recovered the use of the limb in so far as to wield his pen with his wonted energy, but with no little pain. He married in middle age, and left a family. He was elected a Fellow of the Linnean Society in 1865, and of the Royal in 1886.

J. D. H.

SOPHUS LIE. 1842–1899.

Marius Sophus Lie, the son of a Norwegian pastor, was born on the 17th of December, 1842, at Nordfjordeide.* Educated in a private gymnasium in Christiania, he entered the University of that city in 1859. He does not appear to have displayed any special predilection for mathematical pursuits in his student days; and even after passing, in 1865, his qualifying examination as a teacher, he remained in doubt as to the particular direction of his life. He gave private instruction in mathematics, and, after spending some time on astronomy, he turned to the consideration of the foundations of geometry—a subject to which he devoted more special attention in later years.

It was only in 1868, at an age much later than the average at which great mathematicians are wont to settle down to their life-work, that Lie found his true bent. In that year the works of Plücker on modern geometry first gave him the impulse towards research, and inspired him with original ideas, the gradual development of which gave him the first indication of his possession of mathematical powers. Thenceforward his life was one stretch of industry and activity, and the only interruptions to his creative work were the illnesses which overshadowed his later years, and were, without doubt, largely due to the exceeding strenuousness of his devotion to his investigations.

* The writer of this notice wishes to acknowledge his indebtedness to two articles on Lie by Professor Dr. F. Engel: one in the 'Jahersb. d. deutschen Mathematiker-Vereinigung,' vol. 8 (1900), pp. 30—46; the other in 'Bibliotheca Mathematica,' 3rd ser., vol. 1 (1900), pp. 166—204. Reference should also be made to an appreciation by Noether, 'Math. Ann.,' vol. 53 (1900), pp. 1—41.

The publication* of his first paper, "Repräsentation der Imaginären der Plangeometrie," in 1869, led to a travelling scholarship which enabled him to visit foreign universities. The succeeding winter was spent at Berlin, where Lie and Klein met and contracted their long friendship, which was based, in the first instance, upon their common interest in the range of ideas associated with Plücker's geometry. Thence in the early summer of 1870 they went to Paris, coming into personal relations with Darboux and Jordan: and their growing interest in the theory of groups, first awakened in Lie by Sylow's lectures, which he had attended as a student, was stimulated for both of the two friends, as they recognised different modes in which that theory could be newly applied to branches of mathematics. In particular, Lie then discovered his now famous transformation, which makes a sphere correspond to a straight line, and by which, therefore, theorems on aggregates of lines can be translated into theorems on aggregates of spheres. The paper† containing this result was communicated to the Academy of Sciences by Chasles.

The visit to Paris was brought to an abrupt end by the outbreak of the war between France and Prussia. Klein was of course, as a German, bound to leave Paris; but the same obligation did not rest upon a Norwegian, and Lie remained until August, when he conceived a plan of walking through France to Italy. He had gone only as far as Fontainebleau, when a misadventure suspended his journey for a time. Let it be told in the words of Darboux‡:—

> "Occupé sans cesse des idées qui fermentaient dans sa tête, il allait chaque jour dans la forêt, s'arrêtant dans les sites les plus éloignés des sentiers battus, prenant des notes, dessinant des figures au crayon. Il n'en fallait pas tant à cette époque pour éveiller les soupçons. Arrêté et incarceré à Fontainebleau, dans des conditions d'ailleurs fort douces, il se réclamait de M. Chasles, de M. Bertrand, d'autres encore; je fis le voyage de Fontainebleau et n'eus aucune peine à convaincre le procureur impérial; toutes les notes que l'on avait saisies et où figuraient des complexes, des systèmes orthogonaux, des noms de géomètres, ne se rapportaient en aucune façon à la défense nationale. . . . "

After a delay of four weeks, he continued his journey to Italy; then, travelling homewards through Switzerland and Germany, he returned to Christiania in December, after a joint note by Klein and himself had been communicated in that month to the Berlin Academy by Kummer.

