WILLIAM BURNSIDE—1852–1927.

WILLIAM BURNSIDE was born on July 2, 1852, the son of William Burnside, a merchant, of 7, Howley Place, Paddington, London. His father was of Scottish ancestry: his grandfather, who had gone to London, was a partner in the bookselling firm of Seeley and Burnside.

Left an orphan at the age of six, Burnside was educated at Christ's Hospital, where he was a Grecian: there, besides his distinction in the grammar school, he attained the highest place in the mathematical school. Having been elected to an entrance scholarship at St. John's College, Cambridge, he went into residence in October, 1871, and was regarded as the best man of his year in the College. In accordance with the general custom of capable students of mathematics in Cambridge, he "coached" for the tripos, his private tutor being W. H. Besant, one of the few rivals of the famous Routh. For some reason, Burnside migrated to Pembroke College in the same university, the change being made late in his second year (May, 1873). He graduated in the Mathematical Tripos of 1875 as second wrangler, being bracketed with George Chrystal, who afterwards was professor at Edinburgh; the fourth wrangler was R. F. Scott, now* Master of St. John's College. In the subsequent Smith's Prize Examination, Burnside was first and Chrystal second.

A fellowship at Pembroke was the worthy sequel of such a degree: he continued a fellow from 1875 until 1886. He was at once appointed to lecture in his college: and he lectured also at Emmanuel in 1876 and at King's in 1877. At that time, college teaching for the best students was sometimes shared by a few colleges, in isolated groups, and included subjects selected from the average normal course for Honours; and Burnside, in addition, gave lectures in hydrodynamics, an advanced course open to all the University. That particular subject was coming into vogue again at Cambridge; attention, regularly paid to the established work of Stokes, was stimulated by the then new work of Greenhill and especially of Lamb. Burnside also examined for the Mathematical Tripos from time to time. Occasionally, he did some private coaching. But later it appeared that, instead of restricting himself mainly to tripos subjects in furtherance of his lectures and an inevitable share in examinations, he had launched himself upon a broad sea of study, then far removed from the tripos domain.

As an undergraduate, he had proved an expert oarsman. While at St. John's College, even as a freshman, he had rowed in the Lady Margaret First Boat

* The writer is indebted to Sir Robert Scott, for several of the personal records in this notice.

which, with the famous Goldie as stroke, went head of the river in 1872. Rather light in weight as an undergraduate, too light (according to the canons of the day) to be considered for the University Boat, he always was rather spare of build and he retained a wonderful power of endurance ; and he kept his rowing form for many years. He rowed in the Pembroke Boat after graduation, as long as he continued in residence ; he was a splendid " 7," and had a full share in its steady rise on the river. For some years after he left Cambridge, his reputation as an oar survived as a tradition in College circles.

After going out of residence, similar opportunities for rowing were not accessible. But in the course of holidays frequently spent in Scotland, Burnside had acquired a zest for fishing ; and for many a summer onwards he continued to go there, pursuing what grew to be his favourite sport. As in rowing, so in fishing, he developed skill and became an expert fisherman ; indeed, with all he undertook, nothing short of his best was sufficient.

In 1885, at the instance of Mr. (afterwards Sir) William Niven, the Director of Naval Instruction—himself a Cambridge man, devoted to natural philosophy, as it was styled by good Newtonians—Burnside was appointed professor of mathematics in the Royal Naval College at Greenwich. The rest of his teaching life was spent in that post. There was a current belief, a belief now known to be justified by fact, that his old college had invited him to return to important office ; but he remained at Greenwich. His work was to his liking. It was a round, well-defined in extent and in demands on time, within a variety of congenial subjects, though only touching in part upon the regions of his constructive thought. The actual teaching, with its incident duties, left him adequate opportunity to keep abreast of progress, even to advance progress, in the subjects of professional duty. It also left him leisure, which was carefully and diligently used, to pursue his own researches, whatever their direction. Best of all to him, he was free from the interruptions and the incessant small demands, business and social, that are inseparable from official administration. For at all times, and in all ways, multifarious detail—whether incidental to the non-scientific side of official duty, or the current presidency of a scientific society such as the London Mathematical, even the purely algebraical garniture and the side-issues in mathematical investigations—such detail was inexpressibly irksome to his spirit.

At Greenwich, Burnside's work was devoted to the training of naval officers. It consisted of three ranges. There was a junior section for gunnery and torpedo officers ; the chief subject of study was the principles of ballistics. There was a senior section for engineer officers : the chief subjects of study were strength of materials, dynamics, and heat engines. The advanced section—perhaps that in which he exercised the greatest influence on his students—was reserved for the class of naval constructors : in that range, Burnside's special mastery of kinematics, kinetics, and hydrodynamics proved invaluable. Records and remembrance declare that he was a fine and stimulating teacher, patient with students in their difficulties and their questions—though elsewhere, as in discussion with equals, his manner could have a directness that, to some, might appear abrupt. He certainly earned the gratitude of his students, as appeared from their spontaneous token of tribute to him when he left in 1919; the address, which they then presented, was treasured by him and his family.

