T. J. I'A. BROMWICH.

THOMAS JOHN I'ANSON BROMWICH, 1875–1929.

Thomas John I'Anson Bromwich, who died on August 24, 1929, was one of the most accomplished and most versatile among English mathematicians of the last fifty years. He was born in Wolverhampton on February 8, 1875, but spent his youth in Natal, and was educated in Durban. He came to Cambridge, as a Pensioner of St. John's College, in October, 1892. A brilliant career as an undergraduate ended when he was Senior Wrangler in 1895, in an exceptionally strong year which included also E. T. Whittaker and J. H. Grace. He obtained a Fellowship in 1897, but left Cambridge in 1902 to be Professor of Mathematics in Galway, returning in 1907 when appointed a permanent lecturer at St. John's. He was also a University Lecturer from 1909 to 1926. He became a Fellow of the Royal Society in 1906 and a Doctor of Science in 1909. He was for many years a most enthusiastic and energetic member of the London Mathematical Society, of which he was Secretary from 1911 to 1919, and Vice-President in 1919 and 1920. He married in 1901, and leaves a widow and one son.

Bromwich's work covers so wide a field that it is hardly possible for any one person to deal with it competently. His later work in mathematical physics is discussed in Dr. Jeffreys' notice in the 'Journal of the London Mathematical Society,' vol. 5, p. 220. Prof. H. W. Turnbull and Prof. A. E. H. Love have very kindly provided me with notes concerning Bromwich's early work, in algebra and in applied mathematics respectively, and what I say about these subjects is very largely based on them.

Bromwich was the author of one large treatise, "An introduction to the theory of infinite series" (Macmillan, 1908), a second edition of which appeared in 1926; of one of the Cambridge Tracts, "Quadratic forms and their classification by means of invariant-factors" (1906), long out of print; and of from seventy to eighty papers in mathematical and physical journals. It is noteworthy that over fifty of these papers had appeared by the end of 1906, and that after 1908 he published comparatively little of importance, except for the work on "normal co-ordinates" and the operational calculus which was his primary interest in his later years of activity. After the war he was never in normal health, but his period of greatest fertility had ended a good many years before. There is no doubt that he had for long been overworked. He was engaged in original work in several different fields; he put a great deal of energy into his college and university lectures, where his passion for working out every point in detail must have added enormously to his labours; and to all this he added a considerable amount of examining and private coaching. The best pure mathematician among the applied mathematicians of Cambridge, and the best applied mathematician among the pure mathematicians, he must have been an ideal coach for a Tripos candidate of sufficient strength, and if there could have been a dons' Tripos, we would all have laid odds on him to be first ; but the cost of all this was heavy both for mathematics and for Bromwich himself, and he never quite fulfilled his early promise. He would have had a happier life, and been a greater mathematician, if his mind had worked with less precision. As it was, even the best of his work is a little wanting in imagination. For mastery of technique in a wide variety of subjects, it would be difficult to find his superior, but he lacked the power of " thinking vaguely."

Bromwich plunged enthusiastically into research immediately after taking his degree. He was, in the first instance, an applied mathematician ; his special subjects in Part II of the Tripos had been Hydrodynamics, Elasticity, Optics, and Electricity and Magnetism. He was, however, an applied mathematician of an extremely " analytical " type. He had no physical training, and does not in his early work at any rate) show any very great interest in physics for its own sake. He does not seem really to care much about the physical world or the light which his analysis may throw on it, but rather to regard the world as created to illustrate his analysis. What he always wanted was the correct answer to the physical question when reduced to a problem of analysis, a convincing proof, and a logical presentation. It is only fair to add that in his later years, when applied mathematics became again the principal object of his thoughts, he showed a good deal more of the spirit of a genuine physicist than ever before.

The result of this was that Bromwich soon found himself engaged primarily in pure mathematical research. He began, no doubt, by finding a good deal of the current physics deficient in accuracy and in technique, and set himself to remedy some of the deficiencies. This made him, first an algebraist, and then an " analyst " in the technical sense. As soon as he began to study analysis seriously he found, as well he might, that the teaching of analysis in England, in Cambridge and elsewhere, was slovenly and incompetent, and he set to work with characteristic energy to improve it. This was the period of his " Infinite series," when he was in the first place an analyst, more from force of circumstances than from any spiritual necessity. It was, however, with algebra that he occupied himself first, and I must begin by saying something about his work in this field, which he abandoned altogether later.