* Reprinted, under a different title, in 'Crelle's Journal,' vol. 70 (1869), pp. 346—353.

† 'Comptes Rendus,' vol. 71 (1870), pp. 579—583.

‡ See the 'Notice sur M. Sophus Lie,' spoken on the occasion when Lie's death was announced to the Academy of Sciences ('Comptes Rendus,' vol. 128 (Feb. 27, 1899), pp. 525—529).

From that date onwards, the history of the man is mainly the history of his ideas; the external incidents of his life are comparatively few.

At the beginning of 1871 he was assigned a junior post in his own University, and in the summer of that year he graduated as Doctor. The thesis then submitted was subsequently amplified and became his famous memoir,* "Ueber Complexe, insbesondere Linien- und Kugel-Complexe, mit Anwendung auf die Theorie partieller Differential-Gleichungen." In this memoir he constructs the theory of tangential transformations† for space; he applies it to partial differential equations of the first order; he develops the transformation of Plücker's line-geometry into a sphere-geometry, which now is regularly associated with his name; and he shows how the results can be applied to ordinary differential geometry, obtaining (among other properties) the result that his transformation of line-geometry into sphere-geometry makes the asymptotic curves of one surface correspond to the lines of curvature of the transformed surface. These are but a few of the results in a paper which is full of powerful methods and novel ideas; they are sufficient to show that the man who, before 1868, was hesitating about his vocation in life, had found an effective vocation by 1871.

In the succeeding year, the Norwegian Storthing was induced to create a special professorship for him in the University of Christiania. His appointment as Professor Extraordinarius in 1872 enabled him for the future to devote himself to his researches, free from the distracting necessity of supplementing the over-modest salary of his earlier post by private teaching.

About this time Lie seems to have made his first discovery as to the relations that can subsist between ordinary differential equations and infinitesimal transformations; the scope of such a relation can be indicated by the simple example of an equation of the first order. A function $\Omega(x, y)$ is said to admit a finite continuous group of transformations represented by

$$x_1 = \phi(x, y, a), \qquad y_1 = \psi(x, y, a),$$

* 'Math. Ann.,' vol. 5 (1872), pp. 145—256.

† The transformation of surfaces adopted makes (not merely a point correspond to a point, but) an element of any surface at a point correspond to an element of the transformed surface at the corresponding point. The property holds over the whole of the two surfaces, and, for instance, in the case of ordinary space leads to the analytical relation

$$dz' - p'dx' - q'dy' = \rho(dz - pdx - qdy),$$

where x, y, z, p, q, define an element of the one surface, x', y', z', p', q', define the corresponding element of the transformed surface, and ρ is a non-vanishing quantity that does not involve differential elements. Such a relation is the basis of the analytical theory of tangential (or contact) transformations.

where a is an arbitrary parameter, when

$$\Omega(x_1, y_1) = \Omega(x, y).$$

Such a group possesses an infinitesimal transformation, which may be represented by

$$x_1 - x = \xi(x, y)\delta t, \qquad y_1 - y = \eta(x, y)\delta t,$$

where δt is arbitrary, and the infinitesimal transformation determines the group. Moreover, the necessary and sufficient condition that the function $\Omega(x, y)$ should admit the above group is that the function should admit the infinitesimal transformation of the group, and the analytical expression of the condition is

$$\mathrm{U}(\Omega) = \xi(x, y)\frac{\partial\Omega}{\partial x} + \eta(x, y)\frac{\partial\Omega}{\partial y} = 0,$$

If the function Ω involves y', where y' denotes dy/dx, say, it is

$$\Omega(x, y, y'),$$

the analytical expression of the condition, that it admits the same group is

$$\mathrm{U}'(\Omega) = \xi\frac{\partial\Omega}{\partial x} + \eta\frac{\partial\Omega}{\partial y} + \left(\frac{d\eta}{dx} - y'\frac{d\xi}{dx}\right)\frac{\partial\Omega}{\partial y'} = 0.$$