Burnside had married Alexandrina Urquhart in 1886, soon after he was appointed professor at Greenwich. She survives him, with their family of two sons and three daughters.

After his work at the Naval College had ended, the whole family retired to West Wickham in Kent. Burnside, happy as he had been in his work and regretting its actual termination, enjoyed his leisure, spending it among his books, in fishing holidays in Scotland and, not least, in his researches, some continued in regions recognised as specially his own, some of them in the systematic development of ideas in still another branch of mathematics upon which his intellectual interests had settled. The last year of his life was marked by failing health: and the proximate cause of his death was a recurrence of cerebral hæmorrhage. He died on August 21, 1927; and he is buried in West Wickham churchyard.

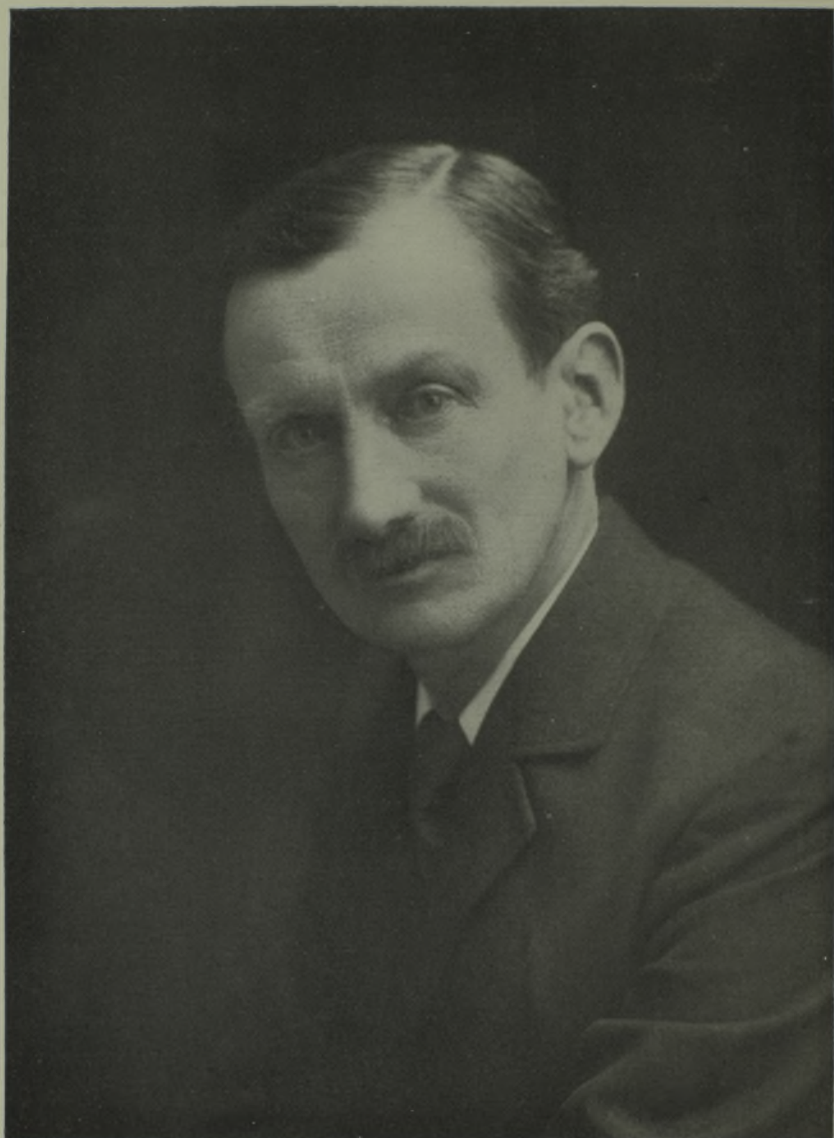
In recognition of his eminence as a mathematician, not a few academic honours came to Burnside during his life. He was never avid of honours: indeed, he was eager to avoid those forms of academic recognition constituted by official positions of dignity, when they demanded the performance of any public duty set in formal pomp or circumstance. He received honorary degrees, Sc.D. from Dublin, LL.D. from Edinburgh. He was elected a Fellow of the Royal Society in 1893, on the first occasion of candidature: he served on the Council of that body from 1901 to 1903; and he was awarded one of the two Royal medals for the year 1904. He was a member of the Council of the London Mathematical Society for the long continuous period from 1899 to 1917: there, he was a tower of strength, in advice during the Council's meetings, and by his many reports as a referee upon a multitude of varied original papers submitted by a small army of authors. He was awarded the De Morgan medal of the Society in 1899. From 1906 to 1908 he served as President: while willingly allowing his name to be submitted for membership of the Council year after year, he accepted their highest office only with grave and characteristic reluctance. The honour, in which he appeared to show most interest, was conferred on him in 1900. In that year he was elected an Honorary Fellow of his old college, Pembroke; and at the time of his death he had become the senior on the small roll of Honorary Fellows. Yet, even in the few and far from fluent remarks of thanks which he made at the College dinner welcoming, by courteous custom, the newly elected honorary members of the foundation, he urged that the happy and successful pursuit of research was its own reward; and the sincerity of his plea was appreciated not least by those who had done their part in recognition of his labours.

