JULES HENRI POINCARÉ, 1854–1912.

Jules Henri Poincaré* was born at Nancy, April 29, 1854. His father was a medical man who is spoken of as enjoying, in an especial degree, the respect of his fellow-townsmen. His uncle was the father of the President of the French Republic. The boy was at the Lycée at Nancy from October, 1862, until August, 1873, leaving with the Prix d'Honneur au Concours Général en Mathématiques Spéciales. At the age of five he had suffered from a severe illness, and is described as a delicate boy, preferring the society of his sister to the games of his schoolmates. But any tendency to devote himself too exclusively to a contemplative view of life must, one feels, have received a rude shock from the experience which came to him at the age of sixteen. Nancy is about thirty miles south of Metz; his father was called upon in 1870, as a medical man, to help with the wounded, and the young Poincaré attended him as secretary. So anxious was he to read the only newspapers that were obtainable that he learned to read German for the purpose, so it is said. In later life he was one of the closest ties between the mathematical world of Germany and that of France.

In 1873 he was first among candidates for the École Polytechnique at Paris. Leaving this in October, 1875, for the School of Mines, he was thence transferred as Engineer to Vesoul, about 80 miles south of Nancy, from April to December, 1879. During this year, in August, 1879, he became Doctor of Mathematical Science in the University of Paris. In December of the same year he was in charge of the Cours d'Analyse à la Faculté des Sciences de Caen. In March, 1881, at the age of nearly twenty-seven, he was honourably mentioned as a competitor for the Grand Prix des Sciences Mathématiques awarded by the Academy of Sciences of Paris. In October, 1881, he became Maître de Conférences d'Analyse à la Faculté des Sciences de l'Université de Paris. In 1886 he was Professeur de Physique Mathématique et de Calcul des Probabilités at the University of Paris, and in 1896 Professeur d'Astronomie Mathématique et de Mécanique Céleste. He was chosen Member of the Academy of Sciences in the Section of Geometry in January, 1887, served as President in 1906, and was elected to the French Academy in 1908. He became a Foreign Member of the Royal Society in 1894, and on the inauguration of the Sylvester Medal for Mathematics in 1901, he received the first award. He died at his house in Paris in July, 1912, during the celebration by the Royal Society of its fifth jubilee. At the funeral ceremony the Society was represented by the Senior Secretary and the Astronomer Royal.

Such, in briefest outline, are the facts of his public career. To give any

* A fairly complete bibliography, with a portrait and various appreciations, is edited by Ernest Lebon (Gauthier Villars, July 1, 1909). To this the writer is much indebted for dates and references.

complete account of his work is a task well nigh impossible on account of its vast range. His writings deal with nearly every branch of analysis, with every part of theoretical astronomy, and with most of the issues of modern mathematical physics. To whatever he deals with he brings a breadth of outlook, a wide generality of conception, which stirs the imagination, though it may puzzle the mind. Of the final value of his applications of mathematics to physics, time will pronounce; of the importance of the influence which his wide knowledge enabled him to exert, especially in his own country, there can be no question.

