F. Klein

CHRISTIAN FELIX KLEIN—1849-1925.

At Düsseldorf, on the night of April 25th, 1849, there was anxiety in the house of the secretary to the Regierungspräsident. Without, the cannon thundered on the barricades raised by the insurgent Rhinelanders against their hated Prussian rulers. Within, although all had been prepared for flight, there was no thought of departure ; on that night was born a son to the stern Prussian secretary. That son was Felix Klein. His birth was marked by the final crushing of the revolution of 1848 ; his life measured the domination of Prussia over Germany, and typifies all that was best and noblest in that domination ; with his last illness came the consummation of its downfall. Gradually, but irresistably, the nervous malady of which from time to time he had had serious warnings, mastered and prostrated him. It was, he thought, partly due to heredity on his mother's side, partly to his own unbridled expenditure of mental energy—an energy which remained so indomitable that, even during the last two years of his life, when he lay helpless, becoming daily weaker and weaker in body, he never complained, and remained clear to the end, working and even correcting proof-sheets. At half past eight on the evening of Monday, June 22nd, 1925, he passed painlessly away.

2. Few mathematicians have left such ample material for forming an opinion of their life and work as Felix Klein. We have his life, written by his own hand two years before his death.* We have his Collected Mathematical Papers, in three volumes,† thoroughly revised by himself, and interspersed with supplementary notes and introductory articles of an autobiographical character. We already have the greater part of his mathematical lectures in print, lectures which had for many years enjoyed a considerable publicity in lithographed form ; we have even a faithful record of lectures given by him in the years just preceding his death, carefully annotated by his colleague and successor, Professor Courant.

After his death there appeared, one after another, a number of sketches of the man and his work from the pens of many of his pupils. But, just as a photograph of a man of unusual personality, or of a place of striking beauty, conveys little to one not personally acquainted with the original, so it is, and

* 'Göttinger Professoren: Lebensbilder von eigener Hand. No. 4, Mitteilungen des Universitätsbundes Göttingen.' Jahrg. 5, Heft 1. This will be referred to as "Autobiography."

† Referred to below as 'Ges. Math. Abh.'

so it must be, with these sketches of Klein. At the same time it behoves the Royal Society to attempt more than merely to chronicle the leading events in the life of him who was its oldest Foreign Member. It is not merely that for 40 years Klein figures in the List of the Society, and received in 1912 the Copley Medal. It is still less that he was personally known to and valued by many of the Fellows. The life of Klein is bound up with the regeneration of mathematical study.

3. We have been told* that Klein was one of the greatest mathematicians of all time, a claim which he would certainly not have countenanced; but his personal influence was as great, or greater, even outside his own country, than that, perhaps, of any mathematician of modern times. He owed this to his forceful and attractive personality, to his wide mathematical outlook and to his objective open-mindedness, only occasionally tinged by the play of personal feeling, for Klein had by no means the cold, calculating nature, supposed typical of the mathematician. He owed it also partly to that intangible agent, so often and so callously cited in England, "Luck," which afforded him, by a succession of unforeseen external events, the opportunity without which a man of the highest intellectual and moral endowments may remain mute inglorious. He owed it partly to his success in interesting his audiences, even in branches of mathematics of which his own command was comparatively slight. And he owed it to his untiring devotion to the cause of education; but his efforts in this direction were rendered effective by the instrumentality of one in whom Klein's ideas aroused a keen interest, Althoff, who from the Curatorship of the University of Göttingen, was transferred to a leading post in the Prussian Ministry.

4. Klein's genius had been precocious. At 17 he was chosen by Julius Plücker as his assistant in his physical laboratory at Bonn; that laboratory where Sommerfeld tells us, Plücker had invented, and Geissler, his glassblower, had fabricated, what in England is known as the Crooke's tube. All through his life Klein was a would-be physicist; like Goethe's Mephistopheles, who aimed at evil and complained that he always did good, Klein was all his life striving to do anything rather than Pure Mathematics, and found himself, with the shortest of interludes, doing nothing else. Indeed, Plücker's own mind had reverted, after an interval of some years, to his early geometrical interests. In his youth he had discovered the famous equations bearing his name, in his later years he became the inspirer of Klein.

It is fitting for the Royal Society to recall that it was Plücker, and not Cayley, who introduced the six homogeneous line-co-ordinates in space, connected by an identity of the second degree, in a paper in the Transactions of this Society (1865, pp. 725–791). All Klein's early work was in line geometry, or was related with that subject. Yet even in writing his Doctor's Disserta-

* F. Pfeiffer.

tion, which he evolved at the age of 19, a year after Plücker's death, he unknowingly introduced, what was so characteristic a feature of his later work, the idea of a Group,* and the same is true of his treatment of one of the five theses which he had to uphold at his examination for the Doctor's degree; indeed, in order to show† that the quadratic complex which Battaglini had previously considered was not the most general quadratic complex, he used an argument based on the existence of transformations of Plücker's co-ordinates, involving three parameters, by means of which transformations the whole set of Battaglini's complexes remained unaltered; these transformations form, of course, a group.

It was in consequence of Plücker's death (22nd May, 1868), leaving unfinished his "New Geometry of Space, founded on the straight line as element," that the task of completing and issuing the second half of the book was entrusted by Clebsch to Plücker's young assistant, the only person, in fact, who, from constant intercourse with the master, was in a position to edit the manuscript and fill in the lacunæ in the spirit of the author.