Burnside was frequently called upon to examine for the Mathematical Tripos and for the open Civil Service examinations of the highest grade. Occasionally, he acted as external examiner for one or other of the English Universities, as well as for the Naval College after his retirement. He was not an easy examiner—before his early days of such duty, the phrase " easy problems " at Cambridge had come to bear a perverse significance. His questions could be of the type which, gathered in one of his papers, might justify the epithet beautiful: they were certainly too beautiful for the candidates in the 1881 Tripos, the first university occasion when he examined. Yet, though they often were difficult and always on a high level, they were set with the design of evoking an examiner's thought, rather than of providing an opportunity for the facile display of trained manipulative skill along familiar lines.

Through many years, Burnside was in constant requisition as a referee, for the Royal Society and for the London Mathematical Society. He could not be called lenient: for, however sympathetic with writers, and especially young writers, he held a high standard of the attainment that was deserving of publication. He was often fruitful in suggestion. He could even be severe on occasion: yet he would mitigate a judgment when grounds for its reconsideration were submitted. Similarly, as a critic of a friend's proof-sheets, he could be severe, yet always objectively so: he obviously assumed, without the possiblity of question, that the friend's standard and his own were alike in practice. Thus, at the end of a discussion, the friend would find that added light had been cast upon the whole matter—surely the best criterion of sympathetic criticism. And if severe with others, he was stern with himself—a mental discipline that exercised its influence towards the directness and the precision both of form and of substance in his writings.

Valuable as were his teaching, his activity as an examiner, and his influence as a referee, it is by the contributions which he has made to his science that Burnside's name will be held in remembrance.

His range was wide; for it branched out, through applied mathematics from the days of his early training, into great tracts of pure mathematics in the years of his matured powers. Yet, even in the later time, when specialisation has tended to become acute, he could specialise with the best. Though of course not comparable with an Euler, a Cauchy, or a Cayley, in the variety or the amount of work he has left, he has delved in many fields and has left his trace in many directions. He published over one hundred and fifty papers, as well as one treatise, the " Theory of Groups," of which a second (and greatly amplified) edition was issued also under his own care. He has also left a manuscript, fairly complete as far as it was carried, on the theory of probability. He himself



W. Burnside.

did not regard this work as finished; on various issues, he was in correspondence from time to time with the present President of the Royal Society, the Astronomer Royal, and others; and he certainly did not consider that he had resolved all his own questions. Had life and health lasted appreciably longer, there is no doubt that he could have attained, as he intended to pursue, further development in a subject which occupied much of the thought of his later years.

In that considerable tale of papers, most are short. Very many of them occupy only a few pages. His longest individual paper—he never used the more ambitious title "memoir"—deals with automorphic functions: it really consists of two parts connected, though not consecutive, in matter; and the whole occupies no more than fifty-three octavo pages. Brief however as his papers are, it can fairly be asserted that each one of them contains some definite and recognisable result or results. He never discussed side-issues; he would not even dwell on the minute details of a main issue. Indeed, he could be intellectually bored by processes that halted in their march to settle subsidiary questions as they arose; with him, auxiliary necessary material was set out before the main advance. When once an issue was attained, he was content to let it stand by its own significance; to others he would leave attempts "to gild refined gold, to paint the lily."

He happily was saved mathematical controversy, which he detested. On one occasion he was surprised, even disturbed, by the receipt of an unseemly letter the very tone of which amazed him (not unjustifiably): it concerned a question of priority which, in so far as it could affect a man punctilious in his acknowledgment of the work of others, to Burnside was as thin as air, though manifestly not so to the writer of the letter. The quiet firmness of Burnside's answer to his ungracious correspondent ended the matter. On occasion, his work has been known to provide ammunition for others. Thus in 1887 and 1888 he wrote papers on the kinetic theory of gases, a subject which at that date led to much disagreement in opinion; stating his assumptions, he dealt with the average exchange of energy during the impact of elastic spheres and with the partition of energy between motions of translation and of rotation. These papers can only have been the outcome of some appeal emanating from Tait. The result was used (but Burnside took no direct part) in an onslaught upon Bolzmann's work made by Tait, a "bonnie fechter," never reluctant in the use of the controversial tomahawk.