His contributions to pure analysis may be classed under differential equations, automorphic functions, general theory of functions, Abelian functions, Analysis Situs, arithmetic. The work on differential equations includes studies in extension of the general existence theorems given by Cauchy, and consideration, on the lines of Riemann and Fuchs, of the theory of linear differential equations. Also a systematic consideration of the utility, as solutions of differential equations, of series which are divergent, and yet asymptotic; this last, forced upon Poincaré's attention, presumably, by his astronomical studies, has had a wide development. It is, however, the general consideration of an infinite discontinuous group, in connection with which he refers explicitly to Fuchs, and the associated automorphic functions, which are the best known results of his study of differential equations. Historically an automorphic function arises among the formulæ for elliptic functions which are found in Jacobi's 'Fundamenta Nova.' Jacobi obtains a series whereby the square of the modulus, k^2, can be expressed as a single-valued function of the ratio K'/K of the two so-called quarter periods. These are solutions of a linear differential equation of the second order whose independent variable is k^2. Putting $\zeta = K'/K$, and $k^2 = \phi(\zeta)$, it is then natural to consider values of ζ of the form $\zeta' = (a\zeta + b)/(c\zeta + d)$, in which a, b, c, d are constants. More simply and precisely, the facts are thus: Let 2ω, $2\omega'$ be two arbitrary quantities whose ratio $\tau = \omega'/\omega$ is not real, but has its imaginary part positive. Let $p(u)$ be Weierstrass's doubly periodic function with these quantities as fundamental periods. Then the function which is the ratio of $2p(\omega') + p(\omega)$ to $p(\omega') - p(\omega)$ is evidently a single-valued function of τ. It is in fact unaltered by substituting in it, in place of τ, the quantity $\tau + 2$; or by substituting $\tau/(2\tau + 1)$; or more generally by replacing τ by $(p\tau + q)/(r\tau + s)$ in which p, q, r, s are any integers of which q and r are even, such that $ps - qr = 1$, so that p and s are odd. There is an infinite number of substitutions of this form; any two of them performed in succession give rise to a substitution of the same form, so that the aggregate of them constitutes a *group* of substitutions. Starting from an arbitrary value of τ, whose imaginary part is positive, all the values of τ arising by these substitutions have their imaginary part positive; representing such values of τ on the upper half of a plane, in the usual way, a fundamental region of this half-plane can be named which is analogous to the fundamental

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parallelogram used in the discussion of doubly periodic functions. Namely, every point of the upper half-plane which does not lie in this fundamental region can be obtained from one, and only one, point of this region by one, and only one, substitution of the group above described. If $\tau = \rho + i\sigma$, we may take for such a fundamental region the part lying between the lines $\rho = \pm 1$ and above the semicircles $(\rho \pm \frac{1}{2})^2 + \sigma^2 = \frac{1}{4}$, $\sigma > 0$. Poincaré's aim was to study the general properties of such infinite groups of (fractional) linear substitutions, and to obtain single-valued functions of the independent variable unaltered when this variable undergoes any substitution of the group. In both respects he obtained a brilliant success. It is not easy in a few sentences to give an account of his general theory of the groups; but it may be possible to make clear the way in which he constructs functions unaltered by a given group. Let

$$\zeta' = \vartheta_r(\zeta) = (a_r\zeta + b_r)/(c_r\zeta + d_r)$$

be any one of the substitutions of this group, the constants being chosen so that $a_r d_r - b_r c_r = 1$. Denote the denominator $c_r\zeta + d_r$ by $\Delta_r(\zeta)$. Let $\mathrm{H}(\zeta)$ be a rational function of ζ, and μ_r a constant such that $\mu_{rs} = \mu_{sr} = \mu_r\mu_s$; let m be an integer. Consider the sum

$$\Theta(\zeta) = \sum_r \mu_r^{-1}\mathrm{H}[\vartheta_r(\zeta)] \cdot [\Delta_r(\zeta)]^{-m},$$

which is to contain a term corresponding to every substitution of the group. It is then easy to prove that if for ζ we substitute $\vartheta_s(\zeta)$, any one of the transformations of ζ arising in the group, we obtain

$$\Theta[\vartheta_s(\zeta)] = \mu_s[\Delta_s(\zeta)]^m \cdot \Theta(\zeta).$$

The proof requires the assumption that the original series converges irrespective of the order in which the terms are taken; when we consider the generality of the ideas involved, Poincaré's proof that this may be so for proper values of m is one of the most striking portions of the work. Taking now another such sum as $\Theta(\zeta)$, having, however, in place of $\mathrm{H}(\zeta)$ a rational function $\mathrm{K}(\zeta)$, we shall have a similar equation. Thus the quotient of the two functions $\Theta(\zeta)$ is unaltered when the independent variable ζ is changed by any substitution of the group.