5. Clebsch had recently been installed at Göttingen; thither Klein followed, and was thus brought into daily intercourse with him.‡ Under the influence of Clebsch, Klein developed his love for geometrical models, already excited by Plücker, who told him, by the way, that it was chiefly through Faraday that he had come to appreciate the importance of models. Faraday, as a non-mathematician, had found the construction of models invaluable in enabling him to comprehend that part of the mathematics of his day which he needed in his work. In this connexion we remark that Plücker, by his intimacy with English scientists, seems to have been instrumental in leading Klein to study the English language and the writings of English mathematicians, and so led to that friendship with the English-speaking nations and individuals to which Klein was loyal to the end.

It was at a meeting of mathematicians in the Bergstrasse at Göttingen in 1868 that Klein first saw an actual model (Wiener's) of the cubic surface with the 27 real straight lines the existence of which had been discovered in 1849 by Cayley, with whose work Klein was already familiar.§ This incited Klein to pursue the subject of models further himself, in which resolution he had Clebsch's approval, and he actually constructed (1871) models in zinc of the Kummer surface of the fourth degree with sixteen double points and sixteen double planes, as well as other particular cases of the same surface, which occur in line geometry. Throughout his life his interest in the subject of the real forms of curves and surfaces remained lively, and he believed intensely

* 'Ges. Math. Abh.,' vol. 1, p. 18.
† *Ibid.*, p. 3.
‡ *Ibid.*, p. 51.
§ *Ibid.*, vol. 2, p. 3.

in the utility of models and actual constructions for didactic purposes. He was proud to be able to say at the end of his life that, thanks to his efforts and those of his friends, the Brills and Dyck, among others, no German university was without a proper collection of mathematical models.*

6. Klein's first stay at Göttingen was from the beginning of 1869 to the middle of August of the same year. His second stay was from the beginning of 1871 to the middle of September, 1872, and his final migration thither was in 1886. Between the first and the second stay there fall three important periods, his stay in Berlin, his stay at Paris, to which we shall return later, and the Franco-German war, in which he only took part in the ambulance corps, being refused for active service, and from which he was sent home invalided with typhoid fever.

In August, 1869, against the advice of Clebsch, which agreed with the caution Plücker had formerly given, Klein went to Berlin and stayed there till March, 1870. The auguries of Clebsch and Plücker were so far verified that Klein did not succeed in impressing his personality on the Berlin authorities, as he had done at Bonn and at Göttingen. Indeed, in spite of his work on Kummer's surface, and although he spoke on line geometry several times at Kummer's Seminar, he says himself,† and expresses surprise at the fact, that no link established itself between himself and Kummer's researches into algebraic systems of rays. In the mind of Weierstrass, a veritable antagonism to Klein seems to have arisen. The word went round among Weierstrass's pupils that Klein was "anathema maranatha," and that what he did was not mathematics at all. Yet, at the suggestion of Lipschitz, to whom he had submitted his Dissertation, Klein had grafted on the original sketch a discussion of degenerate cases, based on Weierstrass's Elementarteile; and, with his ardent desire for breadth of outlook and his determination not to be confined to any one school of thought, it must have been largely in hopes of profit from personal contact with Weierstrass that he insisted on going to Berlin. Yet he himself tells us† that he hardly attended any lectures, and that it was only through a fellow student, Kiepert, that he came across, and studied, Weierstrassian Elliptic Functions. He spoke once, and only once, at Weierstrass's Seminar; his subject, was Cayley's Theory of the Absolute, and, in conclusion, he threw out the suggestion that there was here a connection with Non-Euclidian Geometry, he having just heard of Non-Euclidian Geometry for the first time from another fellow student, Otto Stolz. Weierstrass absolutely rejected this suggestion, laying it down that the concept of distance was at the basis of all geometry. Later, Klein developed his idea in a series of papers extending over several years; he

* 'Ges. Math. Abh.,' vol. 1, p. 51.
† Autobiography, p. 15.

was able also, on the occasion of his attending the Meeting of the British Association at Bradford, in 1873, to discuss the question face to face with Cayley himself, as well as with Clifford and Sir Robert Ball. He never succeeded in convincing Cayley and Ball that he was not arguing in a circle,* and, indeed, to make the argument really convincing required a mind more patient and diffident than that of the impetuous and daringly intuitive Klein.

7. It was the influence of Clebsh that obtained for Klein, at the early age of 23, the full Chair of Mathematics at the University of Erlangen. This would, however, not have afforded him sufficient opportunity of distinguishing himself, but for what seemed at first a disaster for the rising young geometer, the premature death of his chief supporter Clebsch from diphtheria on November 7th, 1872. Two days previously, Klein had given his first lecture to an audience of two, only one of whom reappeared, and he only once or twice. But the mantle of Clebsch could not fall otherwise than on the shoulders of Klein, and a number of advanced students, some older than Klein himself, migrated at once from Göttingen to Erlangen. The young professor found himself suddenly the centre of an intense mathematical life, such as had previously existed at Göttingen.†

It was at Erlangen that Klein produced what he himself regarded as his most notable achievement, the so-called Erlangen Programme. It would be no exaggeration to say that it has revolutionised the treatment of Geometry. Only in details has exception been taken to any part of it, which is the more remarkable when we consider the age of the author and the state of mathematics at the time when it was composed. The notes which Klein himself has added, and the account which he has given‡ of its production, have greatly increased its interest. The new idea which lies at its basis was that all the various species of Geometries which, during the 19th century in particular, had multiplied exceedingly (metrical geometry, projective geometry, line geometry, etc.) could be regarded from a single standpoint, that of the Theory of Groups, each different geometry being conceived as the theory of the invariants of an appropriate group. This idea, which as then formulated, seemed, even to Sophus Lie, "very astonishing," §, was an abiding one in the mind of Klein, who thought he foresaw its extension to regions of mathematics other than geometry. In particular, the development by Einstein of the Theory of Relativity revived in his later years Klein's interest in his Erlangen Programme, and it was with some mortification that he found among his physicist friends no interest whatever in the possibility of a physical interpretation of the most general group of conformal transformations of four dimensional space, to

