be unable to attend on the above occasion the Degree may be conferred upon you at one of the subsequent Ceremonials, which take place annually about the middle of April and the end of July. According to the Regulations of the University, Honorary Degrees are not conferred in absentia except in very special circumstances.

Figure A.7: Notification of Tait's honorary degree from the University of Edinburgh. Sourced from Tait's scrapbook. Reproduced with the kind permission of the J.C.M. Foundation. The ceremony was scheduled for 27 July 1901. Tait died just a few weeks prior on 4 July.

B TAIT'S POEM ON THE FRANCO–PRUSSIAN WAR (1870)

This is an unpublished poem by Tait which has been sourced from his pocket notebook. Written in October 1870, the untitled poem finds its context in the height of the Franco–Prussian War (July 1870 to May 1871). Tait writes that he has sent the poem on to 'Russel'. This is, presumably, Alexander Russel (Russell) (1814–1876), editor of *The Scotsman* newspaper: his identity is suggested by evidence from Tait's scrapbook.[507] A search of the Scotsman Digital Archive (1817–1950) indicates that Tait's poem was not published in the newspaper.[508]

Sent to Russel 11/11/70.

Oct^{r} 31/70.

A Who is this G fat but quick?[509]
The hound that crouched 'neath B's stick[510]
What time the plunderers of the Dane
Quarrelled about their shameless gain.
Beery & fat and scant of wind

---

[507] Tait refers to Russel and *The Scotsman* in an annotation accompanying the original draft of a poem by William Robertson Smith in the scrapbook. Smith's poem is adapted from John Milton's poem, 'On Shakespeare' (1630). In his note, Tait writes that Russel had agreed to publish Smith's poem in *The Scotsman*.

[508] Scotsman Digital Archive (1817–1950) is a digital resource provided by the National Library of Scotland <http://www.nls.uk>.

[509] *G*: Tait's shorthand for 'German'.

[510] *B*: Otto von Bismarck (1815–1898); Prusso–German diplomat and statesman; chief architect of the German Empire; held responsible for having provoked the Franco–Prussian War.

He puffs along the battle plain
For is not B's "stick" behind?
Who's dead to honor, lives to pain.
This is your G, fat yet quick
Driven to war by B's stick.

—

What is this G's lawful prize?
Whate'er finds favor in his eyes.
The accursed one who hounds him on
Knows well his self-respect is gone.
He fears his reckless discontent,
And so in devilish mood
Delighted sees it find a vent
In rapine, lust and blood.
That is this German's lawful prize
Whate'er finds favor in his eyes.

B What does his master hope to gain?
That does not seem so very plain.
To inscribe in each historic tome
Another rush of Goths to Rome?[511]
Seeks he the immortality
Of him who fired Diana's shrine,[512]
Or with the ambition cursed is he
With Caliph Omar's fame to shine?[513]

---

[511] *Goths*: a Teutonic people originating from South Sweden (Gotland) who encroached on the Roman Empire during the fourth century. They split into two divisions: the Visigoths and the Ostrogoths. Under Alaric, the Visigoths devastated Greece and sacked Rome in 410 A.D.

[512] *Diana*: the Italian goddess of the woods, women, childbirth and the moon.

[513] *Caliph Omar* (581–644): adviser to Mohammed; succeeded Abu Bakr as 2nd Caliph. During his reign, Islam became an imperial power. He died at the hands of a foreign slave.

What then does B hope to gain?
I give it up—my quest is vain—

—

But what then will this G gain?
The answer is both full & plain—
Contempt from every honest man
The thief's reward, the murderer's ban,
When Europe's slow but sure police
Are set upon his bloody track
And all shall feel that lasting peace
Requires he should be beaten back.
These will the rabid Germain gain
Fettered at length in Europe's chain.

C But are not Gs civilized?
Is justice not among them prized?
These statements which have long been made
But yesterday were not gainsaid—
But he who runneth now may read
Unlikely as it may seem
This quiet content, devoid of greed
Is but an empty dream.
For Germans are not civilised
Say rather they are brutalized.

—

What should the wretched Fman feel,[514]
Downtrodden by the G's heel?
Glad that the veil is drawn aside
Which did so long the monster hide
That lust of Blood & Rapine rife

---

[514] *Fman*: Tait's shorthand for 'Frenchman'.

Are plainly now revealed
Which secretly preparing strife
Were but by Tartuffe's cant concealed.[515]
This satisfaction he may feel
Though crushed beneath that brutal heel.

D Say what shall be the wretches fate
Who finds this monster at his gate?
Dares he to act the part of man
And shoot the murderer if he can?
Dares she her honor to defend
Who[se] face has pleased some Gman boor,[516]
Or dare the starving peasant tend
His little stock, his winter's store?
The gallows is the wretches fate
Behold this monster at his gate.

—

Death and Dishonour, that is all.
In vain for mercy do ye call.
Hell is abroad—his hounds obscene
Are loosed on every village green—
The fairest spots on earth that smiled
Are soiled by murderer's tread
The grey-beard and the sucking child
Heighten the piles of dead.
Pity has fled, & right is wrong,
Nature aghast—Oh Lord how long?

E But, Frenchman, though thou feel the curse,

---

[515] *Tartuffe*: a comedy by Moliére.

[516] *Gman*: a variation on Tait's shorthand for 'German'.

Rejoice—thy foeman's case is worse.
When from his hordes thy land is free
Thou shalt enjoy thy liberty—
He, crushed beneath an iron hand,[517]
With none from "stick" to save,
May yell in praise of Vaterland[518]
But is not less a Slave!
Hurrah—each mangy skulking hound
In Bismark's leash is firmly bound.

—

All honor, Bismark, to thy stick[519]
Which makes thy beery slaves so quick—
But act with caution—have a care—
And dread the vigor of despair!
Even Germans may at last feel shame
The "stick" so long to bear—
Syne play to thee this pleasant game
For "turn about" is fair.
And Frenchmen will pronounce it "chic"
When Bismark's slaves give him the "stick."

[517] *Iron hand*: Bismarck was known as "The Iron Chancellor".

[518] *Vaterland*: German homeland or fatherland.

[519] *Bismark*: a variation in the spelling of 'Bismarck'.

C SCHOOLDAYS AT THE EDINBURGH ACADEMY

Included in this Appendix is the following:

— the syllabus in 1846–1847 for the 6th and 7th classes;

— a list of prizes and medals won by Tait at the Academy;

— a reproduction of the results of the 1846 Academical Club Prize competition;

— and a transcription of the mathematics examination paper from the 1847 Academical Club Prize competition.

The Academical Club Prizes were instituted in 1831 by the "Accie Club"—an association of Academy old-boys founded in 1828.[520] Initially, the prizes were awarded for Latin verses but in 1846, under the influence of the Rector, John Williams, they came to be awarded for the best performance in voluntary written examinations across all subjects: the competition was open to classes 5–7 and the examinations took place in school, over a period of three days.[521] In the 1846 competition, Tait beat Maxwell in mathematics and overall; however, in 1847 the situation was reversed as Maxwell placed first in mathematics that year and Tait came second.

The 1847 mathematics exam paper was attempted by both Tait and Maxwell. It constituted the mathematical component of the Academical Club Prize competition. I am grateful to Andrew McMillan, Honorary Archivist at the Academy, for his kind permission to publish the paper.[522] In Section 4 on geometry, the questions are based on propositions found in Euclid's *Elements*. They require either a proof or a ruler and compass construction.

520 [5, p162]

521 [5, pp162–163]

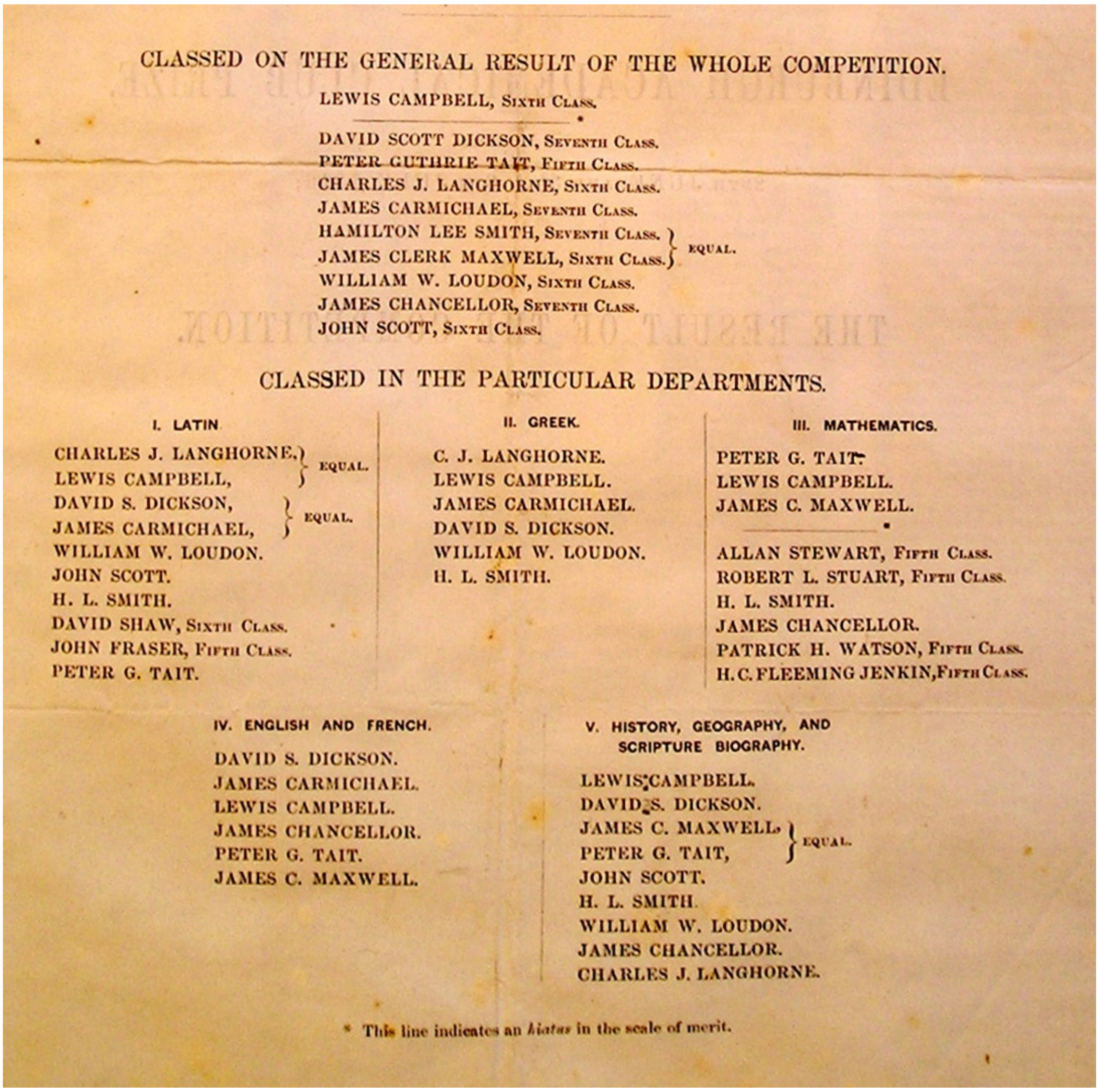
522 The examination paper was published in the Academy's *Prize List* for 1847 [206, pp15–17] which is preserved in the archives of the Edinburgh Academy.

	6th class	**7th class**
English	Shakespeare, Campbell, Irving's Elements of Composition. Weekly Themes. (4 hrs)	Composition and Elements of Physical Science. (2 hrs)
Latin	Portions of Horace, Virgil and Livy. Exercises, Verse and Prose. Elements of Physical Science. Antiquities.	Portions of Cicero, Tacitus, Juvenal, Horace, Catullus. Exercises, Prose and Verse.
Greek	Portions of New Testament, Homer, and Euripides, Sandford's Exercises. Greek Prose Composition. Ancient Geography. Evidences of Christianity. (18 $\frac{3}{4}$ hrs, incl. Latin)	Portions of New Testament, Homer, Sophocles and Xenophon, Dunbar's Exercises, Antiquities. Exercises, Prose and Verse. (16 $\frac{1}{2}$ hrs, incl. Latin)
Arithmetic/ Mathematics	*Arithmetic*: Cube Root. Geometry, four books. Algebra, to Quadratic Equations. (5 hrs)	*Mathematics*: Six Books of Euclid, Trigonometry, Mensuration, Algebra, Quadratic Equations, &c. (6 hrs)
French	Grammar, Phrases, Charles XII. (3 hrs)	Grammar, Portions of Voltaire and Racine. (3 hrs)
German		Grammar, Extracts, Prose and Verse. (2 hrs)

Table C.1: Syllabus in 1846–1847 for the 6th and 7th classes. Information abstracted from the Directors' *Report* for 1847 [59, p7], preserved in the archives of the Edinburgh Academy. Tait was in the 6th class in 1846, Maxwell was in the 7th. There is a heavy emphasis on the classics because the Edinburgh Academy was founded in order to fulfil a need for a school in Edinburgh which could offer a high standard of classical education (higher than that offered by the Scottish seminaries). Edinburgh's Royal High School provided a classical education but the founders of the Edinburgh Academy felt that greater provision was needed for the teaching of Greek, to compete with England's public schools.