In his writings, Burnside had a style which precisely, and habitually (as if it were an instinct), contributed to efficiency of presentation. Even while an undergraduate, he had been noted for the style of his mathematical work; he was reputed to be the most "elegant", though not the most widely read (Chrystal was thus reputed), among the young mathematicians of his own standing. In pure literature, critics, whether analytic or constructive, do not always agree upon the necessary essentials of general style, though they can select individual characteristics. In scientific productions, the task is assuredly no easier than in the humanities. Burnside had two of the essential secrets of an effective style: he exercised a power of clear and precise thinking that was maintained until the achievement of a definite issue; and he possessed a faculty of lucid (if condensed) expression of the whole course of a constructive argument. He was intolerant of approach to vague meandering: "words, words" would be his caustic comment on an unconstructive passage. The elusive charm of the sudden thought, that in itself is a revelation, is rare in mathematics, though it can be found in a Fourier or a Salmon. But such was not Burnside's aim, perhaps never his dream; he did not seek for aught else than clearness, directness, terseness most of all. He would practise no art in trying to secure the attention of an inexpert beginner. In exposition, conciseness was his rule. Once, the attempt of a friend, to obtain from him a more expanded treatment of some early stages in his Theory of Groups, was met by a declaration of regret that he had been unable to effect further condensation. The consequence is that all Burnside's published work is close and firm in texture; yet, to an attentive reader, it is never lacking in clearness and movement.

Throughout Burnside's residence at Cambridge, the University had been in the finest flower of her activity in applied mathematics. Stokes, Cayley, Adams, were long-established professors; Maxwell's appointment had been more recent. The staple subjects for the most capable mathematical students were physical astronomy, dynamics, light, sound and heat. The range of electricity and magnetism, except for a slight infusion of some of the work of Sir William Thomson (afterwards Lord Kelvin), was academic and unconnected with laboratory knowledge; and Maxwell's presentation, based on the researches of Faraday, had still to make its place in the Cambridge course, men scarcely even dreaming of the revolution it was to accomplish later. Pure mathematics, save for the rare appearance of a Clifford, a Pendlebury, or a Glaisher, was left to Cayley's domain, unfrequented by aspirants for high place in the tripos. Much of the original thought of her mathematicians in those years found its expression in problems, a veritable mine of isolated results propounded as conundrums in the Senate House and in College examinations. Even so, the worship of the mathematical spirit at the shrine of natural philosophy was maintained in a well-defined conservative range.

At the beginning of his work, Burnside could hardly fail to conform to this Cambridge use; indeed, as regards the subjects (though not as regards all methods for the subjects) in applied mathematics, he largely remained in the older round to the end. Yet even while he continued in Cambridge, he was gradually emerging into his own domain. Bred an applied mathematician in the Cambridge school of natural philosophy, which tended to regard all mathematics as a useful tool—no more than a tool—in so-called practical applications, he came to find that there was a world of pure mathematics different from that which filled the receptive stage of his student days. In the creative stage of thinking for himself beyond the range of learning and of teaching for the tripos, he gradually made his way into that new world. He took rank with the constructive pure mathematicians, without losing hold of his earlier studies. Indeed to him, as to others with a similar experience, the new knowledge shed fresh light upon the older interests; but any effective combination of the old and the new could only be made by an intellect of the type such as Burnside happily possessed.

Thus, as already stated, Burnside's earliest advanced lectures were devoted to hydrodynamics. Elsewhere, the old-fashioned methods for conjugate functions, stream-lines, and velocity-potential, were being analytically transformed through the introduction of functions of a complex variable. For many a day, Cambridge had preserved an almost invincible repulsion to the then objectionable $\sqrt{-1}$, cumbrous devices being adopted to avoid its use or its occurrence wherever possible. But some teachers could show that, in two-dimensional fluid motion, simplicity and new results alike were easily attainable by its means; and its formal debut within the Cambridge enclosure was made in Lamb's treatise. To Burnside's intellect the new calculus appealed; and as a matter of record, his first published paper (1883) is concerned with elliptic functions, not with hydrodynamics.

Three examples will suffice to indicate the development in Burnside's thought, thus indicated.