Bromwich's most important contributions to algebra are to be found in his Tract and in the papers 4, 6, 8, 17, 22, and 24*. The most striking of these papers to the ordinary reader is certainly 22, and the result is easily stated and stands apart from the rest of his work, though the proof depends on theorems first proved explicitly by Bromwich himself in 6, 8, and 23. The

---

* See the references at the end of the notice.

problem is the old problem of finding bounds for the real and imaginary parts of the roots of the characteristic equation of a linear substitution. If the substitution is

(A) $$\xi_s = \Sigma_r a_{sr} x_r \quad (r,\, s = 1, 2, \ldots, n),$$

then the equation is

$$|\mathrm{A} - \lambda\mathrm{E}| \equiv \begin{vmatrix} a_{11}-\lambda & a_{12} & \ldots \\ a_{21} & a_{22}-\lambda & \ldots \\ \ldots & \ldots & \ldots \end{vmatrix} = 0,$$

where $|\mathrm{A}|$ is the matrix of A and $|\mathrm{E}|$ is the unit matrix. The classical results are as follows. The roots are all real when A is real and symmetrical (Cauchy, 1829), or, more generally, when A is Hermitian, *i.e.*, when a_{rs} and a_{sr} are conjugate (Christoffel, 1864). They have unit modulus when A is real and orthogonal (Brioschi, 1854). Finally, they are pure imaginaries when A is real and alternate, *i.e.* when $a_{rs} = -a_{sr}$ (Weierstrass, 1879). The general case was considered first by Bendixson (1900) and Hirsch (1902), by the former for real and by the latter for complex A. Bromwich, using a different method, completes their results and arrives at a final theorem. Suppose that $\lambda = \mu + i\nu$ is a root and write

$$b_{rs} = \tfrac{1}{2}\,(a_{rs} + \bar{a}_{sr}), \quad b_{sr} = \tfrac{1}{2}\,(a_{sr} + \bar{a}_{rs}),$$
$$ic_{rs} = \tfrac{1}{2}\,(a_{rs} - \bar{a}_{sr}), \quad ic_{sr} = \tfrac{1}{2}\,(a_{sr} - \bar{a}_{rs}),$$

where the bar denotes the conjugate, so that $|\mathrm{B}|$ and $|\mathrm{C}|$ are Hermitian. Then, after Christoffel, the roots of $|\mathrm{B} - \lambda\mathrm{E}| = 0$ and $|\mathrm{C} - \lambda\mathrm{E}| = 0$ are real, say μ_1^*, μ_2^*, …, and ν_1^*, ν_2^*,…, respectively; and Bromwich's theorem is that *μ lies between the least and the greatest of the μ^* and ν between the least and the greatest of the ν^*.*

The content of the other algebraical papers is more difficult to characterise. They are not at all easy to read, for Bromwich shows defects as an expositor which he never wholly overcame. He is clear enough in detail, but, as Berry said very justly later, in a review of his "Infinite series," he does not help the reader by laying emphasis at the right moment on the really fundamental idea. His first object in all this work is to expound and extend the ideas and methods of Kronecker in the theory of quadratic and bilinear forms, which he was the first to introduce to English readers. Yet one will search in vain for any quite clear and explicit statement of the fundamental difference between Kronecker's methods and those of writers before him.

The difference shows itself particularly clearly in the very simplest case, that of the reduction of a single quadratic form

$$\mathrm{A}\,(x_1, x_2, \ldots, x_n) = \Sigma a_{rs} x_r x_s$$

neral case

to a sum of squares. Lagrange and Kronecker each reduce step by step, making

$$A(x_1, x_2, \ldots, x_n) = c\xi_1^2 + B(\xi_2, \xi_3, \ldots, \xi_n),$$

clI2 + B

to take the simplest case. In Lagrange's reduction $\xi_2, \xi_3, \ldots$ are $x_2, x_3, \ldots$; the coefficients in B have changed but not the variables. In Kronecker's, $B = \Sigma a_{rs}\xi_r\xi_s$, the coefficients are the same but the variables have changed[†]. This divergence of procedure naturally persists and widens throughout the whole theory which Bromwich is occupied in developing.