Prizes for scholarship and prizes for particular merits.	
Geits class 1841–1842	Dux, Best English Reader, 2nd Best English Scholar.
Second class 1842–1843	Dux, 2nd Best English Scholar.
Third class 1843–1844	Dux, 4th Best Arithmetician.
Fourth class 1844–1845	Dux, Best Arithmetician, Best Latin Verse ('Aeneas in tumulo cereris').
Fifth class 1845–1846	Dux, Silver Medal for Geometry, 2nd Best English Scholar, Best French Scholar.
Sixth class 1846–1847	Dux, Mitchell Medal for Best Mathematician, 2nd Best French Scholar, 2nd Best Examination Papers on Physical Science, Best Latin Verse ('Sit felix proventibus annus: Rusticus loquitur').
Academical Club Prizes.	
1846	*Tait's results*: 3rd overall; 8th in Latin; 1st in Mathematics; 5th in English and French; equal 3rd in History, Geography, and Scripture Biography. *J. C. Maxwell's results*: equal 6th overall; 3rd in Mathematics; 6th in English and French; equal 3rd in History, Geography, and Scripture Biography.
1847	*Tait's results*: 3rd overall; equal 3rd in Latin; equal 4th in English; 2nd in Mathematics. *J. C. Maxwell's results*: 2nd overall; 2nd in Latin; equal 4th in Greek; 1st in English; 1st in Mathematics. (First overall in the 1847 competition was Lewis Campbell, later Professor of Greek at St Andrews and co-author of *Life of James Clerk Maxwell* (1882).)

Table C.2: Prizes and medals won by Tait at the Edinburgh Academy (1841–1847). Information collated from the prize lists for 1842–1847, issued on Exhibition Day which was the annual prize-giving event in July. Sourced from the archives of the Edinburgh Academy.



CLASSED ON THE GENERAL RESULT OF THE WHOLE COMPETITION.

LEWIS CAMPBELL, Sixth Class.

---*

DAVID SCOTT DICKSON, Seventh Class.
PETER GUTHRIE TAIT, Fifth Class.
CHARLES J. LANGHORNE, Sixth Class.
JAMES CARMICHAEL, Seventh Class.
HAMILTON LEE SMITH, Seventh Class. } EQUAL.
JAMES CLERK MAXWELL, Sixth Class. }
WILLIAM W. LOUDON, Sixth Class.
JAMES CHANCELLOR, Seventh Class.
JOHN SCOTT, Sixth Class.

CLASSED IN THE PARTICULAR DEPARTMENTS.

I. LATIN

CHARLES J. LANGHORNE, } EQUAL.
LEWIS CAMPBELL, }
DAVID S. DICKSON, } EQUAL.
JAMES CARMICHAEL, }
WILLIAM W. LOUDON.
JOHN SCOTT.
H. L. SMITH.
DAVID SHAW, Sixth Class.
JOHN FRASER, Fifth Class.
PETER G. TAIT.

II. GREEK.

C. J. LANGHORNE.
LEWIS CAMPBELL.
JAMES CARMICHAEL.
DAVID S. DICKSON.
WILLIAM W. LOUDON.
H. L. SMITH.

III. MATHEMATICS.

PETER G. TAIT.
LEWIS CAMPBELL.
JAMES C. MAXWELL.

---*

ALLAN STEWART, Fifth Class.
ROBERT L. STUART, Fifth Class.
H. L. SMITH.
JAMES CHANCELLOR.
PATRICK H. WATSON, Fifth Class.
H. C. FLEEMING JENKIN, Fifth Class.

IV. ENGLISH AND FRENCH.

DAVID S. DICKSON.
JAMES CARMICHAEL.
LEWIS CAMPBELL.
JAMES CHANCELLOR.
PETER G. TAIT.
JAMES C. MAXWELL.

V. HISTORY, GEOGRAPHY, AND SCRIPTURE BIOGRAPHY.

LEWIS CAMPBELL.
DAVID S. DICKSON.
JAMES C. MAXWELL, } EQUAL.
PETER G. TAIT, }
JOHN SCOTT.
H. L. SMITH.
WILLIAM W. LOUDON.
JAMES CHANCELLOR.
CHARLES J. LANGHORNE.

* This line indicates an *hiatus* in the scale of merit.

Figure C.1: Academical Club Prize results, 1846. Tait places third overall but is top in mathematics. Sourced from Tait's scrapbook. Reproduced with the kind permission of the J.C.M. Foundation.

Mathematics Examination Paper, the Academical Club Prize, 1847

FOURTH DEPARTMENT.

MATHEMATICAL PAPER.

SECT. I.—ARITHMETIC.

1. What fraction of a pound is $\frac{7}{9}$ of a guinea?
2. When .365 of a day have elapsed, what is the exact time?
3. Find to 5 places of decimals what fraction of the year had elapsed at 10 a.m. on the 1st of June.
4. Find the square roots of 11675889, 82369, 0.16, and .000625.
5. Find the fourth root of 28561.
6. Multiply 34.643 by 8.1068.

SECT. II.—MENSURATION.

1. Find the area of a quadrilateral whose diagonal is 97 feet, and the perpendiculars 17 and 23 yards.
2. Find, approximately, in square feet, the area of a regular hexagon, the length of a side being 10 yards.
3. Find the area of the whole figure included in the construction of Euclid I. 47; if the base of the right angled triangle measures 20 feet, and the perpendicular from the vertex to the hypotenuse 8 feet.

SECT. III.—ALGEBRA.

1. Prove that $a^m \times a^n = a^{m+n}$; also that $\frac{a^m}{a^n} = a^{m-n}$.

2. Prove that if 2 be divided into any two parts, the difference of their squares is always equal to twice the difference of the parts themselves.

3. Multiply $a+x+x^2+x^3+x^4$ by $a-x$; collecting the coefficients of like powers of x.

4. Find the value of $(a-b)(a+b-c)+(b-c)(b+c-a)+(c-a)(a+c-b)$.

5. Find the greatest common measure of $18x^3-75x^2+83x-20$, and $36x^3-96x^2+13x+5$.

6. Find the product of the fractions $\frac{ax}{(a-x)^2}$ and $\frac{a^2-x^2}{ab}$.

7. Find the square root of $4x^4+4x^3-11x^2-6x+9$, and $4x^4-12x^3+11x^2-3x+\frac{1}{4}$.

8. Solve the equation $\frac{3x+7}{14} - \frac{2x-7}{21} + 2\frac{3}{4} = \frac{x-4}{4}$.

9. Divide 21 into two parts, such that 10 times one of them may exceed 9 times the other by 1.

10. Shew that the difference between the sum of the cubes of two quantities, and the cube of their sum is equal to three times their product multiplied by their sum.

11. Prove algebraically that the square of half a given straight line is greater than the rectangle contained by any two *un-equal* parts into which the line can be divided.

12. Solve the equations

$$(1) \quad \left.\begin{aligned} x+y&=a \\ x^2-y^2&=b \end{aligned}\right\} \qquad (2) \quad \left.\begin{aligned} 3x-4y&=5 \\ 5x-8y&=7 \end{aligned}\right\}$$

SECT. IV.—GEOMETRY.

1. Define a point,—a plane superficies,—a circle,—parallel straight lines.[523]

2. The greater side of every triangle has the greater angle opposite to it.

3. If two triangles have two sides of one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them of the other; the base of that which has the greater angle shall be greater than the base of the other.

4. If the square described upon one of the sides of a triangle be equal to the squares described upon the other two sides of it; the angle contained by these two sides is a right angle.

5. Divide a straight line into two parts, such that the rectangle contained by the whole and one of the parts, shall be equal to the square of the other part.

6. If two sides of a quadrilateral be parallel; the triangle contained by either of the other sides, and the straight lines drawn from its extremities to the bisection of the opposite side, is half the quadrilateral.

7. Prove that the opposite angles of any quadrilateral figure, inscribed in a circle, are together equal to the two right angles. State and prove the converse of this. What mode of demonstration is usually adopted by Euclid in proving converse propositions?

8. On a given straight line describe a square, and, on the side opposite to the given line, describe equilateral triangles lying in opposite directions; circles described through the extremities of the given line, and through the vertices of these triangles are equal.

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[523] *superficies*: in geometry, 'a continuous extent having only two dimensions (length and breadth, without thickness); an entity such as forms the boundary or one of the boundaries of a solid object, or separates two adjacent portions of space; a surface'. [207]

9. If a straight line bisecting the vertical angle of a triangle also bisect the base, prove that the triangle is isosceles.

10. In an obtuse angled triangle, if perpendiculars be drawn from the points bisecting the sides, they will all pass through the same point. May the same thing be proved in an acute-angled triangle? and if so, what difference occurs in the situation of the point of intersection? If the triangle were inscribed in a circle, prove that the point of intersection is the centre.

11. If the diameter of a circle be one of the equal sides of an isosceles triangle, the base will be bisected by the circumference.

12. Given the diagonals and perpendicular breadth of a parallelogram, to construct it.

13. Divide a circle into two parts, such that the angle contained in one segment shall be equal to twice the angle contained in the other.

14. If an equilateral triangle be described about a circle, the line joining any angle and the middle of the opposite side has its third part intercepted between the angle and the circumference.

15. Given one angle, the side opposite, and the sum of the other two sides, to construct the triangle.

D TRANSCRIPTION OF TAIT'S NOTES FROM TERROT'S PAPER

Images to accompany the text—comprising photographic extracts of the Tait–Maxwell school-book—appear in Figure D.1 (pages 245–253). The few editorial corrections that I have made to the transcription are recorded in Table D.1 (page 254).

On the
Imaginary roots of Negative Quantities.
By the Right Reverend Bishop Terrot.
1847

1. $\sqrt{-1}$ is called impossible or imaginary $\because$ no ordinary algebraic quantity which must be either $+$ or $-$ can give when squared a negative result. Considering however the common application of Algebra to Geometry we easily see, that the assumption that every line must be either $+$ or $-$ is inconsistent with the possibility of drawing a line in any direction. $+1 \times a$ means a line whose length is a drawn in one direction, $-1 \times a$ means the same length of line but drawn in a different direction, and to say that a line of the length of a cannot be drawn in any other direction than one of these is absurd. $\sqrt{-1}$ $\therefore$ is not impossible any more than $-$ or $+1$ and shows only the direction of the line to which it is affixed.

2. If from C [Fig 1] we draw any number of lines such that they shall be in continued proportion and make at the same time $\angle ACA_1 = A_1CA_2 = A_2CA_3$ &c., then calling $CA = 1$, $CA_1 = a$, $CA_2 = a^2$ or the lines are in this series a^0, a^1, a^2, a^3 &c., while the angles which they make with the line CA are $0, \vartheta, 2\vartheta, 3\vartheta$ &c., being the angle $ACA_1\times$ exponent of that radius vector (CA_a for example) from which to CA they are measured. Thus the line whose angle of inclination is on $n\vartheta$ has its length $= a^n$ & vice versâ.

3. If we now assume the several lines CA, CA_1, CA_2, &c., [Fig 2] all equal or radii of a circle the case will not be altered. Let n be a divisor of $2r\pi$ or let $\vartheta = \frac{2r\pi}{n}$. Thus the Radius $a^n = a^{\frac{2r\pi}{\vartheta}}$ is the same in length & position as CA $\therefore a^1 = 1^{\frac{1}{n}} = 1^{\frac{\vartheta}{2r\pi}}$. We know from ordinary Algebraical principles that the several nth roots of unity may be expressed by the series a, a^2, a^3, &c. It therefore follows that we may take the successive Radii of a circle at equal angles for the several roots of unity & conversely. If R be the numerical length of radius that radius inclined to the first at $\angle\vartheta$ is $= R \times 1^{\frac{\vartheta}{2r\pi}}$. We $\therefore$ call $1^{\frac{\vartheta}{2r\pi}}$ the coefficient of direction because it refers only to the direction, never to the length of a line. Thus, $a \times \frac{1+\sqrt{-3}}{2}$ is a line $= a$ simply.

4. Let us next suppose $n = 2$, AB will be a diameter & if $CA = 1$, $CB = -1$. But $a^2 = 1 \therefore a = \pm 1$. But the radii being a, a^2, a must evidently be $= -1$ & $a^2 = +1$. Next let $n = 4$, CA, CD, CB, CE are the 4 roots of the equation $a^4 - 1 = 0$. But the roots are ± 1 & $\pm\sqrt{-1}$. Here CA & CB are symbolized by $+1$ & -1 respectively $\therefore CD$ & CE must be symbolized by $+\sqrt{-1}$ & $-\sqrt{-1}$ respectively, it being however quite optional which direction from C we account positive or negative either in the horizontal or perpendicular lines.

5. It appears from the foregoing Props. that if a line is symbolised by $= a \cdot 1^{\frac{\vartheta}{2r\pi}}$ we know both its length & direction. $a \cdot 1^{\frac{\vartheta}{2r\pi}} \therefore$ represents the actual transference of the point in space by moving from A to C [Fig 3]. But it is also clear that its actual transference in space though not its distance travelled would be the same did it move from A to B & then from B to C. Thus $\therefore$ ($AC\times$ its coefficient of direction) $=$ ($AB\times$ its coefficient of direction) $+$ ($BC\times$ its coefficient of direction). Therefore also the sum of any two lines making an angle with each other is $=$ the diagonal of their parallelogram completed. Even in this startling form it is only the general assertion of a proposition particular cases of which we admit when we say $AB_1 + B_1C = AC$ or that $AC + CB_1 = AB_1$.

1. As examples to elucidate this let ABC (Fig 4) be an isosceles right angled triangle described on the radius AD. If we call AB the radius or Hypotenuse a each of the sides will be in length $\frac{a}{\sqrt{2}}$ & AB is symbolized by $a \times 1^{\frac{45}{360}} = a \times 1^{\frac{1}{8}} = a \times \frac{1+\sqrt{-1}}{\sqrt{2}}$.

But $AC = \frac{a}{\sqrt{2}}$. CB being perpendicular to original position is $= \frac{a}{\sqrt{2}} \times \sqrt{-1}$ (Prop. 4) $\therefore AC + CB = a \times \left[\frac{1}{\sqrt{2}} + \frac{\sqrt{-1}}{\sqrt{2}}\right] = a \times \frac{1+\sqrt{-1}}{\sqrt{2}} = AB$.