* 'Ges. Math. Abh.,' vol. 1, p. 242.
† Autobiography, p. 18.
‡ 'Ges. Math. Abh.,' vol. 1, pp. 411—414.
§ Sophus Lie, by F. Engel. 'Jahresber. d. deutschen Math. Ver.,' vol. 8, p. 39.

which attention had been called by Bateman and Cunningham as leaving invariant the Maxwell-Lorentz equations.*

It was at Erlangen that Klein married.† His wife's beauty was conspicuous even in later life; her sweet and cultivated personality as Hausfrau, and the sad affliction of her growing deafness, which, when combined with the abnormally acute hearing of her husband, formed a sensible factor in the home-life, will always be remembered by those who had the privilege of enjoying the simple, old-fashioned hospitality of Klein's house. She was Anna Hegel, daughter of an Erlangen professor and grand-daughter of the philosopher Hegel.

8. At Erlangen, Klein worked on the lines which he had drawn up in his Programme, taking as guiding star a simple principle:—" Algebraic forms, which are transformed into themselves by finite groups of linear substitutions, have special properties which are in consequence easily perceived." This he illustrated, in the first instance, by the binary forms of the third and fourth degree, using Riemann's interpretation of the complex variable $x + iy$ on the sphere. We feel here at once the influence which Clebsch had exerted on the mind of Klein, in directing his attention to linear transformations and the theory of invariants and covariants of binary forms. But Klein had seized on this subject-matter with characteristic breadth of view, and he tells us‡ that he had all along asked himself, with a mixture of hope and fear, what Clebsch would say to a presentation of the subject of geometry so essentially different from his own systematically projective method; but Clebsch never saw the Erlangen Programme.‡

In the peroration of the Programme, Klein, who had come to some knowledge of the work of Galois during his stay in Paris, alludes to the connection with the theory of equations, and the application which Galois had devised of the theory of groups to the solution of algebraic equations. He suggests that similar considerations in the theory of linear substitutions would, when Riemann's interpretation is employed, open out illuminating geometrical relations on the sphere connected with the regular solids. Indeed, the linear substitutions, when so interpreted, become rotations of the sphere round its different diameters, combined with reflexions at its diametral planes, so that the vertices of each inscribed regular solid, determining as they do rotations which transform them into themselves, give us binary equations in $x + iy$ with appropriate groups of linear transformations leaving them unaltered. In this way, Klein was able to prove that he had obtained all possible finite groups of linear transformations of $x + iy$ by means of the five ancient Platonic

* F. Klein, 'Entwickelung,' vol. 2, p. 96.

† The only son, Otto, is an engineer. Two of the three daughters have children. The third, a pupil of her father's, and a war widow, is a teacher in a gymnasium.

‡ 'Ges. Math. Abh.,' vol. 1, p. 412.

figures, together with the flat disc bounded by a regular polygon as degenerate form of a regular solid. And further, applying Galois's ideas, he was able to determine completely the solution of the quintic equation, which was at that time occupying the attention of several of the best mathematicians of the day.

9. From the small Bavarian university of Erlangen, Klein passed in 1875 to the Technische Hochschule at the Bavarian capital, Munich. In Germany, where the different levels of society were kept rigidly distinct by what was termed the "Kastengeist," the level of professors at the technical schools was regarded as lower than that of university professors; but Klein looked at things from a broader standpoint. He says himself that there hovered before his mind's eye the vision of a polytechnic school like those of Paris and Zurich, and, in accepting this "call," he felt that he was taking a great step in advance.* The realisation of his plans was rendered easier by the fact that, at the same moment, Brill was called from Darmstadt to Munich. Brill's experience and thorough knowledge of the working and the best traditions of a Technische Hochschule were invaluable to Klein. He enjoyed the advantage, too, of having Dyck as his assistant and Hurwitz as pupil. The latter was also his chief collaborator in the study of the Elliptic Modular Functions, one of the monuments of Klein's fame.†

The years at Erlangen had been only a preparation; it was at Munich that Klein began to feel his feet; here, to use his own words, he "worked himself through to a real mathematical individuality."‡ But, in doing so, he undermined his health, and, at Leipsic, to which university he was called in 1880 as Professor of Geometry,§ he became seriously ill. His breakdown was probably accelerated by the antagonism he experienced at Leipsic. He was much younger than his colleagues, and they resented his innovating tendencies. In particular, there was opposition to his determination to avail himself of the vaunted German "Lehrfreiheit," and to interpret the word "Geometry" in its widest sense, beginning his lectures with a course on the Geometric Theory of Functions.

10. Klein was now at the acme of his mathematical powers. The theorem, which Klein himself prized highest among his mathematical discoveries, known as the "Grenzkreistheorem" in the theory of automorphic functions, flashed upon his mind suddenly at the seaside, in the small hours of the morning, as he sat, propped up on a sofa, because of his asthma.|| This was at Easter, 1882. The stimulus to this important extension of his earlier work

* "Einen grossen Sprung," Autobiography, p. 19.

† Adolf Hurwitz, by W. H. Young, 'Proc. L.M.S.,' Ser. 2, vol. 20, p. xlviii.

‡ Autobiography, p. 20.