In detail Bromwich's work is impressive, and his contributions to the theory substantial. In 4 he considers the reduction of the linear substitution A to a "canonical" substitution[‡], by a method similar to that of Netto. In 5 he begins by considering the corresponding problem for the more general system

$$\sum_r b_{sr}\xi_r = \sum_r a_{sr}x_r, \quad (r, s = 1, 2, \ldots, n),$$

with special reference to the "singular" case in which $|A - \lambda B| = 0$ for all λ. Then, after a dynamical application, he passes to the central problem of the theory, that of the simultaneous reduction of the two bilinear forms

$$A(x, y) = \Sigma a_{rs}x_r y_s, \quad B(x, y) = \Sigma b_{rs}x_r y_s.$$

It is familiar that a single bilinear form A of "rank" p is reducible to the form

$$A^* = X_1 Y_1 + X_2 Y_2 + \ldots + X_p Y_p$$

by an infinity of substitutions $x = PX$, $y = QY$. The most interesting case is that in which the substitution is "congruent," *i.e.* when the form of the substitutions for X and Y is the same. The necessary and sufficient condition that A should be reducible to A* by a congruent substitution is that A should be symmetric. An alternate form may be reduced by such a substitution to[§]

$$X_1Y_2 - X_2Y_1 + X_3Y_4 - X_4Y_3 + \ldots .$$

Kronecker gave a general process for the congruent reduction of any A, which Bromwich, in 23, works out for symmetric and alternate forms.

The next problem is that of two forms A, B, or a "Schaar" $A - \lambda B$. If $|B| \neq 0$, then $A - \lambda B$ can be reduced to

$$C(X,Y) - \lambda(X_1Y_1 + X_2Y_2 + \ldots + X_nY_n),$$

where C is a sum of forms like[||]

$$aX_1Y_1 + (aX_2 + X_1)Y_2 + \ldots + (aX_r + X_{r-1})Y_r.$$

[†] See Bôcher's "Introduction to higher algebra," 131.

[‡] See Hilton's "Linear substitutions," 29.

[§] For all these theorems, see Hilton, 69, 73, 75.

[||] Hilton, 173.

Thus, in the simple case when the roots of $|A - \lambda B| = 0$ are distinct, there are n such forms each consisting of the first term only. The general problem has been attacked by many writers. In 5, Bromwich gives a method of his own, again paying special attention to the singular case when $|A - \lambda B| \equiv 0$, and illustrating it on Darboux's example

$$A = a_1 (x_1 y_1 + \dots + x_{n-1} y_{n-1}) + x_2 y_1 + \dots + x_n y_{n-1},$$
$$B = x_1 y_1 + \dots + x_{n-1} y_{n-1}.$$

In all this there is no question of congruent reduction, a special problem which Bromwich considers in 6 and 23. Kronecker gave a process for the congruent reduction of two quadratic forms, and Bromwich, in 20, applies this method to the four cases in which each form is symmetric or alternate, and also works out the reduction of a pair of Hermitian forms by conjugate substitutions.

In his Tract (and in 17, which is practically a sketch for it), Bromwich confines himself to *quadratic* forms, again using Kronecker's methods. He does not follow Kronecker at all slavishly, and, indeed, the very friendly critic (W. F. Meyer) in the "Jahrbuch" takes him to task for his neglect of Kronecker's rational invariants. In many ways the Tract is exceptionally clear and illuminating, especially for readers who like general processes illustrated by particular applications at every stage, and it has been widely read and quoted while his more general work has been neglected. The geometrical illustrations are very full and clear; there is a complete statement on pages 46–47 of the results for two quadratics in four variables, in terms of the geometry of quadrics; and the applications in the last chapter include, besides the obvious applications to the metrical classification of quadrics (in point or line co-ordinates), applications to bicircular quartics, to cyclides, to quadratic line complexes, and to dynamics. This richness in detail, within the narrow limits of a Tract, may well excuse some neglect of certain sides of the theory.

It was perhaps as an algebraist that Bromwich showed the highest natural talent, but it is no doubt as an analyst, as the author of "Infinite series," that he is most widely known. This book, and the various investigations into which he was led when writing it, must have occupied the principal energies of a good many years of his life.