2. Let $BAC = 60°$, $BCA = 90°$, then AB in length & direction is $a \cdot 1^{\frac{60}{360}} = a \cdot 1^{\frac{1}{6}} = a \cdot \frac{1+\sqrt{-3}}{2}$, $AC = \frac{a}{2}$, CB in length $= a \cdot \frac{\sqrt{3}}{2}$ $\therefore$ in length & direction jointly $= a \cdot \frac{\sqrt{3}\sqrt{-1}}{2} = a \cdot \frac{\sqrt{-3}}{2}$ $\therefore AC + CB = \frac{a}{2} + a \cdot \frac{\sqrt{-3}}{2} = a \cdot \frac{1+\sqrt{-3}}{2} = AB$.

3. Let the triangle (Fig 5) be Equilateral & let AB be the original position. Let $AB = a$, $AC = a \cdot 1^{\frac{1}{6}}$, $CB = a \cdot 1^{\frac{-1}{6}}$ $\therefore AC + CB = a \cdot \left[1^{\frac{1}{6}} + 1^{\frac{-1}{6}}\right] = a \cdot \left[1^{\frac{1}{6}} + \frac{1}{1^{\frac{1}{6}}}\right] = a \cdot \left[\frac{1^{\frac{1}{3}}+1}{1^{\frac{1}{6}}}\right] = a \cdot \left[\frac{-1+\sqrt{-3}}{2} + 1\right] \times \frac{2}{1+\sqrt{-3}} = a \cdot \left[\frac{1+\sqrt{-3}}{2} + \frac{2}{1+\sqrt{-3}}\right] = a = AB$

6. In the foregoing Props. & Examples it has been taken for granted that we know not only the several nth roots of unity but also their proper order; that is the order in which as coefficients they express the radii drawn to the extremities of the arcs $\vartheta, 2\vartheta, 3\vartheta$, &c., with the original radius. But when we determine the roots of $x^n - 1 = 0$ we obtain them in no fixed order. To discover this order we must observe that two roots are always of the form $a \pm \sqrt{-b}$ comparing which with (Fig 6) a is evidently the part symbolical of the cosine, $+\sqrt{-b}$ that of the sine because it is affected by $\sqrt{-1}$ and is $\therefore$ perpendicular to original radius. Thus $\therefore$ in $a \pm \sqrt{-b}$, $+$ refers to radii in the upper semicircle & $-$ to those in the under; and the two radii whose symbols differ only in the sign of $\sqrt{-b}$ are at equal angles to the original radius on opposite sides of it. $\therefore$ the root in which a is greatest is nearest to the original radius. Thus the roots of $n^6 - 1$ arranged properly are $1, \frac{1+\sqrt{-3}}{2}, \frac{-1+\sqrt{-3}}{2}, -1, \frac{-1-\sqrt{-3}}{2}, \frac{1-\sqrt{-3}}{2}$ symbolizing the radii drawn respectively to the ends of the arcs 0° or 360°, 60°, 120°, 180°, 240°, 300°. For if $+1$ be first -1 having no sinal part must be in the middle. Next $\frac{1+\sqrt{-3}}{2}$ & $\frac{-1+\sqrt{-3}}{2}$ must be in the upper half of the circle and $\frac{1+\sqrt{-3}}{2}$ must come first because its cosine is in CA. And so with the rest.

7. It appears from Props. 4, 5 that the radius drawn to the end of an arc ϑ is $= 1^{\frac{\vartheta}{2r\pi}}$ and this again by $a \pm \sqrt{-b}$ where a is what is trigonometrically called the cosine & $\sqrt{b}$ the sine of ϑ. Now (Fig 6) let $\angle ACA_1 = \vartheta$, $\angle ACA_2 = 2\vartheta$, &c., $\angle ACA_p = p\vartheta$, then $CA_1 = CD + \sqrt{-1} \cdot DA_1 = \cos\vartheta + \sqrt{-1} \cdot \sin\vartheta$, $CA_p = \cos p\vartheta + \sqrt{-1} \cdot \sin p\vartheta$. But by prop. 2, $CA_p = \overline{CA_1}^p = \left(\cos\vartheta + \sqrt{-1} \cdot \sin\vartheta\right)^p \therefore \left(\cos\vartheta + \sqrt{-1} \cdot \sin\vartheta\right)^p = \cos p\vartheta + \sqrt{-1}\sin p\vartheta$, which is Demoivre's Theorem.

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cor. If $p\vartheta = 2\pi$, $\cos p\vartheta + \sqrt{-1} \cdot \sin p\vartheta = 1$. Hence $\left(\cos\vartheta + \sqrt{-1} \cdot \sin\vartheta\right)$, $\left(\cos 2\vartheta + \sqrt{-1} \cdot \sin 2\vartheta\right)$ &c., represent the several pth roots of unity. If we arrange the angles, instead of $\vartheta, 2\vartheta, 3\vartheta$ &c., in pairs thus ϑ & $\overline{p-1} \cdot \vartheta$, 2ϑ & $\overline{p-2} \cdot \vartheta$ &c., the several expressions for x — the several pth roots of unity or the simple factors of $x^p - 1 = 0$ taken in pairs corresponding with the above will be $\left(x - \cos\vartheta - \sqrt{-1} \cdot \sin\vartheta\right)$ & $\left(x - \cos\overline{p-1}\,\vartheta - \sqrt{-1} \cdot \sin\overline{p-1}\,\vartheta\right)$ which last is $= \left(x - \cos\overline{p\vartheta - \vartheta} - \sqrt{-1} \cdot \sin\overline{p\vartheta - \vartheta}\right) = \left(x - \cos\overline{2\pi - \vartheta} - \sqrt{-1} \cdot \sin\overline{2\pi - \vartheta}\right) = \left(x - \cos\vartheta + \sqrt{-1} \cdot \sin\vartheta\right)$. In the same way the next pair must be $\left(x - \cos 2\vartheta + \sqrt{-1} \cdot \sin 2\vartheta\right)$ & $\left(x - \cos 2\vartheta - \sqrt{-1} \cdot \sin 2\vartheta\right)$. Multiplying these together for the quadratic factors of $x^p - 1$, we obtain when p is even $x^p - 1 = (x^2 - 1)(x^2 - 2x\cos\vartheta + 1) \cdot (x^2 - 2x\cos 2\vartheta + 1)$ to $\frac{p}{2}$ terms. But when p is odd $x^p - 1 = (x - 1)(x^2 - 2x\cos\vartheta + 1)$ &c., to $\frac{p+1}{2}$ terms, where ϑ it may be observed is $= \frac{2\pi}{p}$.

8.

$$\sin\overline{A + B} = \sin A \cdot \cos B + \cos A \cdot \sin B$$

$$\cos\overline{A + B} = \cos A \cdot \cos B - \sin A \cdot \sin B$$

Let arc AB (Fig 7) $= \mathcal{A}$, BD_2 & AD_1 each $= \mathcal{B}$. Then by Prop. 3, $CB = r \cdot 1^{\frac{\mathcal{A}}{2\pi}}$, $CD_1 = r \cdot 1^{\frac{\mathcal{B}}{2\pi}}$, $CD_2 = r \cdot 1^{\frac{\mathcal{A}+\mathcal{B}}{2\pi}} \therefore CD_2 = r \cdot 1^{\frac{\mathcal{A}}{2\pi}} \cdot 1^{\frac{\mathcal{B}}{2\pi}}$. But by Prop. 7, $1^{\frac{\mathcal{A}}{2\pi}} = \cos\mathcal{A} + \sqrt{-1} \cdot \sin\mathcal{A}$, $1^{\frac{\mathcal{B}}{2\pi}} = \cos\mathcal{B} + \sqrt{-1} \cdot \sin\mathcal{B} \therefore 1^{\frac{\mathcal{A}+\mathcal{B}}{2\pi}} = \cos\mathcal{A} \times \cos\mathcal{B} - \sin\mathcal{A} \times \sin\mathcal{B} + \sqrt{-1}\left(\sin\mathcal{A} \cdot \cos\mathcal{B} + \cos\mathcal{A} \cdot \sin\mathcal{B}\right)$, but $1^{\frac{\mathcal{A}+\mathcal{B}}{2\pi}} = \cos\overline{\mathcal{A} + \mathcal{B}} + \sqrt{-1}\sin\overline{\mathcal{A} + \mathcal{B}}$. Equating then the sinal & cosinal parts of these, we have, $\cos\mathcal{A} \cdot \cos\mathcal{B} - \sin\mathcal{A} \cdot \sin\mathcal{B} = \cos\overline{\mathcal{A} + \mathcal{B}}$, $\sin\mathcal{A} \cdot \cos\mathcal{B} + \cos\mathcal{A} \cdot \sin\mathcal{B} = \sin\overline{\mathcal{A} + \mathcal{B}}$.

Definition. It should be observed that in the following propositions a line expressed by letter simply as AB must be considered both as to length & direction while when in brackets thus (AB) its length alone is referred to. Thus $(AB)1^{\frac{\vartheta}{2\pi}} = AB$.

9. In any right angled triangle the sum of the squares of the sides is = square of hypotenuse. Let CA (Fig 6) $= r$, then $CA_1 = r \cdot 1^{\frac{\vartheta}{2\pi}}$, & $CA_{n-1} = r \cdot 1^{\frac{-\vartheta}{2\pi}}$ $\therefore CA_1 \times CA_{n-1} = r^2 \times 1^{\frac{\vartheta}{2\pi}} \times \frac{1}{1^{\frac{\vartheta}{2\pi}}} = r^2$. Also $CA_1 = (CD_1) + \sqrt{-1}(D_1A_1)$, $CA_{n-1} = (CD_1) - \sqrt{-1}(D_1A_1)$ for $(D_1A_1) = (D_1A_{n-1})$ $\therefore CA_1 \times CA_{n-1} = (CD_1)^2 + (D_1A_1)^2$ which is $\therefore = r^2 = (CA)^2 = (CA_1)^2$ its equivalent in area.

10. ***<u>Cotes'</u> Properties of the Circle.*** Let the circumference be divided into n equal parts and join OP_1, OP_2, OP_3, &c., (Fig 8) and also join P_1, P_2, P_3 with C any point in the Diameter. Then $CP_1 = OP_1 - OC$, $CP_2 = OP_2 - OC$ &c., $\therefore$ $CP_1 \cdot CP_2 \cdot CP_3 \cdots\ CP_n = \Sigma_n \cdot (OA)^n - \Sigma_{n-1} \cdot (OA)^{n-1} \cdots \pm OC^n$, where Σ_n is the product of all the coefficients of direction for OP_1, OP_2, &c., Σ_{n-1} the sum of $\wedge$ (*the product sq? P.G. Tait*) these coefficients taken $\overline{n-1}$ together & so on. But these coefficients are also the roots of the Equation $x^n - 1 = 0$. Now the product of the roots of this Equation with their signs changed is -1 & Σ_n is = the product with their signs unchanged. Therefore if n be even $\Sigma_n = -1$ but if odd $+1$, and in either case Σ_{n-1}, Σ_{n-2} &c., each $= 0$. Hence $CP_1 \cdot CP_2 \cdot CP_3 \cdots CP_n = \pm(OA)^n \pm (OC)^n$; the upper signs to be used when n is even, the lower when odd. Here CP_1, CP_2 &c., consider the lines both as to length and direction, we must $\therefore$ divide the first or multiply the second by the product of all their coefficients of direction. If n be even the several pairs as CP_1, CP_{n-1} are evidently of the form $(CP_1) \cdot 1^{\frac{\vartheta}{2\pi}}$ and $(CP_{n-1}) \cdot 1^{\frac{-\vartheta}{2\pi}}$ $\therefore CP_1 \times CP_{n-1} = (CP_1) \times (CP_{n-1})$ and this is true for every pair except $CA = (CA) \cdot +1$ & $CB = (CB) \cdot -1$ $\therefore (CP_1) \cdot (CP_2) \cdots (CP_n) = (-OA^n + OC^n) \cdot -1 = OA^n - OC^n$. But if n be odd the several pairs remain as before only no P falling on B, -1 is not a coefficient of direction $\therefore$ $(CP_1) \cdot (CP_2) \cdot$ &c., $= OA^n - OC^n$ as before.

Cor.1. If C be on the opposite side of O from A, the other conditions remaining the same OC is negative. If n be even the deduction in the prop. remains unchanged. But if n be odd, $(CP_1) \cdot (CP_2) \cdot \cdot$ &c., $= OA^n + OC^n$. Here it may be remarked that when lines as OA are in the original direction, since the coefficient of direction in that case is unity it is immaterial whether we write OA or (OA).

Ex. Let $n = 3$ & $OC = \frac{1}{2}$, then, $(AC) = \frac{3}{2}$, $(CP_1) = (CP_2) = \frac{\sqrt{3}}{2} \therefore (CA) \cdot (CP_1) \cdot (CP_2) = \frac{3}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{9}{8} = 1 + \frac{1}{8} = \overline{1}^3 + \overline{\frac{1}{2}}^3 = OA^3 + OC^3$.

Cor.2. If C be in OA produced the reasoning & result will be the same as in the prop., only, that now CA & CB being of the same affection -1 is not a divisor of the second member of the Equation, &, $(CP_1) \cdot (CP_2) \cdot$ &c., $= (OC)^n - (OA)^n$.