§ *Ibid.*

|| 'Ges. Math. Abh.,' vol. 3, pp. 584, 627.

on the icosahedron and the modular functions had been given by Poincaré's three notes in the 'Comptes Rendus' of February and March, 1881, on Fuchsian Functions, a name given by him, in honour of Fuchs, to a particular type of what Klein afterwards denominated Automorphic Functions.*

Poincaré was only five years younger than Klein,† but had been relatively retarded by the customary somewhat lengthy stages of the mathematical profession in France. In 1880 and 1881 he was *chargé de cours* at Caen. The complete absence of an adequate library there left him quite unaware that, in the grand theory which his three notes foreshadowed, preliminary steps had already been taken. His chivalrous action, when Klein wrote to apprise him of this omission,‡ was to give Klein's name to the remaining Automorphic Functions. The theory of these functions is now marked for all posterity by the intertwined names of Klein and Poincaré. The correspondence that ensued, published in the 'Acta Mathematica,' vol. 39, shows us how the two mathematicians acted on one another as the foundations of the theory were gradually laid.§ After Klein's breakdown, Poincaré continued his work for some time.

11. Obliged by his illness to give up work for a time, and to reduce it materially in the long run, Klein undertook the publication of the one book‖ which he produced single-handed, 'The Icosahedron.' The subject-matter of this little volume, in so far as it originated in Klein's own brain, had been published, in the papers already referred to, in the 'Mathematische Annalen.' We cannot do better than quote Klein's own summing up of the relation in which the book stands to previous work on the subject :—

"In the book many details have been more exactly discussed, and much has been simplified, in which the works of Gordan and Kiepert were of special use to me. On the other hand, much has been suppressed, especially in the theory of invariants. . . . For the rest, the matter is presented, at least as far as regards the equation of the fifth degree, in historical sequence.

* See Poincaré's own account, 'Acta Math.,' vol. 38, pp. 44, 45.

† The impression which might be given by Klein and others, *e.g.*, 'Ges. Math. Abh.,' vol. 3, p. 582, that Poincaré was a very much younger man, and very young at the time, is evidently erroneous.

‡ See Poincaré, 'Math. Ann.,' vol. 20, p. 53. "En ce qui concerne les fonctions Kleinéennes, j'aurais dû commettre une injustice si je leur avais donné un autre nom que le vôtre. C'est M. Schottky qui a découvert la figure qui faisait l'objet de votre lettre, mais c'est vous qui avez : 'ihre principielle Wichtigkeit betont,' comme vous dites à la fin de votre savant travail 'Ueber eindeutige Functionen mit linearen Transformationen in sich.'"

§ Reprinted, 'Ges. Math. Abh.,' vol. 3, pp. 587–621. *See* also Klein's historic account, *ibid.*, pp. 577–586.

‖ Klein's classical paper on "Riemann's Theory" ('Ges. Math. Abh.,' vol. 3, pp. 499–574), was also published as a little book.

Hence, apparently, the impression arose in mathematical circles that the object of my method was to render intuitive the accepted theory of the icosahedron. This way of regarding it does not in the least tally with what I still maintain, looking backward. I believe rather that the real foundation of the previous researches of Hermite, of Kronecker, and of Brioschi is only to be found in a preliminary study of the theory of the icosahedron, and in the theory of the elliptic modular functions. The proof of this is that from 1876 to 1880 not only all the points were cleared up which in their theory had remained nebulous; but also, in our rapid advance, questions were attacked which had not even been raised till that moment. I have always looked on those years, in which the momentous steps forward were taken, as the happiest period of my mathematical productivity. Externally, it was characterised by my daily meetings with Gordan. The spot chosen for these meetings was generally Eichstadt, which is half-way between Munich and Erlangen, and we usually spent Sunday there. Gordan used in later years often to speak of the 'Mathesis quercupolitana,' as he called it."*

12. In tracing Klein's career it is striking to contrast it with that of Gordan, the friend just referred to. For eleven years, in spite of his high merit as a mathematician, and in spite of his friendship and collaboration with Clebsch, Gordan had to hold out as Privat-Dozent at the University of Giessen, and only obtained the next step, to an extra-ordinary professorship at Erlangen, a few months before Klein, twelve years his junior, vacated the full chair, which Gordan then occupied till two years before his death in 1912. He seems to have had a rare gift of what we may call mathematical sympathy. In a few words let fall on his 70th birthday, Gordan characterised his ten years' relation to Clebsch as "a happy kind of partnership in which his own task was frequently that of transforming the ideas of that fertile genius."† Noether says‡ of Gordan that no difficulty seemed to him insurmountable and that he threw light on any subject in a truly Socratic manner. This was just the friend for the man of whom Lie said,§ and not without knowledge, "I rank Klein's talents high and shall never forget the ready sympathy with which he always accompanied me in my scientific attempts; but I opine that he does not, for instance, sufficiently distinguish induction from proof, and the introduction of a concept from its exploitation."

13. The moment at which Gordan joined Klein at Erlangen was also opportune. Klein had been more and more progressing from the algebraic and geometric points of view of Plücker and Clebsch to the wider outlook

* 'Ges. Math. Abh.,' vol. 2, p. 259.

† Paul Gordan, by Noether, 'Math. Ann.,' vol 75, p. 7.

‡ *Ibid.*

§ Vorrede to the third volume of the 'Transformationsgruppen,' by Lie and Engel (1893).

of Riemann, that of the geometrical Theory of Functions, which in the end dominated his mathematics. Klein had never known Riemann,* but he had begun in his Göttingen days to read Riemann's original papers. Gordan, on the other hand, had just met Riemann and sat at his feet for a few weeks in the autumn of 1862, immediately before Riemann's health broke down. The book on Abel's Functions by Clebsch and Gordan had appeared in 1866, three months after Riemann's death, and Klein says of it†: "Essentially this important book succeeded in deducing Riemann's results about algebraic curves by the methods of Analytical Geometry. Riemann's own methods were at that time a sort of Arcanum of his immediate pupils, and were regarded by the remaining mathematicians almost with distrust. I can only repeat, on the other hand, what I myself realised at once for curves, that in course of time the progress of mathematics must evidently and inevitably lead to the methods of Riemann himself becoming part of the heritage of every mathematician." If this is the case to-day, and in so far as it is the case, it is chiefly due to the apostleship of Klein. But the chief tenet in Klein's cult of Riemann was only given verbally to his choice disciples, and must remain for ever esoteric: Read Riemann!