It is to misjudge Bromwich to imagine him an "analyst" in the fullest sense of the word. He was a very able analyst, but an analyst made and not born. He was no *Funktionentheoretiker*. He knew the theory of series extremely well, and the theory of functions very well up to a point, but there both his knowledge and his inclination stopped short rather abruptly, and none of his work shows any interest in the most modern developments of the theory of functions either on the real or the complex side. No one will blame for that a man whose fault was to be interested in too many things rather than too few.

The book is unquestionably a very fine one. It is not merely a good and an interesting book; it has a character and a distinction which show at once that it is written by an exceptional mathematician. It is easy to criticise it, and many people have done so. It is excellent in detail; there are singularly few actual mistakes, when the richness of the content is considered, and it is easy to find a clear and convincing proof of any particular theorem; but the general plan of the book is open to obvious objections, and, considered as a reasoned and comprehensive text-book, it is not so satisfactory as Knopp's. There is a clear and temperate statement of these objections in the review by Berry which I have already quoted. The continual forward references to the elaborate appendices are, as Berry points out, very trying, confusing to a student and irritating to an expert; the arrangement suggests that Bromwich "originally intended to confine himself almost entirely to series and to use only the elements of the infinitesimal calculus, but that in the course of writing he became interested in the allied theories of definite integrals, and added his Appendix III rather as an afterthought." One might add that it is really impossible to justify the position and arrangement of Appendix I; part of it should come at the beginning, and the rest should be split up and incorporated in different chapters. It is also very difficult to defend the logical attitude adopted by Bromwich towards the elementary transcendental functions. To use the properties of logarithms from the beginning, justifying their use later by methods based on the integral calculus, may perhaps be defensible in an elementary course of analysis, but not in a systematic treatise on the theory of infinite series. "Algebraical analysis" as such has a right, in a book like this, to a more serious treatment.

It would be unfair to lay too much stress on these undeniable defects, in view of the great and striking merits of the book. It has two supreme merits, rare indeed in English text-books of its time; it is thoroughly interesting, and the detailed analysis is almost always sound and clear. Whatever one may think of the general scheme, individual sections are hardly ever obscure. Finally, among all treatises on infinite series, this book stands by itself as a work of reference and a storehouse of information; it really is "a book that no mathematician can do without."

One particular merit is that, when Bromwich has to prove a rather difficult theorem, he is so careful to separate any general difficulty of principle from the small complications due to the special functions considered. A very good example of this is his treatment of the product for sin$x$. The difficulties of principle are disposed of by a general theorem ("Tannery's theorem" of § 49), and all that remains is to verify that the conditions are satisfied in the special case. In such proofs as those in Hobson's "Trigonometry" or Chrystal's "Algebra" the two sources of trouble are encountered simultaneously, with the result of a much heavier strain on the comprehension of a moderate student.

The best chapters have always seemed to me to be Ch. 5, on double series, and Ch. 11, on divergent and asymptotic series. In each case Bromwich was practically introducing the subject to English readers. Ch. 5 is quite short, but it contains all the essentials and is admirably clear. The examples are most instructive, and it cannot be said here, as it can in some parts of the book, that they are so numerous and so difficult that they overshadow the general theory.

Ch. 11 must be judged by a different standard. The subject matter of Ch. 5 was already almost "classical," and there would be no reason even now for any very substantial change; while Ch. 11 is essentially a "pioneer" account of a theory which has changed almost out of recognition since. It would be difficult now, for example, to justify the proportion of space allotted to Borel's method, while the account of the "Cesàro" methods is obviously inadequate. For these reasons much of this chapter has been struck out of the second edition, a decision which is perhaps correct, but which takes away a good deal from the attractiveness of the book.

The book contains the results of a great deal of original research, especially of the papers which Bromwich published in the 'Proceedings of the London Mathematical Society' from 1903 to 1908. Large parts of these papers are incorporated in the text, the appendices, or the examples, and give the book a refreshing air of liveliness and actuality. If we contrast it, for example, with Chrystal's "Algebra," there can be no doubt that it is incomparably a better book. There are many directions in which it has suggested problems for further research. Thus in Ch. 11 Bromwich raises for the first time the problem of the relations between Borel's and Euler's methods of summation; it was not till 1922 that Knopp and Rademacher gave a definite solution.[*] What Bromwich actually proved was very little, but it is only one of many cases in which his book initiated a very interesting discussion.