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11. If from A the extremity of the Diameter (Fig 8) the circumference be divided into n equal parts & if these several extremities be joined, then $(AP_1) \cdot (AP_2) \cdots (AP_{n-1}) = nCA^{n-1}$. As in former prop. $AP_1 = CP_1 - CA$, $AP_2 = CP_2 - CA$ & so on $\therefore AP_1 \cdot AP_2 \cdots AP_{n-1} = \overline{CP_1 - CA} \cdot \overline{CP_2 - CA}$ &c., to $\overline{n-1}$ factors $= R^{n-1} \cdot \{S_{n-1} - S_{n-2} \cdots \pm S_1 \pm 1\}$ where S_1, S_2 &c., are the sum, sum of products two & two, &c., of all the values of $1^{\frac{1}{n}}$ except unity there being no line drawn from A to the circumference in the direction CA. S_1, S_2 &c., are $\therefore$ the coefficients of the Equation $\frac{x^n - 1}{x - 1}$ or of $x^{n-1} + x^{n-2} + \cdots + 1 = 0$ with the signs changed for the products of odd numbers of roots, unchanged for even ones. If $\therefore \overline{n-1}$ be even $S_{n-1} = +1$, $S_{n-2} = -1$, & so on. If $\overline{n-1}$ be odd $S_{n-1} = -1$, $S_{n-2} = +1$ & so on. $\therefore AP_1 \cdot AP_2 \cdot$ &c.,$= R^{n-1} \times \pm\{1 + 1 + 1$ to n terms$\} = \pm nR^{n-1}$ according as $\overline{n-1}$ is even or odd. If $\overline{n-1}$ be even, $AP_1 \cdot AP_2 \cdot$ &c., $= (AP_1)(AP_2) \cdot$ &c., the several pairs of coefficients giving unity for their products. If $\overline{n-1}$ be odd, then the several pairs give as before their product unity but there remains the factor $-AB$ which has for its coefficient -1. $\therefore$ in either case $(AP_1)(AP_2)$ &c., $(AP_{n-1}) = nR^{n-1}$.

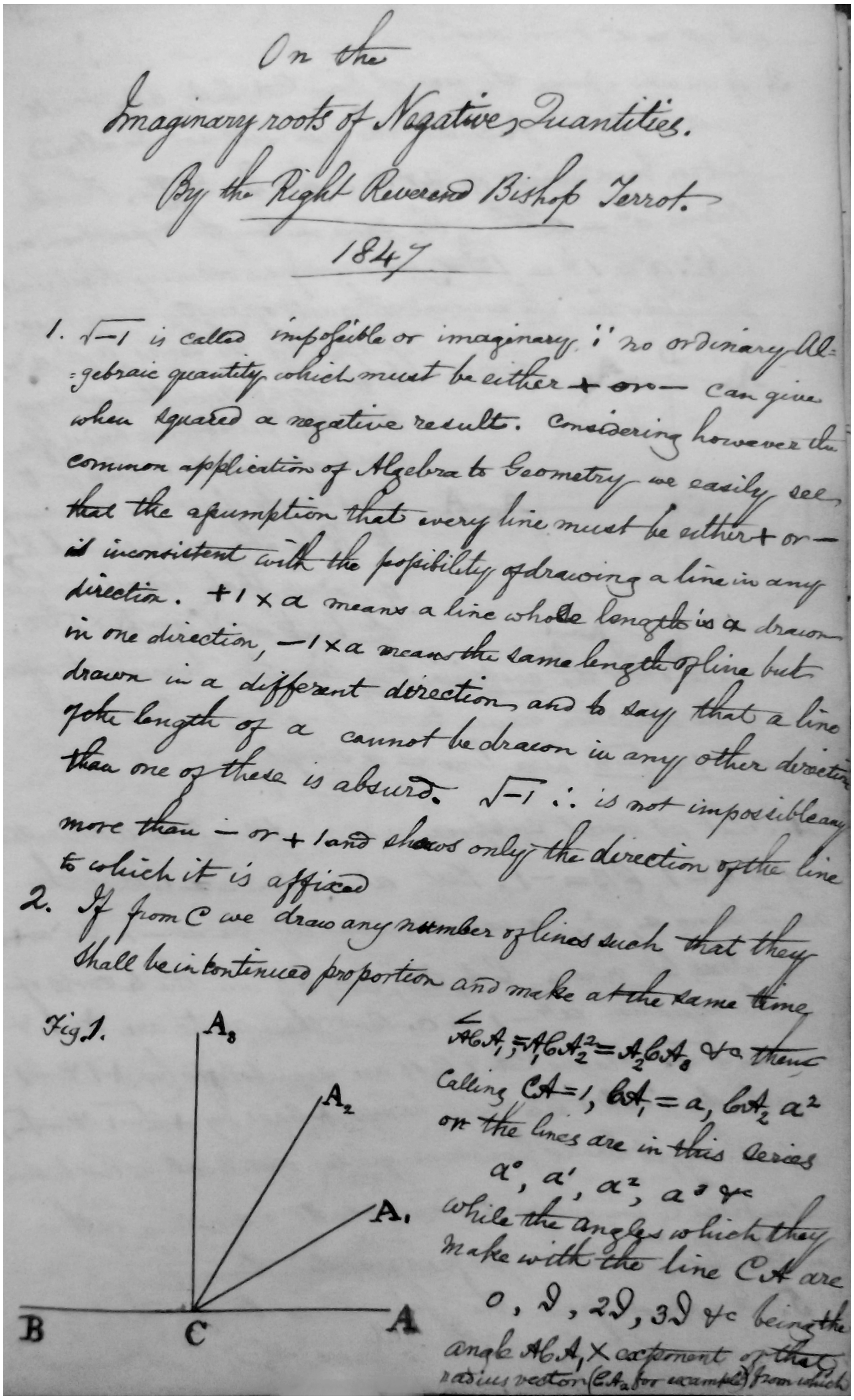
12. If by this method we undertake to prove that the angles at the base of an Isosceles triangle are = eachother we have $(AC) = (BC)$ (Fig 5). But $AC = (AC) \cdot 1^{\frac{A}{2\pi}} = (AC) \cdot [a + \sqrt{-b}]$, $CB = AD = (AC) \cdot 1^{\frac{-B}{2\pi}} = (AC) \cdot [a' + \sqrt{-b'}]$. But $AC + CB = AB$. $\therefore$ $(AC) \cdot (a + a' + \sqrt{-b} + \sqrt{-b'}) = AB =$ a positive quantity consequently the sinal parts destroy one another or $\sqrt{-b} = -\sqrt{-b'}$ or $b = -b'$. Therefore the angles A & B have their sines of equal length but of different affections. The angles themselves $\therefore$ being together less than π are geometrically equal to each other.

Cor. Much in the same way we might prove that in every triangle the greater side has the greater angle opposite to it & vice versâ that the greater angle has the greater side opposite to it.

May 27th 1847.

P. G. Tait.

Figure D.1: Photographic reproduction of Tait's notes from Terrot's paper. Sourced from the Tait–Maxwell school-book. Reproduced with the kind permission of the J.C.M. Foundation.



On the
Imaginary roots of Negative Quantities.
By the Right Reverend Bishop Terrot.
1847

1. $\sqrt{-1}$ is called impossible or imaginary ∵ no ordinary Algebraic quantity which must be either + or − can give when squared a negative result. Considering however the common application of Algebra to Geometry we easily see that the assumption that every line must be either + or − is inconsistent with the possibility of drawing a line in any direction. $+1 \times a$ means a line whose length is a drawn in one direction, $-1 \times a$ means the same length of line but drawn in a different direction, and to say that a line of the length of a cannot be drawn in any other direction than one of these is absurd. $\sqrt{-1}$ ∴ is not impossible any more than − or + 1 and shows only the direction of the line to which it is affixed

2. If from C we draw any number of lines such that they shall be in continued proportion and make at the same time $\angle ACA_1 = A_1CA_2 = A_2CA_3$ &c then calling $CA = 1$, $CA_1 = a$, CA_2 a^2 or the lines are in this series

a^0, a^1, a^2, a^3 &c

while the angles which they make with the line CA are

$0, \vartheta, 2\vartheta, 3\vartheta$ &c being the angle ACA_1 × exponent of that radius vector (CA_2 for example) from which to CA they are measured. Thus the line whose angle of inclination is $n\vartheta$ has its length $= a^n$ & vice versâ.

3. If we now assume the several lines CA, CA_1, CA_2 &c all equal or radii of a circle the case will not be altered. Let n be a divisor of $2r\pi$ or let $\vartheta = \frac{2r\pi}{n}$. Thus the Radius $a^n = a^{\frac{2r\pi}{\vartheta}}$ is the same in length & position as $CA \therefore a' = 1^{\frac{1}{n}} = 1^{\frac{\vartheta}{2r\pi}}$. We know from ordinary Algebraical principles that the several n^{th} roots of unity may be expressed by the series a, a^2, a^3 &c.

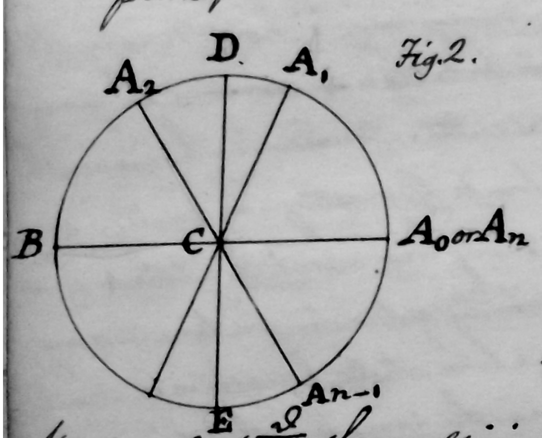


Fig. 2.

It therefore follows that we may take the successive Radii of a circle at equal angles for the several roots of unity & conversely. If R be the numerical length of radius that radius inclined to the first at $\vartheta°$ is $= R \times 1^{\frac{\vartheta}{2r\pi}}$.

We $\therefore$ call $1^{\frac{\vartheta}{2r\pi}}$ the <u>coefficient of direction</u> because it refers only to the direction, never to the length of a line. Thus $a \times \frac{1+\sqrt{-3}}{2}$ is a line $= a$ simply.

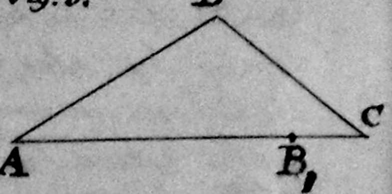
4. Let us next suppose $n = 2$, AB will be a diameter, & if $CA = 1$, $CB = -1$. But $a^2 = 1 \therefore a = \pm 1$. But the radii being a, a^2, a must evidently be $= -1$ & $a^2 = +1$.

Next let $n = 4$, CA, CD, CB, CE are the 4 roots of the equation $a^4 - 1 = 0$. But these roots are ± 1 & $\pm\sqrt{-1}$. Here CA & CB are symbolized by $+1$ & -1 respectively $\therefore CD$ & CE must be symbolized by $+\sqrt{-1}$ & $-\sqrt{-1}$ respectively, it being however quite optional which direction from C we account positive or negative either in the horizontal or perpendicular lines.

5. It appears from the foregoing Props that if a line is symbolized by $a \cdot 1^{\frac{\vartheta}{2r\pi}}$ we know both its length & direction.

$a \cdot 1^{\frac{9}{360}}$ ∴ represents the actual transference of the point in space by moving from A to C. But it is also clear that its actual transference in space though not its distance travelled would be the same did it move from A to B & then from B to C. Thus ∴ (AC × its coefficient of direction) = (AB × its coefficient of direction) + (BC × its coefficient of direction). Therefore also the sum of any two lines making an angle with each other is = the diagonal of their parallelogram completed. Even in this startling form it is only the general assertion of a proposition particular cases of which we admit when we say $AB_1 + B_1C = AC$ or that $AC + CB_1 = AB_1$.

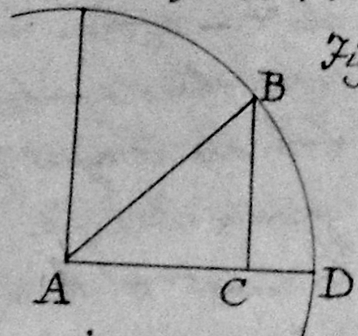
Fig. 3.



1. As examples to elucidate this let ABC (Fig 4) be an isosceles right angled triangle described on the radius AD. If ~~we~~ we call AB the radius or Hypotenuse a each of the sides will be in length $\frac{a}{\sqrt{2}}$ & AB is symbolized by $a \times 1^{\frac{45}{360}} = a \times 1^{\frac{1}{8}}$ $= a \times \frac{1+\sqrt{-1}}{\sqrt{2}}$, But $AC = \frac{a}{\sqrt{2}}$ CB being perpendicular to original position is $= \frac{a}{\sqrt{2}} \times \sqrt{-1}$ (Prop 4) ∴

$$AC + CB = a\left[\frac{1}{\sqrt{2}} + \frac{\sqrt{-1}}{\sqrt{2}}\right] = a$$
$$= a \times \frac{1+\sqrt{-1}}{\sqrt{2}} = AB.$$

Fig. 4.



2. Let $BAC = 60°$, $BCA = 90°$, then AB in length & direction is $a \cdot 1^{\frac{60}{360}} = a \cdot 1^{\frac{1}{6}} = a \cdot \frac{1+\sqrt{-3}}{2}$, $AC = \frac{a}{2}$, CB in length $= a\frac{\sqrt{3}}{2}$ ∴ in length & direction jointly $= a \cdot \frac{\sqrt{3}\cdot\sqrt{-1}}{2} = a \cdot \frac{\sqrt{-3}}{2}$.

$$∴ AC + CB = \frac{a}{2} + a \cdot \frac{\sqrt{-3}}{2} = a \cdot \frac{1+\sqrt{-3}}{2} = AB.$$

3. Let the triangle (Fig 5) be Equilateral & let AB be the original position. Let $AB = a$, $AC = a \cdot 1^{\frac{1}{6}}$, $CB = a \cdot 1^{-\frac{1}{6}}$ ∴ $AC + CB$

$$= a \cdot \left[1^{\frac{1}{6}} + 1^{-\frac{1}{6}}\right] = a \cdot \left[1^{\frac{1}{6}} + \frac{1}{1^{\frac{1}{6}}}\right] = a \cdot \left[\frac{1^{\frac{1}{3}} + 1}{1^{\frac{1}{6}}}\right] =$$
$$= a \cdot \left[\frac{-1+\sqrt{-3}}{2} + 1\right] \times \frac{2}{1+\sqrt{-3}} = a \cdot \left[\frac{1+\sqrt{-3}}{2} \cdot \frac{2}{1+\sqrt{-3}}\right] = a =$$
$$= AB.$$

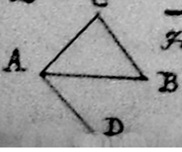


Fig. 5.