Now Lie discovered that, if an equation

$$f = \mathrm{X}(x, y)\, y' - \mathrm{Y}(x, y) = 0$$

admits the infinitesimal transformation just indicated, so that $\mathrm{U}'(f) = 0$ then

$$\frac{\mathrm{X}dy - \mathrm{Y}dx}{\mathrm{X}\eta - \mathrm{Y}\xi}$$

is an exact differential save only in the trivial case $\mathrm{X}\eta - \mathrm{Y}\xi = 0$; so that the transformation determines a factor of integrability, and thus, merely after a quadrature, leads to the integral of the equation. Further, the significance of the result is not thereby exhausted, for it permits the construction of the differential equations of the first order that admit any given finite continuous group of transformations, for instance, a projective group. All that is necessary for this purpose is to construct the infinitesimal transformation which determines the group, and to obtain a couple of independent integrals, say u and v, of the system

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dy'}{\dfrac{d\eta}{dx} - y'\dfrac{d\xi}{dx}};$$

the required equation is

$$u = F(v),$$

where F is any functional form. Manifestly, such an idea is capable of wide application: under Lie's direction, it proved fruitful in succeeding years.

Similarly, the integration of partial differential equations of the first order was discovered by Lie to be bound up with infinitesimal tangential transformations under which they are invariantive. This discovery led him to resume the whole problem of the integration of such equations; and, as the outcome of his investigations, specially built upon the completed analytical theory of tangential transformations, he made two notable advances. One of these consisted in a great simplification of the known method of Jacobi, by affecting a material reduction in the number of quadrature processes; the other led him to a new method for the solution of Pfaff's problem, which, besides being simpler and shorter than preceding methods, indicated the real functional significance of the necessary analysis.

These results, obtained by connecting infinitesimal transformations with widely verging questions in differential equations, prepared the way for the consideration of a problem certain to possess an extensive range, viz., the theory of finite continuous groups of transformations, in general, and without special regard to any particular application. Lie began this work in 1873, and, for the next three years, concentrated upon it all the intensity of his creative enthusiasm: he once spoke of himself as having, during that period, lived only among his groups of transformations. The result was to constitute this theory an independent subject: begun, as already indicated, from its association with differential equations, and finding in its progress some of its most direct applications in that region; but, as the theory grew, it obtained a wider significance, and the geometrical bent of much of Lie's thought gave it applications within the region of geometry.

Towards the close of 1877 Lie had completed one stage of these investigations. His conclusions were embodied in a number of memoirs; many of them were published in a new journal in Christiania, edited by Sars, Müller, and himself, some of them in the 'Mathematische Annalen,' most of the latter being revised and extended accounts of earlier papers. Apparently, Lie suffered from severe disappointment at the lack of interest so far shown in his work by mathematicians: his story at this time reads like the occasional experience of the investigator who lives, remote from fellow-workers and unstimulated by eager pupils, voyaging through his sea of thought alone, at the end finding himself weary, isolated, unacknowledged, perhaps therefore discouraged, and certainly left uncheered by any confident satisfaction that others are following him.

At any rate, whatever the explanation may be, Lie sought relief in change of subject, and devoted himself, almost entirely for the next few years and partially for the rest of his life, to differential geometry. In a long succesion of valuable papers, he made masterly additions to our knowledge of minimal surfaces, particularly those which are algebraic; he dealt with surfaces which have their Gaussian measure of curvature equal to a constant, or are determined by other assigned relations between their principal radii of curvature; and he discussed surfaces as generated by the translational motion of a curve. The theory of his groups was frequently applied in these researches, and with considerable effect; thus his papers on the classification of surfaces according to the groups of transformations of their geodesics are of high importance. Darboux, in the 'Notice sur M. Sophus Lie,' already quoted, indicated his sense of the value of these contributions to differential geometry: no less significant is the testimony in Darboux's great treatise, 'Théorie générale des surfaces,' furnished by the number of references to Lie's name in its index.