In 1888 he investigates three main questions connected with deep-water waves resulting from a limited initial disturbance, a research probably suggested by certain phenomena noted in the Krakatoa eruption. In that paper he proceeds by analysis which belongs to what would now be called the classical methods of Fresnel, Poisson, and Stokes; it requires much elaborate work in definite integrals with real variables, without any reference to the (happily satisfied) convergence of those integrals; and Burnside arrives at direct results of observable significance, which relate to the greatest amplitude of displacement, the range of propagation, and the governance of the wave-length. It is not without interest, in connection with his increasing grasp of newer methods, to note that in this paper he "justifies" the use of a complex value for a constant—while, in a paper two years later which deals with streaming motion, he uses complex variables without a word of prelude to superfluous justification.

The problem of the two-dimensional potential, as envisaged by the applied mathematicians in the middle third of the last century, such as Green, Stokes, Thomson and Tait, has been completely changed by the ideas of the theory of functions. Old assumptions have had their significance and their limitations revealed, the earlier physicists not always in sympathy with exacting refinements which to them smack of pedantry, the later mathematicians not always respectful to the intuitions content with a semblance of proof. Burnside knew both attitudes of mind—the earlier from his training, the later from his continued study ; and so he could bring old results to new issues. Thus in a paper (1891) on the theory of that two-dimensional potential, satisfying the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,$$

and determined by prescribed conditions within an area and assigned values along a boundary, he returns to the old property—the possession of every undergraduate—that the potential can have no maximum or minimum within the boundary. Pointing out that maxima and minima must therefore lie on the boundary and that conditions of continuity require their aggregate to be an even integer, he obtains a relation between that integer, the integer denoting the number of distinct portions of the boundary, and the integer representing the number of double points on the equipotential contour lines as they pass from a boundary arc over the area back to another boundary arc. Moreover, he obtains the relation for the most general case when the conditions are extended so as to admit discontinuities (in the form of logarithmic or algebraic infinities) within the boundary ; and he indicates the bearing of the relation on the graphs of these contour lines.

In 1894 he published a paper discussing Green's Function for a system of non-intersecting spheres. There, beginning with the known result for two spheres, he transformed it by a property he had deduced from a geometrical interpretation of homographic substitutions. He extended the transformed result to any number of spheres. By inversions which are represented by point transformations, and by sets of inversions which accumulate into a group of transformations, he obtains a pseudo-automorphic function, in the form of a series where the coefficients of the successive terms are powers of the magnification at the successive inversions. Lord Kelvin would not have recognised his theory of images in that final form : yet the development into that form is only a continued amplification of the theory. Burnside, moreover, carried it further, by connecting the application with any solution of Laplace's equation, instead of the inverse distance alone as in the theory of images. Here, as in all his investigations, it was only too evident that he had wandered far from the ancient Cambridge fold.

Various well-marked stages in the progress of Burnside's knowledge almost indicate themselves, from the evidence of his original papers.

Apparently, the first large new subject, of which he made a profound study, was elliptic functions : its rudiments had hardly been admitted to his Cambridge course. At every turn he devised something novel—Is it the transformation of the simplest elliptic differential element ? Noting the general characteristic of the four critical points in the Riemann interpretation, he deals with the successive possibilities of the transformation: (*a*) into itself, by interchanging these four points in pairs, with the obvious inference that there are three modes, which are explicitly obtained; (*b*) into the Weierstrass normal form, with one of the critical points sent to infinity, and the remaining three practically arbitrary; (*c*) into the Legendre normal form, with the four points symmetrically arranged round the origin along an axis; and (*d*) into the Riemann normal form, with 0, 1, ∞ as three canonical points for all, and the fourth defined by the parametric invariant of the element. Is it so simple an issue as the division of the periods by 3 or by 9? Even for the simplest form of that issue, he treats it by a general method and not by any special artifice: a short paper in 1883 achieves the trisection for the Jacobian elliptic functions; a later paper in 1887 achieves the same problem for the Weierstrass elliptic functions; a still later paper uses the same method, supplemented by the introduction of resolvents, to obtain the results for division by 9.—Is it the extension of Jacobi's expression of the apparently hyperelliptic integral

$$\int \{x(1-x)(x-\lambda)(x-\kappa)(x-\kappa\lambda)\}^{-\frac{1}{2}}\,dx,$$

under the (quadratic) transformation

$$z = x + \frac{\kappa\lambda}{x},$$

as the sum of two elliptic integrals? Burnside deals with the cubic and the quintic transformations in odd degree, with the quartic transformation in even degree, and obtains the respective types of degenerate hyperelliptic integrals; characteristically leaving other instances as "exercises" (though, not "easy" exercises) in the method expounded. And, almost as an incident, he notes a case when an apparently elliptic integral

$$\int \frac{x-p}{x-q}\{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\}^{\frac{1}{2}}\,dx,$$

where the relation

$$\frac{y-p}{y-q} = -\frac{x-p}{x-q}$$

transforms the elementary elliptic differential into itself, is only simply periodic. Or, to take only a last example in this range, he completes the known proposition that the co-ordinates of a point on the intersection of two quadrics are expressible in terms of elliptic functions, by constructing the actual arguments; and he shows that the two invariants in the Weierstrass form are the quadrinvariant and the cubinvariant of the customary quartic equation occurring in the reference of the quadrics to their common self-conjugate tetrahedron.