6. In the foregoing Props. & Examples it has been taken for granted that we know not only the several n^{th} roots of unity but also their proper order; that is the order in which as coefficients they express the radii drawn to the extremities of the arcs ϑ, 2ϑ, 3ϑ &c with the original radius. But when we determine the roots of $x^n - 1 = 0$ we obtain them in no fixed order. To discover this order we must observe that two roots are always of the form $a \pm \sqrt{-b}$ comparing ~~which~~ with (Fig 6) a is evidently ~~the~~ part symbolical of the cosine & $\sqrt{-b}$ that of the sine because it is affected by $\sqrt{-1}$ and is ∴ perpendicular to original radius. Thus ∴ in $a \pm \sqrt{-b}$, + refers to radii in the upper semicircle & − to ~~those~~ in the under; and ~~that~~ the two radii whose symbols differ only in the sign of $\sqrt{-b}$ are at equal angles to the original radius on opposite sides of it. ∴ the root in which a is greatest is nearest to the original radius. Thus the roots of $x^6 - 1$ arranged properly are

$$1,\ \frac{1+\sqrt{-3}}{2},\ \frac{-1+\sqrt{-3}}{2},\ -1,\ \frac{-1-\sqrt{-3}}{2},\ \frac{1-\sqrt{-3}}{2}.$$

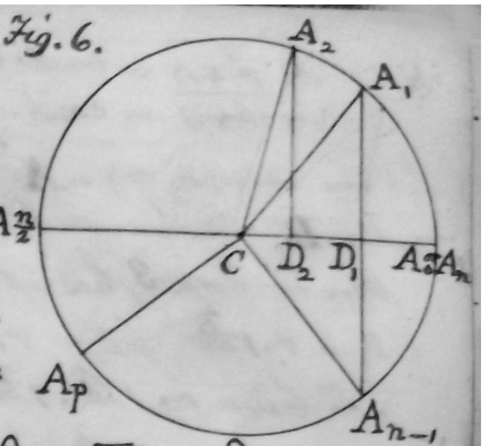
symbolizing the radii drawn respectively to the ends of the arcs

$$0^\circ \text{ or } 360^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ.$$

For if $+1$ be first -1 having no sinal part must be in the middle. Next $\frac{1+\sqrt{-3}}{2}$ & $\frac{-1+\sqrt{-3}}{2}$ must be in the upper half of the circle and $\frac{1+\sqrt{-3}}{2}$ must come first because its cosine is in 6A. And so with the rest.

7. It appears from Prop^s. 4, 5 that the radius drawn to the end of an arc ϑ is $= 1^{\frac{\vartheta}{2\pi}}$ and this again by $a \pm \sqrt{-b}$ where a is what is trigonometrically called the cosine & $\sqrt{b}$ the sine of ϑ. Now (Fig 6) let $\angle A_0CA_1 = \vartheta$, $\angle A_0CA_2 = 2\vartheta$ &c $\angle A_0CA_p = p\vartheta$, then

Fig. 6.



$$CA_1 = CD + \sqrt{-1}\cdot DA_1 = \cos\vartheta + \sqrt{-1}\cdot\sin\vartheta,$$
$$CA_p = \cos p\vartheta + \sqrt{-1}\,\sin p\vartheta,$$

But by prop 2,

$$CA_p = \overline{CA_1}\big|^p = (\cos\vartheta + \sqrt{-1}\,\sin\vartheta)^p$$
$$\therefore (\cos\vartheta + \sqrt{-1}\,\sin\vartheta)^p = \cos p\vartheta + \sqrt{-1}\,\sin p\vartheta,$$ which is

<u>Demoivre's</u> Theorem.

Cor. If $p\vartheta = 2\pi$, $\cos p\vartheta + \sqrt{-1}\cdot\sin p\vartheta = 1$.

Hence $(\cos\vartheta + \sqrt{-1}\cdot\sin\vartheta)$, $(\cos 2\vartheta + \sqrt{-1}\cdot\sin 2\vartheta)$, &c represent the several p^{th} roots of unity. If we arrange the angles, instead of $\vartheta, 2\vartheta, 3\vartheta$ &c, in pairs thus ϑ & $\overline{p-1}\cdot\vartheta$, 2ϑ & $\overline{p-2}\cdot\vartheta$ &c the several expressions for x — the several p^{th} roots of unity or the simple factors of $x^p - 1 = 0$ taken in pairs corresponding with the above will be

$$(x - \cos\vartheta - \sqrt{-1}\cdot\sin\vartheta) \;\&\; (x - \cos\overline{p-1}\,\vartheta - \sqrt{-1}\cdot\sin\overline{p-1}\cdot\vartheta)$$

which last is $= (x - \cos\overline{p\vartheta - \vartheta} - \sqrt{-1}\cdot\sin\overline{p\vartheta - \vartheta}) =$

$$= (x - \cos\overline{2\pi - \vartheta} - \sqrt{-1}\cdot\sin.\overline{2\pi - \vartheta}) = (x - \cos\vartheta + \sqrt{-1}\cdot\sin\vartheta)$$

In the same way the next pair must be

$$(x - \cos 2\vartheta + \sqrt{-1}\,\sin 2\vartheta) \;\&\; (x - \cos 2\vartheta - \sqrt{-1}\cdot\sin 2\vartheta)$$

Multiplying these together for the quadratic factors of $x^p - 1$, we obtain when p is even

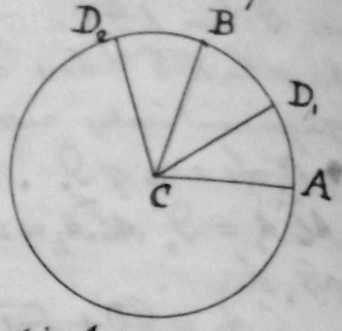
$$x^p - 1 = (x^2 - 1)\cdot(x^2 - 2x\cos\vartheta + 1)\cdot(x^2 - 2x\cos 2\vartheta + 1) \text{ to } \frac{p}{2} \text{ terms}$$

But when p is odd $x^p - 1 = (x-1)(x^2 - 2x\cos\vartheta + 1)$ &c to $\frac{p+1}{2}$ terms

where ϑ it may be observed is $= \frac{2\pi}{p}$.

8. $\sin \overline{A+B} = \sin A \cdot \cos B + \cos A \cdot \sin B$.
$\cos \overline{A+B} = \cos A \cdot \cos B - \sin A \cdot \sin B$.

Fig. 7



Let arc AB (Fig 7) = **A**, BD_2 & AD_1 each = **B**.

then by Prop. 3, $CB = r \cdot 1^{\frac{A}{2\pi}}$

$CD_1 = r \cdot 1^{\frac{B}{2\pi}}$, $CD_2 = r \cdot 1^{\frac{A+B}{2\pi}}$

$\therefore CD_2 = r \cdot 1^{\frac{A}{2\pi}} \cdot 1^{\frac{B}{2\pi}}$;

But by Prop 7, $1^{\frac{A}{2\pi}} = \cos A + \sqrt{-1} \sin A$,

$1^{\frac{B}{2\pi}} = \cos B + \sqrt{-1} \sin B$,

$\therefore 1^{\frac{A+B}{2\pi}} = \cos A \times \cos B - \sin A \times \sin B + \sqrt{-1}$ (##) $(\sin A \cdot \cos B + \cos A \cdot \sin B.)$,

but $1^{\frac{A+B}{2\pi}} = \cos \overline{A+B} + \sqrt{-1} \sin \overline{A+B}$.

Equating then the sinal & cosinal parts of these, we have

$\cos A \cdot \cos B - \sin A \cdot \sin B = \cos \overline{A+B}$
$\sin A \cdot \cos B + \cos A \cdot \sin B = \sin \overline{A+B}$

Definition.

It should be observed that in the following propositions a line expressed by letters simply as AB must be considered both as to length & direction while when in brackets they (AB) its length alone is referred to. Thus $(AB)\, 1^{\frac{\theta}{2\pi}} = AB$.

9. In any right angled triangle the sum of the squares of the sides is = square of hypotenuse.

Let CA (Fig 8) = r, then $CA_1 = r \cdot 1^{\frac{\theta}{2\pi}}$ & $CA_{n-1} = r \cdot 1^{-\frac{\theta}{2\pi}}$

$\therefore CA_1 \times CA_{n-1} = r^2 \times 1^{\frac{\theta}{2\pi}} \times \frac{1}{1^{\frac{\theta}{2\pi}}} = r^2$,

Also $CA_1 = (CD_1) + \sqrt{-1} \cdot (D_1A_1)$.

$CA_{n-1} = (CD_1) - \sqrt{-1}\, (D_1A_1)$ for $(D_1A_1) = (D_1A_{n-1})$

$\therefore CA_1 \times CA_{n-1} = (CD_1)^2 + (D_1A_1)^2$ which is $\therefore = r^2 = (CA^2) = (CA_1)^2$ its equivalent in area.

Cotes' Properties of the Circle.

Let the circumference be divided into n equal parts and join OP_1, OP_2, OP_3 &c (Fig. 8.) and also join P_1, P_2, P_3 with C any point in the Diameter. Then

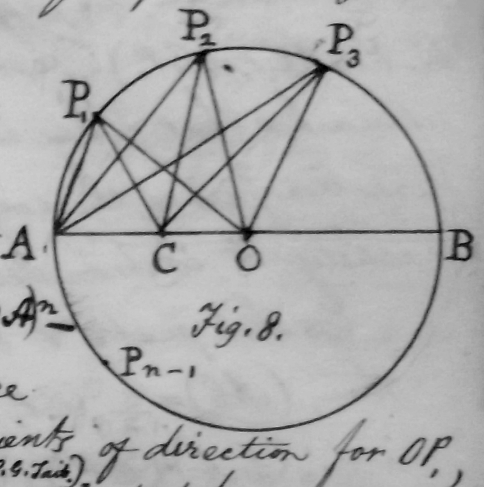


Fig. 8.

$CP_1 = OP_1 - OC$, $CP_2 = OP_2 - OC$ &c

$\therefore CP_1 ; CP_2 ; CP_3 \cdots CP_n = \Sigma_n \cdot (OA)^n - \Sigma_{n-1}(OA)^{n-1} \ldots \pm OC^n$ where

Σ_n is the product of all the coefficients (the product of ? P. G. Tait) of direction for OP_1, OP_2 &c, Σ_{n-1} the sum of these coefficients taken $\overline{n-1}$ together & so on. But these coefficients are also the roots of the Equation $x^n - 1 = 0$. Now the product of the roots of this Equation with their signs changed is -1 & Σ_n is = the product with their signs unchanged. Therefore if n be even $\Sigma_n = -1$ but if odd $= +1$, and in either case Σ_{n-1}, Σ_{n-2} &c each $= 0$. Hence $CP_1 \cdot CP_2 \cdot CP_3 \cdots CP_n = \pm(OA)^n \pm (OC)^n$; the upper signs to be used when n is even, the lower when odd.

Here CP_1 CP_2 &c consider the lines both as to length and direction, we must $\therefore$ divide the first or multiply the second by the product of all their coefficients of direction. If n be even the several pairs as CP_1 CP_{n-1} are evidently of the form $(CP_1) \cdot 1^{\frac{\theta}{2\pi}}$ and $(CP_{n-1}) \cdot 1^{\frac{-\theta}{2\pi}}$ $\therefore CP_1 \times CP_{n-1} = (CP_1) \times (CP_{n-1})$ and this is true for every pair except $CA = (CA) \cdot +1$ & $CB = (CB) \cdot -1$ $\therefore (CP_1) \cdot (CP_2) \cdots CP_n = (-OA^n + OC^n) \cdot -1 = OA^n - OC^n$

But if n be odd the several pairs remain as before only no P falling on B, -1 is not a coefficient of direction $\therefore (CP_1) \cdot (CP_2)$ &c $= OA^n - OC^n$ as before.

Cor. 1. If C be on the opposite side of O from A, the other conditions remaining the same OC is negative. If n be even the deduction in the prop. remains unchanged. But if n be odd, $(CP_1)(CP_2)\ldots$ &c $= OA^n + OC^n$: Here it may be remarked that when lines as OA are in the original direction, since the coefficient of direction in that case is unity it is immaterial whether we write OA or (OA).

Ex. Let $n = 3$ & $OC = \frac{1}{2}$,

then, $(AC) = \frac{3}{2}$, $(CP_1) = (CP_2) = \frac{\sqrt{3}}{2}$

$\therefore (CA)\cdot(CP_1)\cdot(CP_2) = \frac{3}{2}\cdot\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} = \frac{9}{8} = 1 + \frac{1}{8} =$

$\overline{1}|^3 + \overline{\tfrac{1}{2}}|^3 = OA^3 + OC^3$:

Cor. 2. If C be in OA produced the reasoning & result will be the same as in the prop; only, that now CA & CB being of the same affection -1 is not a divisor of the second member of the Equation, &,

$$(CP_1)\cdot(CP_2)\cdot \&c = (OC)^n - (OA)^n:$$

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11. If from A the extremity of the Diameter (Fig. 8.) the circumference be divided into n equal parts & A ~~their~~ several extremities be joined, then,

$$(AP_1)(AP_2)(AP_{n-1}) = n\,\overline{CA}^{n-1}$$!