But now arises another consideration. Reverting to the function before considered,

$$\lambda(\tau) = [2p(\omega') + p(\omega)]/[p(\omega') - p(\omega)],$$

for purposes of illustration, we may regard τ, or ω'/ω, as the quotient of two independent solutions of a linear differential equation of the second order whose independent variable is λ. This is, in fact, a hypergeometric equation with singular points only at $\lambda = 0, \lambda = 1, \lambda = \infty$. If we have any function of the unrestricted complex variable λ, of which every existing branch is expressible, in the neighbourhood of any value λ_0 other than $\lambda_0 = 0$, or 1, or ∞, as a power series in $\lambda - \lambda_0$, then it can be proved that, by regarding λ as that function of the new independent variable τ which is given by the function above, the function under consideration becomes a single-valued function of τ. A particular case of this result is that the dependent variable of any hypergeometric differential equation is a single-valued function of τ, if the independent variable be identified with $\lambda(\tau)$. These are evidently results of wide scope. Quite early (in 1883) Poincaré formulated a demonstration that every analytical function, $w = \phi(z)$, of an independent variable z, is such that both w and z may be regarded as single-valued functions of an independent variable ζ. And both Klein (in 1882) and he (in 1883) have sought to make it clear that any rational algebraic equation $f(y, x) = 0$, connecting x and y, can be satisfied by regarding x and y as single-valued functions of another variable ζ, the suggestion of particular cases being that the functions can be taken to be automorphic functions in the sense already explained. To enunciate such a theorem, even though its exhaustive proof is a matter for subsequent investigation as it was in this case, may be to exert a great stimulus to the development of a theory. It ought, however, perhaps, to be mentioned that unless the equation $f(y, x) = 0$ is capable of being satisfied by rational functions, or by elliptic functions, of a parameter, the functions x, y of the new parameter, which are contemplated by the theorem, must possess an infinite number of essential singularities. The deductions to be drawn from the result must then, it would seem, be of a general character, and independent of the precise form of the functions. Poincaré himself returned to the matter in a paper on the uniformisation of analytic functions as late as 1907 (Acta Math., vol. 31), and the proof of the theorem has called forth an extensive literature. There is one matter of subsidiary importance to which a word may be given in connection with Poincaré's theory of automorphic functions. The division of the upper half of the plane of the complex variable into regions corresponding to the substitutions of a group may be made, as it was in the particular case previously taken for illustration, by means of circles having their centres on the real axis. Such circles have obviously at least some of the properties of straight lines in a plane; two such circles intersect in one point (in the upper half-plane); one such circle can be drawn through two given points. As straight lines are the curves which render the integral $\int ds$ stationary, taken between two given points, where $ds = (dx^2+dy^2)^{\frac{1}{2}}$, so the circles in question are the curves which render the integral $\int ds/y$ stationary, y being the ordinate to the real axis and x the abscissa parallel to this axis. The relations of these circles are, in fact, those of the so-called straight lines in the geometry of Lobatchewski, the integral $\int ds/y$, which we may call the separation of its two extreme points, replacing the distance of the Euclidian geometry. Similarly the integral $\iint dx\,dy/y^2$, which we may call the extent of the region over which it is taken, may be used instead of the area. Every one of these elements, the circles with centres on the real axis, the separation, and the extent is unaltered by transformations

$$\zeta' = (a\zeta+b)/(c\zeta+d),$$

in which a, b, c, d are real, just as straight lines, lengths and areas are unaltered by movements in Euclidian geometry. It is thus convenient to make use of these elements in discussing the groups of such linear substitutions. It is in this sense that Poincaré employed non-Euclidian geometry in his discussion of these substitutions.