It is interesting to note that the turning point in Klein's recovery from his breakdown at Leipsic was marked by his being invited to Baltimore, to take the chair vacated by Silvester in 1884. Indeed, he attributed to this invitation, and to the attractive vistas which it opened to his fancy, the tonification of his mind enabling him to throw off his malady. He eventually declined the invitation, which was not the last he received from an American university. But the days of his great productivity were over, although he had only reached what Dante calls "the middle of our walk in life"; from this time, for many years, he devoted himself to the development of his mathematical school.‡

In 1886 he was called to Göttingen. The great city was indeed no home for the true German, and it was with enthusiasm that the Kleins migrated to "the little garden-town."§ Time has so touched with the pencil of prosperity the features of Göttingen, that to-day it is difficult for the visitor to realise what it was in the primitive simplicity of those days. Göttingen was then still a medieval poem. It was here that the happiest home-days of the family were passed, in the comfortable villa, all their own, built by themselves, in its cool shady garden. On one side it was close to the Auditorium, on the other, near

* 'Ges. Math. Abh.,' vol. 3, p. 477.

† *Ibid.*, p. 490.

‡ Autobiography, p. 22. It was in the excellent historical seminar of Waitz at Göttingen in 1868–9 that Klein first got the idea of the importance of a well-organized department, Autobiography, p. 15.

§ *Ibid.*

by, was the magnificent forest, the Göttingerwald, stretching to the Harz Mountains. Every day, after midday dinner and a short siesta, Klein used to walk up into the forest in mathematical converse with his colleagues and friends, after which he received his special students upstairs in his book-lined* study, and discussed their work, or he prepared his own lectures.

16. Göttingen did not disappoint Klein and he refused every "call" elsewhere. He saw the university raised once more to a world-celebrity which even in the days of Gauss had never been surpassed, and to a material magnificence which had never been even imagined. The turning-point came when, as he himself humorously pointed out at a Christmas gathering of his students in 1893, the centre of the universe veered round to Göttingen, following as it always must, the advent of Eve's daughters. In fact, Prussian university life had been gently invaded. Two American women, inspired by Klein's visit to Chicago that year, had followed him to Europe, and a Girton girl had appeared at the same time independently†; it was she who became eighteen months later the first female Doctor of Philosophy at a Prussian University.‡

17. At the Auditorium, on the top floor, was Klein's lecture room; in the days of his best lectures it rarely contained more than a dozen hearers, and these nearly all foreigners,§ but at the beginning of the present century there were near on a hundred students, mostly Germans, in his audience. Hard by was the wonderful mathematical reading-room, with the adjacent room lined with glass-cases full of models; here the professor held, after his lecture, his "Sprechstunde," when any of his pupils could come to him with requests or questions. For a very small fee the student received a pass-key enabling him to enter the reading-room at any hour and avail himself of the books, which, unlike those at the University Library, might not be taken out, and of the manuscript and lithographed notes of lectures; in particular he could consult the written "Ausarbeitung" of Klein's lecture of the day before, made by Klein's assistant and previously submitted to himself; it happened, not infrequently, that rash statements of the professor were in this way controlled and corrected, these assistants, particularly the too early lost Ernst Ritter, being for the most part mathematical collaborators of a high order.‖ All this, which had been organised by Klein, has now been developed into a Mathematical Institute in a separate building.

18. We have here touched only on a very small portion of Klein's activity in the improvement of the university institutions and arrangements. Incidentally

* Klein's magnificent collection of books and papers is now at the University of Jerusalem.

† Autobiography, p. 26, is here misleading.

‡ 'Ges. Math. Abh.,' vol. 3, Anhang, Dissertations No. 38.

§ Here Professor Courant's 'Gedächtnisrede' might give a false impression.

‖ 'Ges. Math. Abh.,' vol. 3, Anhang, p. 14.

this was facilitated by the part he played in bridging over the cleft which existed between scientific and industrial circles. The brilliant idea of the "Göttinger Vereinigung zur Forderung der Angewandten Mathematik" was Klein's own, and was suggested to his mind by his observations in America. This society, founded in 1898, included some of the foremost members from among the manufacturers. The funds which these public-spirited individuals placed at the disposal of the university served to eke out the generous, but necessarily inadequate, contributions of the Prussian Government. In this way new Institutes for Geodesy, Astronomy and Assurance, technical Mechanics and technical Electricity were created, and corresponding chairs founded and filled.*

19. What the university and the town of Göttingen owe to Klein can never be forgotten; the monument is the Göttingen of to-day.† Perhaps the most important influence which he directly exerted—putting aside his constant advocacy of the rights of Applied Mathematics on an equality with Pure—was in the election of his colleagues. A single appointment to a professorial chair based on reasons, openly or tacitly accepted, other than that of *efficiency*, is not only sufficient to undermine the prestige of a university but also to corrupt its ideals for a considerable period of time. Klein's two principles in recommending the call of a new professor were, first, *individual pre-eminence*, and second, *collective representativeness*. In particular, it was Klein who brought Hilbert, and indirectly also Minkowski, to Göttingen, for Hilbert, on the occasion of his refusing a call to Berlin, made it a condition of his remaining at Göttingen that a place should be found there for Minkowski: on similar occasions when Klein refused such calls, he always used the opportunity to obtain concessions in the interest of the university.‡ This point deserves our attention; it is habitual in German university life, and is one of its noblest traits, without anything corresponding to it in our own universities.