As in former prop. $AP_1 = CP_1 - CA$, $AP_2 = CP_2 - CA$ and so on

$\therefore AP_1 \cdot AP_2 \cdot\cdot AP_{n-1} = \overline{CP_1 - CA}\cdot\overline{CP_2 - CA}$ & to $n-1$ factors

$= R^{n-1}\left\{S_{n-1} - S_{n-2} \cdots \pm S_1 \pm 1\right\}$ where S_1, S_2 &c are the sum, sum of products two & two, &c of all the values of $1^{\frac{1}{n}}$ except unity there being no line drawn from A to the circumference in the direction CA. S_1, S_2 &c are $\therefore$ the coefficients of the Equation $\frac{x^n - 1}{x - 1}$ or of $x^{n-1} + x^{n-2} + \cdots + 1 = 0$ with the signs changed for the products of odd numbers of roots, unchanged for even ones.

If $\therefore \overline{n-1}$ be even $S_{n-1} = +1,\ S_{n-2} = -1$ & so on,
if $\overline{n-1}$ be odd $S_{n-1} = -1,\ S_{n-2} = +1$ & so on.

$\therefore AP_1 \cdot AP_2 \cdot$ &c $= R^{n-1} \times \pm \{1+1+1 \text{ to } n \text{ terms}\} = \pm n R^{n-1}$ according as $\overline{n-1}$ is even or odd.

If $\overline{n-1}$ be even, $AP_1^{+} \cdot AP_2 \cdot$ &c $= (AP_1)(AP_2)$ &c the several pairs of coefficients giving unity for their products.

If $\overline{n-1}$ be odd then the several pairs give as before their product unity but there remains the factor AB which has for its coefficient -1.

$\therefore$ in either case $(AP_1)(AP_2)$ &c $(AP_{n-1}) = nR^{n-1}$

12. If by this method we undertake to prove that the angles at the base of an Isosceles triangle are = each other we have $(AC) = (BC)$ (Fig. 5).

But $AC = (AC) \cdot 1^{\frac{A}{2\pi}} = (AC) \cdot \left[a + \sqrt{-b}\right]$,
$CB = AD = (AC) \cdot 1^{-\frac{B}{2\pi}} = (AC) \cdot (a' + \sqrt{-b'})$.

But $AC + CB = AB$.

$\therefore (AC) \cdot (a + a' + \sqrt{-b} + \sqrt{-b'}) = AB =$ a positive quantity consequently the sinal parts destroy one another or $\sqrt{-b} = -\sqrt{-b'}$ or $b = -b'$. Therefore the angles A & B have their sines of equal length but of different affections. The angles themselves $\therefore$ being together less than π are geometrically equal ~~to~~ to each other.

Cor. Much in the same way we might prove that in every triangle the greater side has the greater angle opposite to it & vice versâ that the greater angle has the greater side opposite to it.

May 27th 1847.
P. G. Tait.

Editorial corrections made to the transcription of Tait's notes from Terrot's paper

Reference	Editorial correction
§2	Tait has: '$\angle ACA_1 = A_1CA_2^2 = A_2CA_3$ &c., then calling $CA = 1$, $CA_1 = a$, $CA_2\ a^2$'. I cannot see a reason for the superscript 2 in CA_2^2 so I have removed it. I have also added in an equals sign between CA_2 and a^2.
Ex.1	Tait has: '$\because AC + CB = a \times \left[\frac{1}{\sqrt{2}} + \frac{\sqrt{-1}}{\sqrt{2}}\right] = a = a \times \frac{1+\sqrt{-1}}{\sqrt{2}} = AB$'. I have removed the $= a$ as it appears only because there is a break in the line.
§7	Tait has: '$\because \left(\cos\vartheta + \sqrt{-1}\cdot\sin\vartheta\right)^p = \cos p\vartheta + \sqrt{+1}\sin p\vartheta$'. This is incorrect: it should be -1 under the square-root sign on the right hand side of the equation. The error also appears in Terrot's paper (p.350). Looking carefully at Tait's original notes, it seems that Tait had spotted Terrot's mistake and attempted to correct it.
§9	Tait has: 'which is $\therefore = r^2 = (CA^2) = (CA_1)^2$'. I have repositioned the superscript 2 outside the bracket (CA).
§10	Tait has: 'and this is true for every pair except $CA = (CA)\cdot +1$ & $CB = (CB)\cdot -1 \therefore (CP_1)\cdot(CP_2)\cdots CP_n = (-OA_n^n + OC^n)\cdot -1 = OA^n - OC^n$'. I have added in the bracket around CP_n which Terrot (p.353) and consequently Tait have omitted. Also there is no reason for Tait to have the subscript n in $-OA_n^n$ so I have removed it.
§11	Tait has: 'if these several extremities be joined, then $(AP_1)\cdot(AP_2)(AP_{n-1}) = nCA^{n-1}$'. I have added in $\cdots$ on the left hand side between (AP_2) and (AP_{n-1}) to show that all the AP_i up to $i = n-1$ are being multiplied together.

Table D.1: Editorial corrections made to Tait's notes from Terrot's paper.

E BUÉE'S 1806 PAPER AND GERGONNE'S TWO-DIMENSIONAL TABLE

In 1806 Adrien-Quentin Buée's paper, 'Mémoire sur les quantités imaginaires' [144] was published in French in the *Philosophical Transactions of the Royal Society of London*. It came to feature in many accounts of the history of the geometrical representation of complex numbers; no doubt because the significant and pioneering contributions of Wessel (1799) and Argand (1806) went unnoticed by the mathematical community.[524] The question of priority was raised by Sylvestre François Lacroix (1765–1843) in a note [210] published in volume IV of Gergonne's *Annales*. Lacroix was writing to draw attention to Buée's contribution following the re-discovery of Argand by Jacques Français (1775–1833) in 1813. Argand responded by insisting that he had worked independently of Buée and without knowledge of his contribution.[525]

Reviews of Buée's paper were largely unfavourable. It was even feared that the publication had tainted the reputation of the Royal Society, in whose *Transactions* it had appeared.[526] Nevertheless, the paper remains of historical interest for the following reasons. (i) It was the first non-English-language publication in a nineteenth

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[524]Fortunately, Argand's contribution was brought to light in 1813, by Jacques Français (1775–1833) in his paper [208] published in volume IV of the *Annales de mathématiques*. For an account of the rediscovery of Argand see [209]: it contains an English translation of Argand's 1806 pamphlet and of some of the subsequent related correspondence in the *Annales* between Français, Gergonne (the editor) and Servois (another contributor); it also provides a good account of the historical context in the section entitled, 'Notes' (pages 85–135).

[525]See [211, p209] for Argand's response to the issue of priority raised by Lacroix.

[526]Examples of reviews of Buée's 1806 paper: [212], [213] and [214]. According to [215, pxix] this last anonymous review [214] was written by 'Playfair'. I assume this is John Playfair (1748–1819): Professor of Mathematics (1785–1805), then Natural Philosophy (1805–1819), at the University of Edinburgh; Fellow of the Royal Society (1807) and one of the founding members of the R.S.E. [216]

century British journal; although the contribution is not considered foreign as Buée was living in England at the time as a political refugee.[527] (ii) It was this paper which first drew George Peacock's attention to the subject.[528] (iii) New evidence has come to light which reveals Buée's initial motivation.

According to two sources, Buée's paper was written in response to William Frend's *The Principles of Algebra* (1796). Both sources refer to a letter Buée had written to Frend on 21 June 1801. From the *Annual Report of the Royal Astronomical Society* (1842): 'Among his [Frend's] papers is preserved a letter to him from M. Buée, a Frenchman residing in England, dated June 21, 1801, containing the form in which the perusal of Mr. Frend's work made the writer put together his own views of the subject'.[529] And in an historical note in *The Algebra of Coplanar Vectors and Trigonometry* the author, Hayward writes:

> I have a letter in my possession from M. Buée to Mr. Frend, dated June 21, 1801, by which it appears that the former was desired by a gentleman in whose house he was living (as tutor, perhaps) to write a private reply to Mr. Frend's objections [to negative and imaginary quantities].[530]

Strangely, Buée makes only a single insignificant reference to Frend. One would assume from the paper that Buée's principal influence was Carnot's *Géométrie de position* (1803).

Adrien-Quentin Buée (1748–1826): a biographical sketch

Snippets of biography that I have been able to obtain record Buée as a French Catholic priest who was forced to flee Paris during the Revolution.

The *Nouvelle biographie universelle* describes Buée as a French writer and mathematician. He was born in Paris in 1748 and died there on 11 October 1826. His

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[527] [217, p76f(no.74)]

[528] [218, pxxvii]

[529] [219, p151] William Frend (1757–1841) was a member of the Council of the R.A.S.

[530] [215, ppxviii–xixf]

first ecclesiastical role was as organist at the Basilica of Saint-Martin in Tours. He fled Paris in 1792 and sought refuge in England, where he remained for a period of twenty-one years. On his return to Paris, he became an Honorary Canon of the city.[531] His leisure time was devoted to music and the exact sciences. He left a number of manuscripts in which he considered various mathematical problems.[532]

From *La France littéraire* we learn that Buée had a number of publications through a periodical pamphlet published during the Revolution entitled, *La Mère Duchesne*.[533] *La Mère Duchesne*—a variant of *Le Père Duchesne*—allowed contributors to express their views on the current political situation through the character of la mère Duchesne. Buée wrote to voice his opposition to the Civil Constitution of the Clergy Law of 1790. All Catholic clergy were required to swear an oath of allegiance to the Civil Constitution which made the Roman Catholic Church in France subordinate to the French government. Those who refused—the "non-jurors"—suffered restricted freedom to carry out their ministry, imprisonment and persecution.

According to [222], when Buée escaped Paris and settled in England he established himself in Bath and supported himself financially by publishing pamphlets, articles and leaflets on various aspects of mathematics and science.

More on Buée's 1806 paper

Much of Buée's paper is unintelligible and hardly resembles mathematics at all. Some of what he says in relation to the sign $\sqrt{-1}$ appears insightful (e.g. that it is as a sign of perpendicularity, or a mean proportional between $+1$ and -1); however, credibility is lost when the document is considered as a whole.

Buée has a variety of interpretations of the sign $\sqrt{-1}$. It is: a sign of impossibility in the context of the solution of a problem involving impossible conditions; a mean proportional between $+1$ and -1; a descriptive sign of perpendicularity and a mean

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531 *Honorary Canon of Paris*: an honorary title granted in France, at one time, by the Bishop to other members of the clergy.

532 [220, p730]

533 [221, p555]

'quality' between two opposite qualities $+ \; -$.

Buée has two species of algebra: (i) universal arithmetic and (ii) a mathematical language. In the first, the signs $+ \; -$ denote addition and subtraction. In the second, they are taken as signs of 'quality'; and, according to Buée, any quantity affected by a sign of quality is capable of meaning or interpretation. Thus, when $\sqrt{-1}$ is considered as a mean quality between the qualities $+ \; -$, quantities affected by the sign $\sqrt{-1}$ take on a variety of fantastic interpretations. A few examples suffice: (i) as an amount of books, neither in the possession of a man $(+)$, nor owed by him $(-)$; (ii) with $+t$ signifying the month approaching and $-t$ the month past, $\frac{-t\sqrt{-1}}{2}$ denotes the first half of the current month and $\frac{+t\sqrt{-1}}{2}$ the second half, with their sum (equal to zero) expressing the current month; and (iii) placed before the expression for a cube or parallelepiped, $\sqrt{-1}$ indicates that the cube or parallelepiped is, in fact, a void.

Making a proper assessment of Buée is made difficult by the fact that he does not attempt to prove established results but instead works through a variety of bogus problems. Intermittently, however, we encounter something which we recognize as being definitely incorrect. Consider, for instance, his application of Pythagoras' theorem to the lines $AB = 1$ and $AD = \sqrt{-1}$ (Figure E.1, page 259). He has

$$\overline{BD}^2 = \overline{AB}^2 + \overline{AD}^2 = 1^2 + (\sqrt{-1})^2 = 1^2 + (-1) = 1 - 1 = 0$$

He recognizes that the result is absurd but fails to realize that the apparent paradox arises over his confusion between the concepts of *length* and *vector*: the length of AD is 1, not $\sqrt{-1}$. His explanation for what has happened is bizarre and not worth reproducing.

Amidst all this confusion, however, there appears—out of nowhere—something of substance: Buée gives the direction of a line as $e^{i\theta}$, in which he surely takes inspiration from Euler.[534] He writes:

> let $\sqrt{-1} = 1 \times e^{\frac{\pi}{2}\sqrt{-1}}$ (e being the base of hyperbolic logarithms and π half of the

[534] *Introductio in analysin infinitorum* (1748), a work which is cited by Buée elsewhere in the paper in relation to another matter. See [144, p25].

circumference of a circle of which the radius is 1); $1 \times e^{\frac{\pi}{2}\sqrt{-1}}$ signifies the line $\overline{AD}$ [Figure E.1] of which the direction is $e^{\frac{\pi}{2}\sqrt{-1}}$.[535]

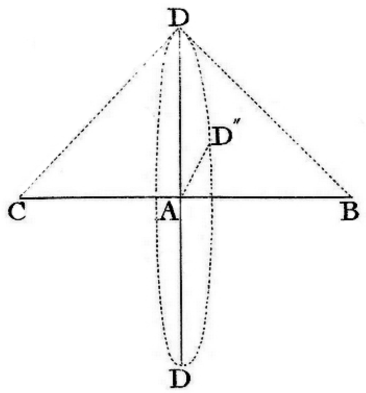

Figure E.1: Buée's application of Pythagoras' theorem. [144, pfacing 88]

Gergonne's two-dimensional table of real and imaginary magnitudes

In 1811 Joseph Gergonne (1771–1859), editor of the *Annales de mathématiques*, conceived of a two-dimensional table of real and imaginary magnitudes. He revealed his conception in an editorial footnote to Jacques Français' paper [208], published in volume IV of the *Annales*, in which Français famously communicated the ideas

[535]'soit $\sqrt{-1} = 1 \times e^{\frac{\pi}{2}\sqrt{-1}}$ (e étant la base des logarithmes hyperboliques et π la demi-circonférence d'un cercle dont le rayon est 1); $1 \times e^{\frac{\pi}{2}\sqrt{-1}}$ signifie la ligne $\overline{AD}$ dont la direction est $e^{\frac{\pi}{2}\sqrt{-1}}$'. [144, pp40–41]

contained in Argand's 1806 pamphlet.[536,537] In revealing his conception it was not Gergonne's intention to strip Français of priority, merely to show that their shared conception was very much "in the air".