I may add a few remarks about one or two of Bromwich's individual contributions to analysis. They begin with 11, in which he makes a remark of considerable importance in the theory of double and repeated limits. The question is that of the equality of the limits

$$\lim_{x\to0}\lim_{y\to0} f(x, y), \quad \lim_{y\to0}\lim_{x\to0} f(x, y): \tag{1}$$

let us denote the inner (simple) limits, which we assume to exist, by $g(x)$ and $h(y)$ respectively. "It is the merit of Bromwich," says W. H. Young,[†] "to have recognized the importance" of the equation

$$\lim_{y\to0}\lim_{x\to0} \{f(x, y) - g(x)\} = 0 \tag{2}$$

(2)

---

[*] See Knopp, 'Math. Zeitschrift,' 15 (1922), 238–242. Actually the region of Borel summability includes that of Euler summability (as is suggested by Bromwich's examples), and is the limit of the region covered by iteration of Euler's method.

[†] 'Trans. Camb. Phil. Soc.', 21 (1910), 368.

in this problem. It is evident that (2) is true whenever the limits (1) exist and are equal. What is less obvious is that (2) implies the existence of the limits (1) and not merely their equality when they exist. Bromwich makes a number of interesting applications of this principle.

Another paper on double limits is 12, about the extension of Abel's continuity theorem to power series in several variables. This is a joint paper, but I am concerned here only with a remark which is Bromwich's, and which illustrates his quickness in seeing curious connections between pure and applied mathematics. The force between two equal electrified spheres of radius a, in contact and at potential V, is $V^2(\frac{1}{6}\log 2 - \frac{1}{24})$. This result was found by Kelvin; Bromwich gives a direct proof in 19. Kelvin, however, uses the method of images, which leads to the series

$$V^2\Sigma\Sigma(-1)^{i+j}\frac{ij}{(i+j)^2},$$

a series which is obviously not convergent as a double series, and does not even converge when summed by squares, though the *repeated* series are convergent and have the value which Kelvin finds. The method of images, in short, fails; it is not true that the force between the spheres is "the force between the two sets of images," which could only be regarded as the limit of the sum over a square.

I mention, in passing, the very interesting papers 13 and 14. The first concerns the theory of the logarithmic potential, and reveals some curious contrasts with Petrini's results for the Newtonian case. In the second Bromwich studies the analogues for series of zonal harmonics of Abel's continuity theorem (a field in which it seems that there is still something to be done). Another important paper on a similar topic is 24, in which Bromwich considers the limit

$$\lim_{x\to 0}\Sigma a_n v_n(x),$$

where $v_n(x) \to 1$ for every n, and Σa_n is a divergent series summable by Cesàro's or Hölder's means. The results which he obtains have been mostly superseded by more comprehensive theorems, but the paper played a considerable part in its time in the development of the theory of divergent series; it is here, for example, that we first find an explicit statement of the critical condition

$$\Sigma n^k |\Delta^{k+1} v_n| < K.$$

It seems odd now that so skilful an algebraist as Bromwich should not have been able to prove the general equivalence of the two kinds of means, but there is a great deal in the theory of series which seems easy now and seemed very difficult in 1908.

It remains to say something about Bromwich's earlier work in applied mathematics. His first published paper was 1, on a subject suggested by Larmor. This paper has become a classic. Later Bromwich found his own subjects, though he often makes it clear that they have been suggested by the work of others. Thus 3 was suggested by a paper of Forsyth, and Bromwich's interest in Forsyth's work arose from a passage in Darboux's chapters on minimal surfaces, referred to in 2. The last paper contains the elegant theorem that a surface defined by a tangential equation $p=p\,(l, m, n)$, where p is homogeneous and of degree one, is minimal where $\nabla^2 p=0$.* This was one of Bromwich's occasional contributions to geometry. There are others in 7, which contains a new proof and generalizations of Liouville's theorem concerning conformal space-transformations, and in 18, a joint paper with Hudson, in which there is a very clear account of some of the more delicate points in the theory of envelopes, and a number of results concerning osculating envelopes difficult to find elsewhere. But Bromwich would never have pretended to be a "geometer" in any very real sense of that difficult word.

Other early papers, such as 9 and 10, are concerned with physical applications of his algebraical work. Bromwich, when a student, worked out for himself simple methods of finding the wave surface in a medium from the general equations of electricity or electrodynamics. A note on this subject by Macdonald appeared in Vol. 32 of the 'Proceedings of the London Mathematical Society,' and Bromwich, having acquired a wide knowledge of algebra in the interval, was in a position to construct, in 10, a much more comprehensive theory.