20. It would be difficult to gauge how far Klein's influence in his own country was enhanced by his entry into the Prussian Herrenhaus in 1907, chosen by the University of Göttingen as its representative. This election had been partly prompted by the desire to see him expend some of his unsleeping energy outside the university. However, though he sat on various committees connected with education, he hardly ever spoke.

21. To mathematicians for generations to come, Klein is likely to be thought of as the principal promoter of the Enzyklopädie der Mathematischen Wissenschaften. He had a remarkable memory, and never forgot what he had once imbibed nor where he had come across it; but he had no unusual facility at any

* 'Ges. Math. Abh.,' vol. 2, pp. 507 et seq.

† See Courant's 'Gedächtnisrede.'

‡ Autobiography, p. 32.

time of his life for mastering the contents of printed books,* and, as time went by, he found himself mathematically stranded, and quite incapable of following, even from afar, the actual course mathematics had taken, except where, as in the Theory of Relativity, there was a distinct connexion with the realms in which his own early successes had been achieved; and, even then, it was only to the extent of this connexion that he was able to follow. The Enzyklopädie was, from Klein's point of view, an effort to render accessible to his pupils, to himself, and to the mathematical public at large, the bulk of existing mathematics. One day in the '90's the concept of the Enzyklopädie was formulated by Klein in the presence of the writer: the progress of mathematics, he said, using a favourite metaphor,† was like the erection of a great tower; sometimes the growth in height is evident, sometimes it remains apparently stationary; those are the periods of general revision, when the advance, though invisible from the outside, is still real, consisting in underpinning and strengthening. And he suggested that such was the then period. What we want, he concluded, is a general view of the state of the edifice as it exists at present.

Klein undertook, for his own part, in accordance with his pronounced taste for Applied mathematics, the editorship of the volume on Mechanics; but over and above this, he was indeed the moving spirit of the whole undertaking. He says in his Autobiography (p. 31) :—" I always emphasised above all things the view that, in the Enzyklopädie, Applied Mathematics must be treated on an equal footing with Pure. But in this connection there were very considerable difficulties to be overcome. Our object was not so much to publish articles illustrating the ordinary routine in the application of mathematics to each different domain, but, rather, by searching discussion with the experts, to get at the mathematical kernel of the subject. With this in view, I was entrusted with the task of obtaining the necessary groundwork for the production of the applied volumes (Mechanics, Physics, Geophysics and Astronomy) by personal contact of a very varied character. In particular this demanded on my part journeys to Austria, Italy, France, England, Holland and Denmark."

It was on one of these journeys that he arrived, on his 50th birthday, at Turin, and was fêted by the mathematicians of that city, where the present writer and his wife—Klein's favourite pupil, as the saying went—were then studying. Placed at dinner next to the lady, he whispered, in answer to her felicitations :—" Ah! I envy you. You are in the happy age of productivity. When everyone begins to speak well of you, you are on the downward road."

The conception of the Enzyklopädie was grandiose, and its mere existence had amazing effects. We can understand this when we reflect that, in England,

* In particular, *see* 'Ges. Math. Abh.,' vol. 1, p. 241, where he explains how he gathered his knowledge of v. Staudt's work.

† Cp., *ibid.*, vol. 2, p. 510.

in 1892, the Theory of Functions of a Complex Variable, due to Cauchy—not to speak of Riemann—had barely been brought within the borderland of knowledge. Yet the Enzyklopädie as a whole cannot be regarded as a success. Collaboration between writers of different articles, living in different lands and writing on subjects apparently widely apart, was impracticable, even when the underlying mathematical scheme was similar. The want of unity of design had the effect of making the construction expand to an inordinate degree, and even so, the articles themselves were sometimes felt to be insufficiently detailed, and had to be supplemented by reports in the 'Jahresbericht der Deutschen Mathematikervereinigung.' We cannot help recalling the claim of Roger Bacon, that he would undertake to explain to a person of ordinary intelligence in three or six months, all that he himself knew, and this, as is almost certain exceeded the accumulated learning of his time. That is the feeling which every competent mathematician must *mutatis mutandis* have with respect to our existing mathematical science. All that is essential should be explainable in a small number of volumes of moderate size. But we shall attain this ideal only by a series of approximations, and the Enzyklopädie, if the first, seems a very rough one indeed. It would appear to be characteristic of the Germanic mentality to generate wide-embracing conceptions and, with admirable perseverance, to carry them through to some sort of realisation, while the Romance mind would hesitate to undertake anything unless it was sure of attaining excellence in form.

22. Klein's appetite for work was such that, long before there was any prospect of the Enzyklopädie being completed, finding he was obliged to leave much of the organisation in other hands, he had turned to the international question of mathematical teaching in the schools. He turned perhaps to something more concrete, which brought him into contact with minds other than those of the mathematical expert, something more comprehensible in fact than the huge colossus which was threatening to remain nothing but a torso. In the matter of the International Mathematical Teaching Commission Klein, as President, formed one of a triumvirate, of which another, the pioneer of the movement, was a world-travelled American, David Eugene Smith, and the third, as General Secretary, a native Swiss professor, Henri Fehr. This international work was checked and rendered abortive by the Great War.