Joseph Diaz Gergonne (1771–1859): a biographical sketch

The life of Joseph Gergonne was a life of military service and mathematics. Between 1791 and 1795 the Frenchman saw action as Captain of the National Guard working towards stability for post-revolution France; for the French army defending Paris against the Prussians and as secretary to the general staff of the Moselle army. In 1794 he attended the Châlons artillery school and was commissioned as a lieutenant. The same year he saw action in Spain. From the time of his arrival at Nimes in 1795, however, Gergonne was able to devote himself to mathematics. He was appointed to the Chair of Transcendental Mathematics at the École Centrale, which had recently been established in the city. In 1810 he founded the first scientific journal devoted purely to mathematics, the *Annales de mathématiques pures et appliquées*, after having experienced difficulties in getting his own work published. The journal ran until 1832 and was succeeded by Liouville's *Journal de mathématiques pures et appliquées* in 1836. In 1816 Gergonne moved to Montpellier to take up the Chair of Astronomy at the university there. In 1830 he became Rector at the university. He retired from his position in 1844 and died in Montpellier in May 1859, aged eighty-seven.

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[536] See [208, pp71–72f]. The inspiration for Français' paper had come from a letter found amongst his late brother's papers. The letter had come from Adrien-Marie Legendre (1752–1833), who had acquired his knowledge of the ideas from an unnamed source (Argand). Français acknowledged his source and wrote of his hopes that the publicity generated by the publication of the paper might bring to light the original author of these ideas (Argand). Argand subsequently wrote to the editors of the *Annales*, declaring himself to be the said author and explaining that in 1806 he had published a pamphlet on the subject which Legendre had looked at in manuscript form. Argand provided the editors of the *Annales* with a summary of his pamphlet to support his authorship claim.

[537] Gergonne's table is referred to in [209, pp95–96, 116–117] but my knowledge of it came directly from the *Annales*.

Gergonne's primary mathematical interest was geometry. In 1813 he was awarded a prize by Bordeaux's Société des Sciences, Lettres et Arts for an essay on the rival methodologies of analytic and synthetic geometry, along with another entrant, M. Armand de Maizière.[538] Gergonne was an advocate for the merits of analytic geometry and it was his position on the rival methodologies which brought him into his first conflict with Jean-Victor Poncelet (1788–1867), a proponent of synthetic geometry; later, in 1826, they entered a serious dispute over priority in the discovery of the principle of duality.[539] Both Gergonne and Poncelet were students of Gaspard Monge (1746–1818), the founder of descriptive geometry.[540]

More on Gergonne's two-dimensional table

The impetus for Gergonne's table (Figure E.2, page 262) was a paper published in volume I of the *Annales* entitled, 'Théorème général sur l'invariabilité de la forme des fonctions' [228]. The paper was written by M. Armand de Maizière, winner of the essay prize, along with Gergonne, in 1813.[541]

Maizière considered the relationship between the independent and dependent variables, x and y, of a function φ. He had to ensure that x and y were subject to the law of continuity; that is, if we take two neighbouring states, x_a and x_{a+1}, of the independent variable x, then the difference between the two corresponding y states, y_a and y_{a+1}, of the dependent variable y should fall below a given limit, however small. Gergonne considered how one would measure the proximity of two neighbouring terms in cases where at least one term is non-real: he proposed that the difference between the two terms be written in the form $p + q\sqrt{-1}$, which tends

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[538] [223, p459] Gergonne's essay was never published, however, he does give a summary of it in [224]. Maizière's essay, according to [223, p459]: *Mémoire qui a partagé le prix en 1813* (Paris, 1814), p. 28.

[539] [225, pp577–578]

[540] Uncited sources of biographical information on Gergonne: [226] and [227].

[541] According to [228, p368], Armand de Maizière was a mathematics teacher at the lycée (high school) of Versailles. No further biographical information is available.

to zero when p and q tend to zero.[542]

$$\begin{array}{lllllll}
..., & -2+2\sqrt{-1}, & -1+2\sqrt{-1}, & +2\sqrt{-1}, & +1+2\sqrt{-1}, & +2+2\sqrt{-1}, & ... \\
..., & -2+\sqrt{-1}, & -1+\sqrt{-1}, & +\sqrt{-1}, & +1+\sqrt{-1}, & +2+\sqrt{-1}, & ... \\
..., & -2, & -1, & \pm 0, & +1, & +2, & ... \\
..., & -2-\sqrt{-1}, & -1-\sqrt{-1}, & -\sqrt{-1}, & +1-\sqrt{-1}, & +2-\sqrt{-1}, & ... \\
..., & -2-2\sqrt{-1}, & -1-2\sqrt{-1}, & -2\sqrt{-1}, & +1-2\sqrt{-1}, & +2-2\sqrt{-1}, & ...
\end{array}$$

Figure E.2: Gergonne's two-dimensional table of real and imaginary magnitudes for integer values, conceived of in 1811. Reproduced from [208, p71f].

In private correspondence on the paper, Gergonne had proposed the idea of a double-entry table:

> About two years ago, I wrote to M. de Maizière about his paper inserted at page 368 in the first volume of this series and informed him that we were probably wrong to expect to include all numerical values in a simple series; and that by their nature, they seemed to form a table with double entries, which, limited to integers, could be represented as follows: [Figure E.2] so that already, like M. Français, I assumed numbers in the form $n\sqrt{-1}$ on a line perpendicular to the line which contains numbers of the form n; and that, like him again, I represented the numbers not belonging to those two lines by the sum of their projections on each of them.[543]

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[542]'Pour cela nous remarquerons que la différence de deux pareils termes peut toujours, en général, être supposée imaginaire et de la forme $p+q\sqrt{-1}$; or il n'y a pas de doute qu'une telle expression ne puisse tendre vers zéro, puisqu'il suffit pour cela que p et q tendent eux-mêmes vers cette limite commune.' [228, pp369–370f]

[543]'Il y a environ deux ans qu'écrivant à M. de Maizière, au sujet de son mémoire inséré à la page 368 du 1.er volume de ce recueil, je lui mandais qu'on avait peut-être tort de vouloir comprendre toutes les grandeurs numériques dans une simple série; et que, par leur nature, elles semblaient devoir former une table à double entrée qui, bornée aux seuls nombres entiers, pourrait être figurée comme il suit: [Figure E.2] en sorte que déjà, comme M. Français, je supposais les

Following publication of Français' paper in the *Annales*, another contributor, François Joseph Servois (1768–1847) proposed a way of extending Gergonne's table into three dimensions. Gergonne's response to Servois was to disclaim credit for originally having designed the table with the intention of extending it into three dimensions: a triple argument table had occurred to Gergonne but only after he had read the memoirs of Argand and Français.[544]

So while Gergonne had hit upon a way of representing complex numbers as coordinates in the plane, his conception lacked a geometrical definition of addition and multiplication for complex numbers as directed lines. However, the table was devised before the rediscovery of Argand and without knowledge of Buée, which serves to elevate Gergonne's achievement and make his table all the more interesting.[545]

nombres de la forme $n\sqrt{-1}$ situés dans une ligne perpendiculaire à celle qui renferme les nombres de la forme n; et que, comme lui encore, je représentais les nombres étrangers à ces deux lignes par la somme de leurs projections sur l'une et sur l'autre.' [208, pp71–72f]

[544] See [192, pp234–235, 235f] for Servois' suggestion and Gergonne's response.

[545] Gergonne had the opportunity to declare any knowledge of Buée in his editorial comments on Lacroix's note [210] in which Lacroix called attention to Buée's contribution.

F HAMILTON'S UNPUBLISHED PAPER ON THE FUNDAMENTAL THEOREM OF ALGEBRA: INSPIRED BY MOUREY

I discovered the following unpublished paper by Sir William Rowan Hamilton in his notebook MS.1492/95 (pages 263–265) in the Manuscripts Library of Trinity College, Dublin, during a research trip to the library in February 2014. I am grateful to the Keeper of the Manuscripts at Trinity for kindly giving their permission to publish here this transcription of the original.

To make the transcription easier to read, I have italicized the text which Hamilton had underlined (except for titles) and rendered bold the text which Hamilton had double underlined. I have made no other changes; notably, I have chosen to keep Hamilton's contractions just as they appear in the original, e.g. 'imag$^{\mathrm{y}}$' (imaginary) and 'w$^{\mathrm{d}}$' (would), as I feel that they help to establish a stronger impression of Hamilton for the reader. I understand, however, that these contractions would be replaced by full words by an editor or publisher who was preparing Hamilton's paper for publication.

Hamilton's notation

- — Where $^{\cos}_{\sin}$ appears, as in equations (a) (6) (9), read along the top first, then the bottom, to give two separate equations. E.g. the first equation (a) gives: $r' \cos v' = r \cos v - a \cos \alpha$ from the top and $r' \sin v' = r \sin v - a \sin \alpha$ from the bottom.
- — Similarly for $^{>}_{<}$ in equation (2).
- — $v^{(n+1)}$ is v with $n+1$ dashes. E.g. $v^{(3)} = v'''$. Similarly for $\alpha^{(n)}$.
- — The $\int$ sign, as in equation (11), denotes *integers* and is not to be confused with the integrals of integral calculus.

Theorem of Mourey. (not stated precisely in this way by him; he writes (p.104),

$$\left.\begin{array}{r|r|l|} x^n + b & x^{n-1} + bc & x^{n-2} + ... + (bcd...)x - g = 0 \\ +c & +cd & \\ +d & +bd & \\ +... & +... & \end{array}\right)$$

"Every equation of the form

$$x(x-a)(x-a')\cdots = b \tag{A}$$

has at least one (real or imag^{y}) root, whatever given (real or imag^{y}) values the const^{s} a, a', ... and b may have."

<u>Modified statement of the theorem.</u>

If a, a', ... and b be any n given positives, and $\alpha, \alpha', \ldots \beta$ any n given reals; it is possible to satisfy the system of the $2n$ eq^{ns},

$$r' \mathop{}_{\sin}^{\cos} v' = r \mathop{}_{\sin}^{\cos} v - a \mathop{}_{\sin}^{\cos} \alpha, \quad r'' \mathop{}_{\sin}^{\cos} v'' = r \mathop{}_{\sin}^{\cos} v - a' \mathop{}_{\sin}^{\cos} \alpha', \quad \ldots \tag{a}$$

$$rr'r''\cdots = b, \quad v + v' + v'' + \cdots = \beta + 2m\pi, \tag{b}$$

(where m is some whole n^{o}) by at least one system of n positives $r, r', r'', \ldots$ & of n reals, $v, v', v'', \ldots$

<u>Modified Demonstration.</u>

Let r_v be the least positive root of the equation,[546]

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[546] Hamilton has substituted $\underline{\mathbf{x}} = re^{iv}$, $\underline{\mathbf{x}} - \underline{\mathbf{a}} = r'e^{iv'} = re^{iv} - ae^{i\alpha}$, $\underline{\mathbf{x}} - \underline{\mathbf{a'}} = r''e^{iv''} = re^{iv} - a'e^{i\alpha'}$, etc. from equations (a) and $\underline{\mathbf{b}} = be^{i\beta}$ into equation (A). He has then multiplied the resulting equation together with its complex conjugate to give $b^2 = r^2\{r^2 - 2ar\cos(v-\alpha) + a^2\}\{r^2 -$

$$b^2 = r_v^2\{r_v^2 - 2ar_v\cos(v-\alpha) + a^2\}\{r_v^2 - 2a'r_v\cos(v-\alpha') + a'^2\}\cdots \qquad (1)$$

in wh. it is supposed that $b > 0$; so that

$$r_v > 0, \quad r_\alpha \substack{> \\ <} a, \quad r_{\alpha'} \substack{> \\ <} a', \quad \dots \qquad (2)$$

and

$$r_{v+2\pi} = r_v \quad \text{[Because all the coeff}^{\text{s}}\text{ of the } \sqrt{(1)} \text{ remain unchanged.]} \qquad (3)$$

Let r'_v, r''_v, ... be the other positive & periodic functions of v, wh. are obtained by subst[g] for r in the eq[n] (a) its value r_v; & in order to render determinate the correspond[g] functions v'_v, v''_v, ... let us suppose that they receive no sudden changes (of the form $2m\pi$), & also (as is allowed by the eq[n]) that each $v^{(n+1)}$ becomes $= v$, when $v = \alpha^{(n)} + \pi$; so that if we introduce these $n-1$ other continuous functions of v,

$$w_v = v'_v - v, \quad w'_v = v''_v - v, \quad \dots \qquad (4)$$

we shall have

$$w_{\alpha+\pi} = w'_{\alpha'+\pi} = \cdots = 0 \qquad (5)$$

To render $w_v = \mp\pi$, it w[d] be necessary to suppose

$$v'_v = v \mp \pi, \quad (r_v + r'_v)\substack{\cos \\ \sin} v = a \substack{\cos \\ \sin} \alpha, \quad r_\alpha + r'_\alpha = a > r_\alpha; \qquad (6)$$

If then we have, on the contrary, $r_\alpha > a$, we must conclude that the continuous function w_v (which may become $= 0$) can never attain either of the limits, $\mp\pi$, but is always included between them, or that

$$w_v > -\pi, \quad w_v < +\pi, \quad \text{if } r_\alpha > a \qquad (7)$$

$2a'r\cos(v-\alpha') + a'^2\}\ldots$, call this (†). Then for each angle v we look for the smallest positive r which satisfies (†). Hamilton calls this r_v. Substituting $r = r_v$ into (†) gives equation (1).