Another subject which absorbed his attention was differential geometry, which also, save for some rarely read sections in Salmon's "Geometry of Three Dimensions," hardly entered into the Cambridge course. Burnside gathers together fundamental propositions, then accessible only by search among widely scattered authorities; and he applies them with effect. Before 1890, the parameters of nul lines on a surface had not appeared (or perhaps, only with Cayley) in English memoirs. In one paper, Burnside uses them, with severe ingenuity, to obtain the different classes of surfaces that possess plane lines of curvature. In another paper, he uses them to construct the differential equation of all confocal sphero-conics, proving that the co-ordinates of points are expressible in terms of elliptic functions of a parametric argument which is obtained explicitly. There, as always in his papers, Burnside's work marches forward to a definite issue and constitutes a contribution to knowledge.

Comparative simple known properties are given a widened significance. Thus he takes the known property that two finite screws compound into a single screw; and (1890) he devises a simple geometrical construction for the axis of the resultant screw. He notes that, as the proof does not require the use of parallels, the result is valid for elliptic space and for hyperbolic space. Five years later, he returns to the matter in a paper on the kinematics of non-Euclidean space; and now he notes that displacements correspond to point-transformations, sets of displacements to groups of transformations. The theory of groups is beginning to affect his work.

He can derive new results from elementary results in ordinary geometry, as well as from the range of abstract geometry. His interpretation of a homographic substitution

$$w = \frac{az + b}{cz + d}$$

as inversion at two fixed circles—this 1891 paper seems the first occasion when the specific mention of a group is made in his published work—is used to assign the criteria, necessary and sufficient, to determine whether a group, formed of assigned fundamental transformations, will or will not contain a loxodromic substitution. Or he will deal with the ancient problem of drawing a straight line between two points, for which the ruler suffices in the Euclidean postulate when the points lie at an implicitly supposed finite distance apart; and he gives a construction for the cases, when one of the points is at infinity, when both of them are at infinity, when one of them is the ideal point required in projective geometry; his construction applies to any space, Euclidean, elliptic, hyperbolic. Or he will take a proposition (analytically established) concerning the four rotations by which a triply orthogonal frame of lines can be displaced into coincidence with a similar frame; by the use of a known (Hamilton) proposition in rotations, he gives a geometrical construction for the displacement, a construction which seems almost obvious—after it has been obtained. Or he will proceed to abstract space: he discusses a configuration of 27 hyperplanes and 72 points in space of four dimensions, such that six of the planes pass through each point and sixteen of the points lie in each of the planes. To him it is a natural extension of the customary configuration of the 27 lines on an ordinary cubic surface in three dimensions.

Burnside's investigations in elliptic functions compelled him to range in the wider field of the theory of functions in general; so thither he had proceeded and, in his progress, he became an investigator.

His contributions are, as ever, varied in range. Fifty years ago, it was a surprise—to-day, it is almost a commonplace—to learn that functions of real variables exist, which are always finite, are always continuous, and never possess a determinate differential coefficient: the now classical example, due to Weierstrass, is that of the series

$$\sum_{n=0} b^n \cos a^n \theta,$$

where a is any uneven positive integer, and b is a real positive quantity such that $ab > 1 + \frac{3}{2}\pi$. Burnside made a step in advance (1894). He showed that there are functions of real variables everywhere finite, everywhere uniformly convergent, everywhere possessing the unrestricted complement of successive differential coefficients, yet never expansible in power-series; and, as an illustration, he constructs the function

$$\sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{1 + a^{2n}(x - \tan n\alpha)^2},$$

where a is real and > 1, and where α/π is not a rational fraction. His proof is concise and demands no acquaintance with elaborate theory; as usual, it leads direct to a definite result that completes the investigation.