Yet during this specially geometrical period, he did not altogether neglect the development of his theory of continuous groups: occasional papers were written from time to time, showing that it still occupied part of his constructive thought. Towards the end of the period, about 1882, his papers gave signs of his having again reverted to differential equations by applying his groups to the classification and integration of ordinary differential equations of any order. Moreover, the publication of Halphen's thesis on differential invariants led Lie to point out that his own earlier work included Halphen's investigations. His attention was thus again turned to the subject, and one consequence was that he gave the general theory of differential invariants, not merely for the projective group, as in Halphen's work, and in the subsequent detailed work of a number of English mathematicians, but for any finite continuous group of transformations.*

Lie's investigations had now extended over a considerable number of years. They had covered a wide range in a variety of subjects, and the results had been published in no consecutive form and in partly inaccessible places. He had from time to time thought of undertaking some treatises dealing with the main topics which had occupied his thoughts for more than fourteen years. But it was not until September, 1884, that any such project took a practical shape. In that month Friedrich Engel came to Christiania, partly in order to make himself acquainted with Lie's work, partly (on the advice of Klein and A. Mayer) to assist Lie, if that were possible, in making a systematic exposition of the whole theory of transformation-groups. It was exceedingly fortunate for Lie that he thus found some active co-operation and steady assistance in the execution of a severe, even

* 'Math. Ann.,' vol. 24 (1884), pp. 537—578.

exhausting, piece of work. The labour lasted for nine years. During that time, Engel's co-operation and assistance were given, without stint and in a loyalty beyond praise, and fully merited the acknowledgment which Lie made in his preface. The result was the 'Theorie der Transformationsgruppen,' a treatise in three volumes, covering over two thousand pages, the contribution to science by which his name will probably best be known. It is a work of great originality, containing many methods and a wide range of development; it exhibits in masterly manner the suggestive application of new methods to fundamental subjects; and it may be described briefly as a systematic exposition of Lie's investigations on groups of transformations that are continuous and finite. Among the subjects to which application is made, may be mentioned the theory of ordinary differential equations; the theory of partial differential equations, both single and in systems; differential invariants and their types; the solution of Pfaff's problem; tangential transformations, especially in spaces of two and three dimensions, and more generally in n dimensions; groups of functions transformable into one another, and a substantial simplification (by the use of their properties) in the integration of systems of partial differential equations; a complete determination of types of the groups of transformation in one, two, and three variables, and a partial determination of those in n variables. It concludes with a profound study of the foundations of geometry from the point of view of Riemann and Helmholtz; and after a critical discussion of the significance of the hypotheses which they made, he propounds a solution of his own, based upon more elementary hypotheses.

While this work was in progress, Lie changed the scene of his life by accepting, in 1886, the Chair of Mathematics at Leipzig, which had been vacated by Klein on his appointment at Göttingen; Engel accompanying him, and soon being nominated a colleague. Such a professorship possessed some obvious advantages for Lie as compared with the somewhat isolated chair at Christiania. It secured him a wider recognition; it gave him an audience; it offered him the chance of able pupils, who would work sympathetically in development of his mathematical theories. Though these advantages did not come early enough to encourage him, still they did come gradually, and some of them in full measure. His work began to be known better and to be appreciated; his methods began to influence mathematicians. Pupils came to him from far and near, and one in particular, George Scheffers, rendered to him offices similar in kind to those rendered by Engel. When once the merit of his work began to be recognised, scientific honours were bestowed upon him freely. He received the honorary or foreign membership of societies and academies in great numbers; in particular, he was enrolled among our Foreign Members in the year 1895.