The function w_v is $\therefore$ in this case periodic, (like r_v, &c) because it is determined without ambiguity by means of its cosine & sine; thus

$$w_{v+2\pi} = w_v, \quad v'_{v+2\pi} = v'_v + 2\pi, \quad \text{if } r_\alpha > a \tag{8}$$

On the other hand,

$$\text{if} \quad v'_v - \alpha = 0 \text{ or } = 2\pi, \quad \text{then} \quad r_v \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} v = (a + r'_v) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} \alpha, \quad r_\alpha = a + r'_\alpha > a; \tag{9}$$

If then the inequality $r_\alpha > a$ be not satisfied, we must regard the continuous function $v'_v - \alpha$ as never attaining either of the limits 0 & 2π, although it receives the intermediate value π, when $v = \alpha + \pi$; hence

$$v'_v > \alpha, \quad v'_v < \alpha + 2\pi, \quad v'_{v+2\pi} = v'_v \quad \text{if } r_\alpha < a \tag{10}$$

In like manner

$$v''_{v+2\pi} - v''_v = 2\pi \text{ or } 0, \text{ all } \int \text{ as } r_{\alpha'} > \text{ or } < a', \tag{11}$$

and similarly for v''', ... While $\therefore$ the quantity v is continuously increased by 2π, the continuous function

$$v + v'_v + v''_v + \dots \tag{12}$$

is increased *at least* by 2π, & may be increased by 4π, or by 6π, ... All $\int$ as we have the system of the $n-1$ inequalities

$$r_\alpha < a, \quad r_{\alpha'} < a', \quad \dots \tag{13}$$

or have one or more of these $n-1$ opposite inequalities,

$$r_\alpha > a, \quad r_{\alpha'} > a', \quad \dots \tag{14}$$

In every case $\therefore$, during this contin$^{\text{s}}$ increase of v, the function (12) passes *at least once* through the stage $\beta + 2m\pi$, whatever assigned constant β may be: & the 2^{nd} eq$^{\text{n}}$ (b) is satisfied, under the form,

$$v + v'_v + v''_v + \cdots = \beta + 2m\pi$$

But the first eq$^{\mathrm{n}}$ (b) is also satisfied, by (1), under the form,

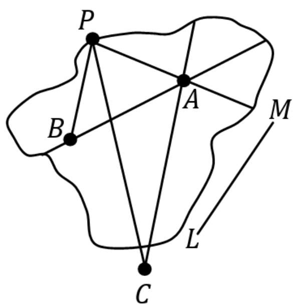
$$r_v r'_v r''_v \cdots = b$$

and the equation (a) has been satisfied, by the choice of the functions of v. The theorem of Mourey is $\therefore$ proved to be true.

(Obs$^{\mathrm{y}}$ July 18th 1854. W.R.H.)

[Presumably, Hamilton meant the following to be an addition to the paper ...]

Admitting the theorem that an eq$^{\mathrm{n}}$ of the form (A) has at least one root, say x_1, so that $x_1(x_1 - a)(x_1 - a') \cdots = b$, where $a, a', \ldots b$ are any real or imag$^{\mathrm{y}}$ const$^{\mathrm{s}}$, & x_1 is some real or imag$^{\mathrm{y}}$ quantity, whose existence is proved as above, we may substitute this last value for b, transpose, develope, and divide, (without remainder), by $x = x_1$. The quot$^{\mathrm{t}}$ will be a polynome in x of the degree $n - 1$, (A) having been of the degree n. Suppose it known, then, that every eq$^{\mathrm{n}}$ of this depressed degree $n - 1$ has $n - 1$ roots, or may be decomposed into $n - 1$ binomial factors of the form $x - a$; it will follow on the one hand that the gen$^{\mathrm{l}}$ eq$^{\mathrm{n}}$ of the n^{th} degree may be put under the form (A), & $\therefore$ that it has *at least one* root; & on the other had, after depression by div$^{\mathrm{n}}$, that it has also $n - 1$ *other roots*. On the whole then, Mourey's theorem proves that *"the gen^{l} eq^{n} of the n^{th} degree has n roots, if the general eq^{n} of the next lower degree have the next lower number of roots"*; that is, in the last analysis, if the gen$^{\mathrm{l}}$ eq$^{\mathrm{n}}$ of the *first* degree have *one* root: but this is manifestly true. Therefore, &c.



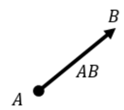
Mourey connects the proof of his own theorem (for such I think it may be properly called) with the prob$^{\text{m}}$ to find a point P, in the same plane with any n given points A, B, C, & such that the product of the n *paths* or *ways* (chemins), to it from them, may be equal to a given path; $AP \cdot BP \cdot CP =$ a given directed line LM. His reasoning appears to me to be correct in the main, but he seems somewhat to spoil his argument by seeking to show (p. 114 at the top), that the eq$^{\text{n}}$ (A) itself has as many as n roots, whereas it was enough to prove (what alone he *does* prove, in my opinion) that *it* has *at least one*. The combination of it with eq$^{\text{ns}}$ of lower degrees is required, & is suff$^{\text{t}}$ to prove finally, as above, the known theorem, that "the general eq$^{\text{n}}$ of the n^{th} degree has n roots". But the proof given in this paper has been entirely *suggested* by Mourey's work.

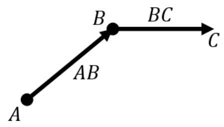
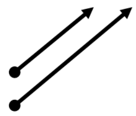
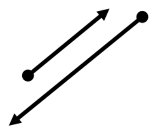
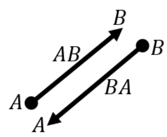
W.R.H.

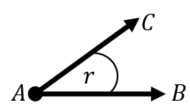
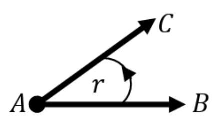
G MOUREY'S TERMINOLOGY AND NOTATION: A LOOK-UP TABLE

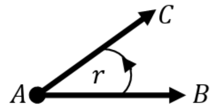
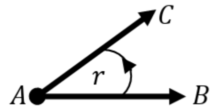
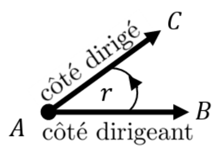
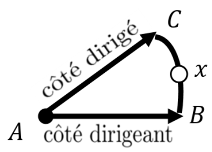
The following table provides a summary of Mourey's original terminology and notation. Its entries are given in the order in which they appear in Mourey (1861) so that it might be used as a companion-guide to the text. In itself the table provides a good summary of Mourey's approach.

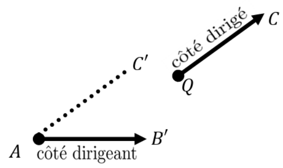
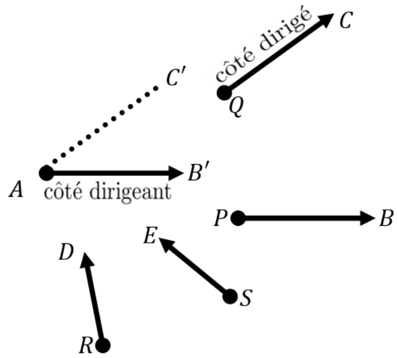
Table G.1: Mourey's terminology and notation: a look-up table.

Term, notion, notation (en Français)	**Definition in English**	**Example of usage or a sketch to aid understanding**
ligne directive ou *chemin* [n.]	directed line or path; a straight line leading in a given direction	
origine [n.]	origin; the initial point of a directed line	
terme [n.]	terminus; the terminal point of a directed line	

(chemins) *de suite*	in succession; the terminal point of one path is the initial point of the other	
(chemins) *concurrents*	concurrent; (paths) leading in the same direction	
(chemins) *opposés*	opposed; (paths) leading in opposite directions	
(chemins) *inverses*	inverse; (paths) of the same length, leading in opposite directions	
unité relative	relative unit; unit of measurement	See *nombre relatif.*
nombre relatif	relative number; quantity measured in terms of the *unité relative*	E.g. Unit of measurement: 1 metre taken from left to right. The numbers $1, 2, 3, \ldots$ represent 1m to the right, 2m to the right etc. The numbers $-1, -2, -3, \ldots$ represent 1m to the left, 2m to the left etc.
concret [adj.]	concrete; describes the *nombre relatif* when the unit of measurement is specified	

abstrait [adj.]	abstract; describes the *nombre relatif* when the unit of measurement is not specified	Note. However, these abstract numbers can be thought of as quantities relative to an abstract unit 1, for example.
nombre directif	directed number	In the context of paths, an abstract number may be regarded as a quantity relative to an abstract geometrical unit, i.e. a path of arbitrary length and direction. E.g. the directed number 2 can be thought of as a path in the same direction as the unit path 1 (the line of length 1 in the direction of the positive real axis) but of twice the length. Non-real quantities are generated by a rotation of the unit path.
(nombre) *positif* [adj.]	positive (number); having the same direction as the abstract unit	E.g. $1, 2, 3, \ldots$ with the unit $+1$
(nombre) *négatif* [adj.]	negative (number); having a direction opposite to that of the abstract unit	E.g. $-1, -2, -3, \ldots$ with the unit $+1$
verser [v.]	to make turn or rotate	'*verser AB de r*': to rotate the directed line AB through an angle r about its origin, expressed by $AB_r = AC$. 
angle directif	directed angle; an angle leading in a particular direction	 Convention. The unit of the directed angle is an anticlockwise rotation of 90°, denoted by a subscript 1, i.e. $AB_{90^\circ} = AB_1$. Similarly, $AB_{180^\circ} = AB_2$, $AB_{-90^\circ} = AB_{-1}$ and $AB_{60^\circ} = AB_{\frac{2}{3}}$.

<table>
<tr><td>verseur (de AB)
[n.]</td><td>the angle through which a directed line is rotated</td><td>'L'angle r est le verseur de AB': r is the angle through which the directed line AB is rotated.

</td></tr>
<tr><td>rapport directeur
(de AC à AB)</td><td>the angle leading from AB to AC, expressed by $AC \because AB$ (read 'AC directeur à AB' or 'AC recteur AB')</td><td>'L'angle r est le rapport directeur de AC à AB': r is the angle leading from AB to AC.

</td></tr>
<tr><td>côté dirigeant</td><td>leading side; the side from which the directed angle originates</td><td>
</td></tr>
<tr><td>côté dirigé</td><td>facing side; the side towards which the directed angle approaches</td><td>See côté dirigeant.</td></tr>
<tr><td colspan="3">How Mourey indicates a directed angle. A directed angle is expressed by a series of letters: the first letter corresponds to the vertex$^{(1)}$, the second to the côté dirigeant$^{(2)}$, and the last to the côté dirigé$^{(3)}$. If no convention is established as to the sense in which the angle turns then it is necessary to place one or several letters between (2) and (3). E.g. the directed angle leading from AB to AC via x is expressed by $ABxC$.

</td></tr>
</table>

égalité spéciale des angles directifs $\doteq$	special equality of directed angles; equality of directed angles modulo 4 right angles (equivalently modulo 2π)	$r \doteq r+4 \doteq r+8 \doteq \cdots \doteq r+4n$ (where n is an integer)
digène [n.]	from the Greek meaning 'two origins'; a geometric figure composed of two paths (not necessarily with the same origin), which we compare in order to find the *rapport directeur* between them. See *rapport directeur.*	 The value of the digène, $QC \because AB'$, is the directed angle $AB'C'$, which has the same *côté dirigeant* AB', and whose *côté dirigé* AC' is concurrent with QC.
équi-digène [n.]	an equation expressing the equality of two expressions of the form $a \because b$, where a and b are paths, i.e. equality of two *digènes.* See *digène* and *rapport directeur.*	E.g. $SE \because RD = QC \because PB$ 
version [n.]	the act of rotation; the operation by which we rotate a path. See *verser.*	
équi-quotient [n.]	an equation expressing the equality of two ratios of directed quantities	E.g. $a : b = c : d$
AB_+	notation expressing a positive path of the same length as AB, no matter the sign of the path AB. Read 'AB *positive*'.	E.g. $(4_2)_+ = 4$
déverseur $+$ [n.]	the sign $+$ used in the manner consistent with the notation AB_+. See AB_+.	

a_r^m	notation signifying a directed line of length a^m rotated through an angle r, i.e. $(a^m)_r$ and not $(a_r)^m$	
σ	notation representing the set of values $0, 4^q, 8^q, \ldots$ i.e. multiples of four right angles	$r \doteq r+4 \doteq r+8 \doteq \cdots \doteq r+\sigma$ See *égalité spéciale des angles directifs.*
(signe) *super-égal* $\stackrel{\vdots}{=}$	super-equals (sign); symbol used to indicate that two quantities are equal under the radical sign	E.g. $-1 \times -1 \stackrel{\vdots}{=} 1_4$ not $\stackrel{\vdots}{=} 1$ as $\sqrt{-1 \times -1} = \sqrt{1_2 \times 1_2} = \sqrt{1_4} = 1_{\frac{4}{2}} = 1_2$ and not $= \sqrt{1} = 1$.
prime-directeur [n.]	the *rapport directeur* of a path in the positive direction	
(chemins) *super-égaux* [adj.]	super-equal; (paths) of the same length, the same direction and with the same *prime-directeur*	
$a_=$	notation expressing a path parallel to unity and of the same length as a. Read '*a parallèle*'.	
$a_{=1}$	notation expressing a path perpendicular to unity and of the same length as a	
mi-déverseur $=$ [n.]	the sign $=$ used in the manner consistent with the notation $a_=$. See $a_=$.	

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