On another occasion he deals with the Schwarz solution of the problem of representing a closed convex polygon in one plane conformally upon the half of another plane—a result that has rendered signal service in mathematical investigations in matters so diverse as heat, hydrodynamics, and electricity. In these last applications, only the simplest examples are used: in the general Schwarz solution, an Abelian integral occurs the use of which is gravely handicapped by its multiplicity of periods, so that additional conditions become necessary to render the analysis specific in application. Burnside, already skilled in polyhedral functions and general automorphic functions, investigates the aggregate of instances where, at the utmost, doubly-periodic functions will suffice. But he goes on to deal with multiply-connected spaces having polygonal boundaries: in particular, he gives the solution for the conformal representation of the doubly-connected area which lies between two concentric similarly placed squares, the side of one square being double that of the other.

He seizes upon the existence-theorem which establishes the possibility of expressing the co-ordinates of a point on an algebraic curve by means of uniform functions that are automorphic under sets of transformation. The lack of determination of the group, appropriate to a postulated equation, has left the solution as one merely of existence without specific determination. Burnside, combining his knowledge of groups, of elliptic functions, and of Klein's icosahedral functions, gives a complete specific resolution of the problem for the (apparently) hyperelliptic equation

$$y^2 = x\,(x^4 - 1).$$

It is unnecessary to accumulate more instances. Burnside's matured development flashed out in his double paper on automorphic functions, published in 1892. The subject belonged to a new section of mathematical knowledge, mainly inaugurated by Henri Poincaré and systematically expounded in a series of memoirs, now classical, in the initial volumes of *Acta Mathematica.* The underlying idea is simple. Trigonometrical functions are singly periodic: that is, each such function is unchanged when its argument suffers an increment or a decrement which is any integer multiple of a single quantity. Elliptic functions are doubly-periodic: that is, each such function is unchanged when its argument similarly suffers an increment or a decrement which is a linear combination of any independent integer multiples of two quantities (the ratio of these quantities must not be real). Jacobi had proved long ago that uniform functions of triple periodicity (and, *à fortiori*, of periodicity higher than triple) in a single variable do not exist. But in every such instance the modification of the argument consists solely of an additive increment or decrement. The question arises: What is the most general type of periodicity for a function of one argument? And it naturally entails the further question: What are the functions possessing that type of periodicity? Isolated results were known, such as Jacobi's elliptic modular functions and Klein's polyhedral functions: their significance as examples of a wider theory had not appeared. It was Poincaré who presented the first general treatment of these questions.

Into this work of Poincaré, Burnside plunged. In it he revelled, and his new results are embodied in his paper on automorphic functions which has just been cited. In particular, Poincaré had overstated an exclusive central result. Burnside detected the overstatement and the fundamental cause; and he devised a new class of automorphic functions, simpler than any of the classes devised by Poincaré. The full theory, even now, remains to be established: it awaits the construction (or the equivalent of the construction) of a central function or functions which, while palpably automorphic, shall be amenable to ordinary analytical manipulation as are the corresponding central theta-functions of purely incremental periodicity. When the history of that theory comes to be written, Burnside's name will hold an honourable place in the record.

The consideration of the very foundation of these automorphic functions led Burnside further afield, along a way already opening out before him in his progress, into a region which he explored with ample discovery. It was to provide the most continuous and most conspicuous of his contributions to his science. The characteristic property of every automorphic function of a single variable is that, without change in the value of the function, its argument is subject to a number of reversible operations, which are independent of one another, are capable of unlimited repetition and reversion, and admit all possible combinations, repetitions, and reversions, in unrestricted sequence. The aggregate of all the operations, which thus emerge, is termed a group, so that a function can be automorphic under a group of transformations (or substitutions). But just as the properties of the integers, which occur in the arithmetic of any calculation, merge into the general theory of number which ignores all specific application, so the properties of transformations in a group merge into a more comprehensive calculus. That calculus deals with the composition, the construction, the resolution, and the essential properties of a group, regarded as an abstract entity whose component elements are subject to mathematical laws of combination. It is no part of that calculus to take account of possible regions of application: instances present themselves in algebraic equations, in analytic functions, in differential equations, in divisions of space of different orders of dimension, in the displacements of a solid body, in invariants and covariants of all kinds :—a selection of subjects manifestly not complete.

The earliest expression of the notion and its initial development are due to Galois: he indicated the kind of relation that could exist between the properties of an algebraic equation and some corresponding group of finite order. The early growth of the theory was due to French mathematicians, Cauchy in particular, then Serret. Somewhat later came the fine exposition by Jordan who, it may be mentioned, had Klein and Lie as pupils at the outbreak of the Franco-Prussian war in 1870. Down to that date, the subject revolved round algebraic equations as its centre.