Another development of his early work on the electromagnetic field is shown in 21 and 16. Bromwich had found a new solution of the general equations as early as 1899, when there was a good deal of discussion of the theory of scattering by a spherical obstacle. This was first published partially as a Tripos question in 1910, and then much later in 21; its value as a means of investigation is exhibited more fully in 16.

15 is another very interesting and valuable paper in which Bromwich returned in later years to his work as a student. He had already published one paper (20) on the determination of the potential of a symmetrical distribution from its values on the axis of symmetry. He returns to this problem in 15, and finds a new solution

$$\mathrm{X}_n = r^n\,\mathrm{P}_n(\mu)\log\frac{r+z}{2a} - 2r^n\left\{\frac{2n-1}{1\,.\,2n}\mathrm{P}_{n-1} - \frac{2n-3}{2\,(2n-1)}\mathrm{P}_{n-2}\cdots + \frac{(-1)^{n-1}}{n\,(n+1)}\right\}$$

---

* The theorem was discovered independently by Richmond. See H. W. Richmond. 'Trans. Camb. Phil. Soc.,' 18 (1899), 324–332, where the idea is applied more systematically to the theory of minimal surfaces.

of Laplace's equation with an axis of symmetry, and applies it to determine potentials of circular discs of variable density. A more obvious form of the solution, viz.

$$X_n = a^n \frac{\partial}{\partial n}\left\{\left(\frac{r}{a}\right)^n P_n(\mu)\right\} - 2r^n P_n(\mu)\left(1 - \frac{1}{2} + \frac{1}{3} - \ldots - \frac{1}{2n}\right),$$

was pointed out afterwards by Watson.

It is, no doubt, on Bromwich's later work, which is dealt with by Dr. Jeffreys, that our judgment of him as a mathematical physicist must primarily depend. His earlier work reveals him not as a physicist but as a very powerful and accomplished mathematician, always on the watch for any application of his analysis to physics, and ready to turn it in almost any direction with singular quickness and versatility. However we may judge him as a physicist, or as a pure mathematician of any particular stamp, there can be no doubt about the substance and the distinction of his contributions to mathematics as a whole. We may feel that a mathematician of Bromwich's powers should have done more than he did, but what he did is a great deal more than the performance of any ordinary man, and it would be very hard to find anyone among his contemporaries who did so many things so well.

G. H. H.

REFERENCES.

The following list contains only papers referred to in the notice :

'Proc. London Math. Soc.,' (1) : 1, vol. 30, pp. 98–120 (1898) ; 2, vol. 30, pp. 276–281 (1899) ; 3, vol. 30, pp. 281–290 (1899) ; 4, vol. 31, pp. 289–297 (1900) ; 5, vol. 32, pp. 79–118 (1901) ; 6, vol. 32, pp. 321–352 (1901) ; 7, vol. 33, pp. 185–192 (1901) ; 8, vol. 33, pp. 197–216, (1901) ; 9, vol. 33, pp. 325–342 (1901) ; 10, vol. 34, pp. 307–321 (1902).

'Proc. London Math. Soc.' (2) : 11, vol. 1, pp. 176–201 (1904) ; 12, vol. 2, pp. 161–189 (1905), (with G. H. Hardy) ; 13, vol. 3, pp. 345–370 (1905) ; 14, vol. 4, pp. 204–222 (1907) : 15, vol. 12, pp. 100–125 (1913).

'Phil. Trans. Royal Soc.' (A) : 16, vol. 220, pp. 175–206 (1920).

'Quarterly Journal' : 17, vol. 33, pp. 85–112 (1902) ; 18, vol. 34, pp. 98–116 (1903), (with R. W. H. T. Hudson).

'Messenger of Math.' : 19, vol. 35, pp. 1–12 (1906).

'Phil. Mag.' (6) : 20, vol. 2, pp. 237–240 (1901) ; 21, vol. 38, pp. 143–164 (1919).

'Acta Mathematica' : 22, vol. 30, pp. 297–304 (1906).

'American Journal of Math.' : 23, vol. 23, pp. 235–258 (1901).

'Math. Annalen' : 24, vol. 65, pp. 350–369 (1908.)