23. It is impossible even to touch on all Klein's schemes, yet we cannot but allude to a vast undertaking, still uncompleted, in which he was the prime mover, the editing and issuing of Gauss's Works. For Klein, Gauss represented the universal in Mathematics, and in this connexion he held Gauss before himself as the master to be copied.* But, if Gauss was here the prototype, it was buried in his individual person; the apostle of universality in Mathematics

* There was hardly a new mathematical idea formulated, but Klein would cite a passage from Gauss in which it had been foreshadowed.

was Klein, and, without hyperbole, he may be termed the founder of what the 20th Century means by "Mathematical Science," raised equally above departmental specialism and narrow nationalism. Indeed, during a period when the forces of material progress were, at first secretly, but afterwards openly, undermining the basis of the unity of civilised nations, generating jealousy, and inciting to war, Klein embodied the contrary principle, and, by his great power in his own country, and his wide-spread influence in the world of intellect, constituted one of the greatest forces tending to international amity and peace. In particular, it was he who revived the practice—prevalent in the Middle Ages, when Latin was the universal tongue of Learning—of inviting distinguished men from other countries to come and give lectures.*

24. In looking through Klein's Autobiography, one is struck by his constant references to his friendships. It was characteristic of him that they all bore directly, or indirectly, on his mathematical development and on his power of organisation. He was indeed a man without a hobby; in particular, and this is curious and interesting from a psychological point of view, although a German, and although endowed with an excessive acuteness of hearing,† he could not distinguish one tune from another.

Social intercourse for Klein meant the interchange of ideas, and for that he was as eager as an ancient Athenian. It was in such give-and-take that his own conceptions took form. There is, perhaps, no contradiction in saying that Klein was never the originator of his own ideas. He had not the generating force of a Cauchy or a Georg Cantor, but he had a phenomenal power of grasping the import of a suggestion, and working it out on a grander scale than any before him had imagined.

We have already mentioned incidentally several of those close friendships that directly influenced and even inspired Klein as a mathematician. The most striking of all, his friendship with Lie, merits a place by itself. It takes us back to his stay at Berlin in 1869. Klein writes‡:—"The most important event of my time in Berlin was certainly that, towards the end of October, at a meeting of the Berlin Mathematical Society, I made the acquaintance of the Norwegian, Sophus Lie. We had, in our work, been led from different points of view finally to the same questions, or, at least, to kindred ones. Thus it came about that we met every day and kept up an animated exchange of ideas. Our intimacy was all the closer, because at first we found very little

* In particular Poincaré came to Göttingen in 1894 or 5, and gave a lecture in French to the mathematical professors and students.

† This enabled him to catch the faintest whisper during his lectures. Also, commonly, when away from home, he could not go to sleep unless he stopped up his ears with cotton-wool; this was the case, for instance, during his short stay at Trinity College in 1897, on the occasion of his receiving his honorary degree at Cambridge.

‡ 'Ges. Math. Abh.,' vol. 1, p. 50.

interest for our geometrical interests in our immediate neighbourhood." At this time it was chiefly with Line Geometry that the two were occupied.

In the summer of 1870 Klein and Lie went together to Paris, where they had rooms side by side, and lived in the closest bonds of friendship. Their intention was to go on to England in the winter, but the outbreak of the Franco-German War prevented it. In this connection Klein writes :—"This striving after the greatest possible breadth of scientific outlook, for which an acquaintance with what was being done in foreign countries seemed important, was very little understood at that time in Germany. Thus, for instance, when, urged by my father, I tried to obtain letters of recommendation from the Minister of Education in Berlin, I received the official answer :—We have no use for French or English mathematics."*

In Paris the two friends did not attend lectures, but worked together. It was, however, undoubtedly by the personal contact with the younger French mathematicians, particularly Camille Jordan, that the attention of the pair was directed to Galois's theory of groups. Jordan's treatise on the 'Theory of Substitutions' had just appeared, and was to the two young men "a book with seven seals."† We continue in Klein's own words :—"It was, however, with Darboux that we were most intimate. At that time the French had been occupied with new investigations in metrical geometry (inversion, constant use of the circle at infinity), and this was quite unknown in Germany. Now it appeared that these were most closely related to our own work in line geometry. . . . For Lie, what was perhaps most interesting was the researches of the French into the geometrical theory of differential equations. It was by the coalescence of the two circles of ideas that Lie came to discover the connection between lines of curvature and minimal lines on a surface."†

And here Klein gives a vivid sketch from this period of his life :—"I had risen early one morning at the beginning of July, 1870, and I was just going out, when Lie, who was still in bed, called me into his room, and began explaining to me what he had found out during the night, the connexion between lines of curvature and minimal lines. I did not understand one word. Anyway, he assured me that it followed that the minimal lines on Kummer's surface must be algebraic curves of the 16th order. In the course of the morning it flashed upon me, while I was seeing over the Conservatoire des Arts et Métiers, that these must be the same curves of the 16th order which had already appeared in my theory of the Line-complexes of the first and second degree, and I succeeded at once in carrying out the geometrical proof, independently of Lie's transformation. When I got back in the afternoon, at 4 o'clock, Lie had gone out, and I left a letter for him containing my proof."‡

* Autobiography, p. 16.
† 'Ges. Math. Abh.,' vol. 1, p. 51.
‡ 'Ges. Math. Abh.,' vol. 1, p. 97.

25. The outbreak of the war, Klein's consequent hasty departure, leaving his Norwegian friend behind, then the continuation of their mathematical conversations by letter, have an interest beyond that of the immediate consequences to Lie. While Lie remained at Paris, all went well; but when, in the heat of August, he started for Italy on foot, he was arrested at Fontainebleau as a spy, and kept in durance for four weeks, till, in fact, their friend Darboux had succeeded in explaining the inoffensive mathematical nature of the mysterious letters, in German, and in Klein's notoriously illegible handwriting, found in Lie's possession.