The interest in the theory began to spread. The next real extension was due to Sylow, in a memoir on groups of substitutions. Then followed a partial construction of its mathematics as a pure calculus, without regard to applications: the contributions of Cayley and of Weber may be noted. The theory soon divided itself into two co-ordinate sections, sometimes advancing as pure calculus, sometimes extending to new regions of application. A theory of continuous groups branched off into complete independence; it became a great body of mathematical doctrine, under the inspired researches of Sophus Lie and his disciples. The theory of discontinuous groups attracted an equally ardent band of investigators: the names of Klein, Burnside, Frobenius, Hölder, and Dyck, recall diverse developments in theory and in use.

It was to the theory of discontinuous groups of finite order that Burnside mainly devoted his attention. Scattered references to such groups occur in some of his papers already cited. At first, their occurrence seems merely incidental; then they almost prove that his thought was gradually accumulating the evidences of a connected theory. From the early nineties onward through much of the remainder of his life, Burnside's constructive thought concentrated on the subject. Paper after paper appeared from him, on a vast variety of associated topics, in ordered development, each providing some fresh contribution, all of them marked by imaginative insight and compelling power. They found their first culmination in his book on the "Theory of Groups," published in 1897. That volume was a systematic and continuous exposition of the pure calculus of the theory as it then stood; and it embodied the researches of other workers in Europe and America (always with ample references) as well as his own. His papers on the theory of groups continued, unhastingly, unrestingly. A second edition of the book, considerably more extended than the first, appeared in 1909. Even so, his activity in the subject still continued, though with a gradually decreasing production. He published over fifty separate papers on this range of knowledge alone; each of them, even the briefest, contained some definite result or results of significance. All this work, original from himself, is a splendid contribution emanating from one mind and, of itself, is sufficient to secure the remembrance of his name.

With the coming of the war in 1914 and during its course, there was a comparative cessation in Burnside's productivity. His frame was almost as lithe as ever and apparently as full of easy spring, as though to belie the passage of years. Some of his constructive activity passed silently into the service of his country in certain naval matters. In those years he undoubtedly continued to produce papers; but the main body of his work could be regarded as verging towards its termination.

One new subject, however, secured some regular attention from him, even amid his unbroken interest in groups. It may have originated from the mathematics of some war problems, and its interest may have been fostered as he pondered over the combinations of diverging results of observations. In the year 1918 he produced a short paper dealing with a question in probability, purely mathematical as propounded; and it was followed, from time to time, by other papers, some suggested by practical problems. Probability, as a mathematical theory, has not yet lent itself to a single process of organised development based on any unique set of ideas, which are generally accepted as fundamental. Even the method of almost universal use in astronomical observations depends upon the Gauss assumption of the arithmetic mean of a number of discordant observations, as the best measure of the unknown quantity. But that assumption stands as only one out of many inferences from the less arbitrary assumption that the probability of an error, in any observation, is some function solely of the deviation from the unknown accurate measure; with that less arbitrary assumption, a more general inference is that the difference between the unknown measure and the arithmetic mean is some symmetric function of the differences between the observed magnitudes. (Of course, the occurrence of the symmetric function modifies the law of facility of error: or the adoption of an admissible law, not inconsistent with the assumption and differing from the exponential law, determines the form of the symmetric function.) Burnside deals only with the arithmetic mean: thus tacitly, with other writers, making the symmetric function to be zero. As indicated earlier, he did not consider that he had resolved all his difficulties. Ever a severe critic, he remained critical of himself; he was not afraid to modify an opinion; he did not hesitate to abandon an opinion, if ever he regarded it as not fully tenable, as indeed happened in fact. The manuscript, which he has left and which will be published by the Cambridge University Press, is the expression of his views so far as they had been framed into a system.

There is one activity in human nature which exercises a perennial lure for living minds. When a worker of recognised distinction in any field has completed his contribution to thought, some survivors delight in assigning him his place in an ordered hierarchy of memorable names. The task demands an easy omniscience which shall gauge all knowledge and all intellect, if its estimate of precedence in relative merit is to be promulgated with authority and received with belief. Yet, somehow, such estimates lack the quality of permanence. Nearly two thousand years ago Lucretius, the brilliant expositor of natural philosophy in an age of culture, described Epicurus as a man

Qui genus humanum ingenio superavit,

a tribute paid two full centuries after the death of the Greek philosopher of the atom: the world to-day, if it ever hears of the name thus lauded, greets the judgment with a smile. Less confident men may, in their own day, render a more modest yet equally sincere homage to a passing spirit, from their reverence for the genius that has striven, and in their remembrance of the worldy task that has been done. Burnside, during a life of steadfast devotion to his science, has contributed to many an issue. In one of the most abstract domains of thought, he has systematised and amplified its range so that, there, his work stands as a landmark in the widening expanse of knowledge. Whatever be the estimate of Burnside made by posterity, contemporaries salute him as a Master among the mathematicians of his own generation.

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