Unhappily, recognition appears to have been, not merely slow in coming, but almost too late when it came. There is no doubt that his ceaseless activity in thought and work had undermined his strength, and his spirit had brooded in its loneliness. He suffered from sleeplessness, and developed nervous symptoms: the result was a complete breakdown in 1889. The direct interruption of his work lasted for a large part of a year; happily he was afterwards able to resume it, and for a time was as fertile in production as he ever had been. But the effect upon the man never completely passed off; it seems to have exercised, upon his attitude towards life and in his personal relations with his friends, a morbid influence which lasted for the remainder of his days. The brighter side of these years is to be seen in the record of his continued work. How great that record is, may be gathered from the tale of his published work.* It includes over 150 memoirs, many of them of considerable length, and six volumes. Reference has already been made to his three-volume treatise on groups of transformations. In a couple of instances, his lectures in amplification and elucidation of portions of his theory were edited and published in volume form by his pupil, Scheffers, whose help is gratefully acknowledged: one of these relates to differential equations that admit of known infinitesimal transformations; the other to continuous groups.

Two other works were promised by him. One of these, to be written in co-operation with Engel, was to deal with the theory of infinite continuous groups and the application of the general group-theory to the integration of differential equations: this work has not appeared. The other, to be written in co-operation with Scheffers, was to be devoted to a systematic exposition of his geometrical investigations; the first volume has appeared under the title 'Geometrie der Berührungstransformationen.'

As his fame grew, placing him in the forefront of the mathematicians of his day, a strong desire was felt by his fellow-countrymen that he should return to Norway, and that some professorship of exceptional dignity should be created expressly for him in Christiania. Such a post was made for him about 1896; but he only returned to his native country to occupy it in September, 1898. The desire of his fellow-countrymen was thus gratified honourably for Lie, but unhappily too late to be effective. His broken health forbade any long tenure of a chair in which, as had been hoped, he would be able to continue his mathematical researches. He was almost a dying man on the day of his return; he lingered through part of the winter; such little strength as was left was undermined by pernicious anæmia; and he passed away on the 18th of February, 1899.

* A full bibliography is given by Engel in the article already quoted in 'Bibliotheca Mathematica,' 3rd ser., vol. 1; see pp. 174—204.

Whatever be any prophetic estimate now made as to the position which the future will assign to Lie among the great mathematicians, his contemporaries and immediate survivors would agree in regarding him as one of the most conspicuous, independent, and original workers in his generation.

A. R. F.

Sir WILLIAM ROBERTS. 1830–1899.

William Roberts was born at Bodedern, Anglesea, on March 18, 1830. He was the youngest son of Mr. David Roberts, of Mynydd-y-gof, and of Sarah, daughter of Mr. John Foulkes, of Machynlleth, Montgomeryshire. Mr. David Roberts farmed his own land in Anglesea, and in addition practised as a surgeon in the neighbourhood, where indeed he was the only medical man. Both of Sir William Roberts's parents lived to a great age, and some of his elder brothers settled in Manchester, where they achieved distinction, and one of them, Alderman J. Foulkes Roberts, was Lord Mayor of Manchester in 1897. William Roberts received his early education in Manchester, subsequently he went to Mill Hill School, and he entered University College, London, as a medical student in 1849. Walsh, Garrod, Jenner, Quain, were amongst his teachers at this period, and Roberts came early under the influence of Sharpey, and the interest which he maintained throughout life in physiological problems was probably aroused by the special influence of this teacher. Roberts had a distinguished career as a student at University College, and he graduated as a B.A. in the University of London in 1851, and took the degree of M.D. in 1854. After completing his studies in London, he studied for some months at Paris and also at Bonn and Berlin. In 1854 he was appointed house surgeon to the Manchester Royal Infirmary, and soon afterwards, when 25 years of age, he was elected, in July, 1855, without opposition, physician to the Royal Infirmary and lecturer on anatomy and physiology in the school of medicine. Subsequently he became lecturer on pathology, and, in 1863, lecturer on the principles and practice of medicine at Owens College. When the Victoria University was established he became Professor of Medicine. In 1883 Dr. W. Roberts resigned his physiciancy at the Royal Infirmary, after serving on the active staff for nearly thirty years. During the whole of this time he was an energetic teacher of clinical medicine, and