26. This episode was one of the manifestations of the unbroken continuance during the Franco-German war of friendly scientific relations between himself and his newly-won French friends, which made such an indelible impression on Klein's mind. We may say that those were still the days of scientific chivalry. When the Great War broke out in 1914, Klein, like most of us, was incapable of conceiving any other state of affairs, but he was, as he had in fact been in 1870, carried away by the wave of patriotic fervour of the moment. When he therefore began to perceive that there was serious danger of what he regarded as a purely political cataclysm affecting international scientific relations, he lost, for one unfortunate moment, that dignified poise of judgment which was one of his most striking characteristics. Invited by telephone to sign a document, which, according to the impression left on his mind* was to make an appeal to the scientists of Europe to maintain an objective attitude during the struggle, he impetuously acceded to the request. He did not see the text of the document until it appeared in the daily papers with his signature, as one of the too well known 93, printed below. It has been very damaging to Klein's international position that he permitted his name to appear without protest at the foot of a document whose tenour was so entirely antagonistic to the whole tone of his life. And he was too proud, even after the Armistice, to withdraw publicly his signature. He desired his friends to know, and to tell, how it had been obtained, but he felt that, for himself, he stood upon his whole career, and not upon an isolated and misinterpreted action.†

* All this is founded on an autograph letter to my wife dated December 7, 1918. See her Obituary notice in the *Times* of July 9, 1925. Ultimately, Klein allowed an extract from this letter to be printed in a Berlin daily paper, together with explanations from most of the 93 of more or less the same tenour.

† In a letter of July 15, 1919, to the same, the conclusion is as follows:—"Allgemein verbietet mir den Sinn für Objektivität mich über Dinge zu aeussern, die ich, wenn überhaupt, nur aus den subjektiven und einander wiedersprechenden Aeusserungen irgend welcher Zeitungen kenne. Also bleibt für mich, was ich die ganze Kriegszeit über getan habe: Schweigen und Arbeiten. Für die wenigen Jahre, die ich noch vor mir habe, werde ich damit auskommen.

"Die Welt aber wird ihren Lauf nehmen und die Völker werden sich eines Tages wieder zusammen finden. Vorläufig ist eingetreten, was einst beim Turmbau von Babel der Fall war, sie verstehen einander nicht mehr."

27. Lie was seven years older than Klein, but their standing in mathematical productivity was about the same. Lie had only begun mathematics seriously at 26,* whereas Klein, at the age of 20, when they first met, had been for three years recognised as a mathematician and a writer. Lie's first publication in which he represents the ∞^4 imaginary lines of the plane by the ∞^4 real lines of space, had been published in 1869, and Klein's Doctor's Dissertation "on the Transformation into a Canonical Form of the General Equation of the Second Degree between Line-co-ordinates" was presented to the University of Bonn in December, 1868. Both works were stimulated by the ideas of Plücker; indeed Engel, who was intimate with Lie, attributes to him a phrase in which Lie calls himself a pupil of Plücker, which he was not. But the printed phrase to which Engel gives the reference,† bears the stamp of being written carelessly, and is in a language other than Lie's native tongue; it hardly bears the interpretation Engel puts on it. When Lie writes "my extensions are quite obvious for a pupil of Plücker's," one cannot help thinking that the pupil of Plücker's referred to in general terms, if specialised at all, was not Lie, but Klein. Indeed it is difficult to believe that Lie, the Norwegian, would have gone as far as Engel, typically German, in attributing even the inspiration of his own ideas to a German. Lie was excessively tenacious of his own rights to his own ideas, and resented, as is well-known, any suggestion that they were in any sense second-hand. He quarrelled with Klein, his early friend, on this very point, and in the introduction to the third volume of the 'Transformationsgruppen'‡ he repudiates the notion that he was a pupil of Klein's; "rather," he says, "the contrary was the case." This irritated Klein, and justly. His generous nature was incapable of making the former of these suggestions, and his frank and healthy self-confidence prevented his accepting the latter of them. It is so rare in this world of ours that misunderstandings are ultimately cleared up, that it is a real pleasure to know that this was the case with Klein and Lie. In the vivid picture which Klein gives in his Autobiography and in the Collected Mathematical Papers of the interweaving of his own work with that of Lie, there is no bitterness. I am so fortunate as to be able to add to what may be there read a little sketch by the hand of Mrs. Klein,§ sent specially for the purpose of the present writing.

"The relation to Lie was an intimate friendship in mathematical and in personal respects. That remained the same also later, when Lie visited us in Leipzig. And I, too, was fond of him, the powerful Northman with the frank

* "Sophus Lie," by F. Engel, 'Jahresber. d. deutschen Math. Vereinigung,' vol. 8, p.30 (1900).

† 'Math. Ann.,' vol. 9, p. 246; 'Engel,' p. 34.

‡ *Loc. cit. supra.*

§ Dated August 15, 1925. She died last year.

open glance and the merry laugh, a hero in whose presence the common and the mean could not venture to show themselves.

"Then he became my husband's successor at Leipzig, and there he was seized with home-sickness. How well I understand that! He, the free, accustomed to his rough but beautiful northern home, how could he stand the great smoky city, the high houses, the narrow streets, the puny Saxon people. He suffered from melancholia, he was taken to a sanatorium, but there things only became worse, for they robbed him of his work and of his freedom. There, in his embittered state of mind, it must have been that he wrote the malicious things about my husband which were both painful and incomprehensible to him.

"But soon he understood that his best friend was ill and that he could not be held responsible for his actions. It was not only magnanimous and good, but also wise of my husband, not to enter into any polemic, but to let the thing simply rest. And he was not mistaken in his friend.

"One summer evening, as we came home from an excursion, there, in front of our door, sat a pale sick man. 'Lie!' we cried, in joyful surprise. The two friends shook hands, looked into one another's eyes, all that had passed since their last meeting was forgotten. Lie stayed with us one day, the dear old friend, and yet changed. I cannot think of him and of his tragic fate without emotion. Soon after, he died, but not before the great mathematician had been received in Norway like a king."

We will only add that, physically, Klein was very tall, erect and slim, with rich brown wavy hair and characteristically sparkling light blue eyes, with a genial glance which has not been completely caught in the photograph here produced.

W. H. Y.