In addition to the systematic development of the theory of automorphic functions, of which we have given some account, Poincaré wrote several papers dealing with questions of Abelian functions. One of the briefest is an application of Kronecker's theory of characteristics to determine the number of pairs of variables for which two theta functions of two variables have each assigned values. The same theory is used also by Poincaré in his great paper on the equilibrium of a rotating fluid mass (Acta Math., vol. 7, p. 268, 1885–6). In Kronecker's hands the theory becomes an extension of Cauchy's theorem for integrals of functions of one complex variable to integrals of functions of several complex variables, and is put into connection with the theory of potential in any number of dimensions. It is interesting, then, to find papers of Poincaré dealing with extensions of Cauchy's theorem to functions of several complex variables, to see the theory of potential in any number of dimensions applied to the theory of integral functions of several variables, and to note how extensive and persistent were Poincaré's attempts to grapple with the problems of Analysis Situs in higher space. One of the problems to which Weierstrass devoted much consideration was to generalise the expression, as a quotient of two integral functions, of a single-valued analytic function of one variable whose only finite singularities are poles. Consider a single-valued analytic function of two variables; assume that about every finite pair of values of these the function is expressible, generally as a power series, but, if not, then as a quotient of two power series, with presumably only a limited range of convergence. The question is whether there exist two power series, each convergent for all finite values of the variables, as the quotient of which the function can be represented for all values of the variables. A difficulty arises from the fact that there are points at which the function has no definite value at all, the expressions which represent it having different limits according to the path by which the variables approach the point—it is desirable that the representation of the function as the quotient of two integral functions should be such that these integral functions do not simultaneously vanish except at points for which the function is actually indeterminate. Poincaré's papers in regard to the connection of the theory of potential with the theory of integral functions furnish a proof that such a representation is possible, and give rise incidentally to a splendid generalisation of Weierstrass's factor expression for an integral function of one variable. The real part of the logarithm of the primary factor $(1-z/c)e^{\psi}$, wherein ψ, which is introduced for convergence, may, for the purposes of our statement, be left out of account, is the logarithmic potential of a mass at the point c. We may thus say that, save for a correction necessary to secure convergence, the real part of the logarithm of the integral function is built up from the potential of masses situated at the zero points of the integral function. When we come to an integral function of two variables, its zero points form a continuum. The integral expressing the potential of this continuum is the guiding portion of the real part of the logarithm of the integral function. The application of this suggestion to Weierstrass's problem requires the establishment of the notion of a definite continuum upon which the given function vanishes, and of another continuum upon which the function becomes infinite, and so furnishes a further incitement to the study of hyperspace. The ideas of which we have attempted to give some account are applicable to another pair of connected problems. One striking result of the manifold study of Abelian functions in the nineteenth century was the emergence of certain integral functions of several variables, known as theta functions, and, intimately connected therewith, of simultaneously periodic functions. A function of n variables $u_1, \ldots, u_n$ may be such that if appropriate constants $\omega_1, \ldots, \omega_n$ be simultaneously added to the variables $u_1, \ldots, u_n$ respectively, the value of the function is unaltered. And there may be $2n$ sets of quantities such as $\omega_1, \ldots, \omega_n$ for which this is true. The theta functions are not so periodic; they are integral functions say of $u_1, \ldots, u_n$, associated with $2n$ sets of constants such as $\omega_1, \ldots, \omega_n$, so that for the values $u_1+\omega_1, \ldots, u_n+\omega_n$ the function is multiplied by the exponential of a linear function like $A_1u_1+\ldots+A_nu_n+B$.

For the theta functions and for the multiply periodic functions which can be formed from them, the $2n^2$ quantities such as $\omega_1, \ldots, \omega_n$ are connected together by certain bilinear relations of equality and inequality. The question then arises whether these relations are necessary for every possible multiply periodic function, and, a connected enquiry, whether the most general periodic function is expressible by theta functions. Even though, as is now the case, these questions have been given affirmative answers, there remains a need of some comprehensive and direct method of arriving at the result. And the suggestion that this will be associated with some greater insight into the possibilities of Analysis Situs (in space of $2n$ real dimensions) seems inevitable. Various lines of enquiry are thus opened; there is evidence that Poincaré gave attention to many of these. With Picard he published a note dealing with the bilinear relations among the periods of a multiply periodic function; to the properties of integral functions whose second logarithmic differential coefficients are periodic functions, and to the problem of sets of n integrals whose periods are expressible linearly by less than $2n$ sets, he devoted a long paper. To his study of the Analysis Situs in any number of dimensions several laborious memoirs bear witness. The surface imagined by Riemann, it is well known, serves the purpose of representing an algebraic function which is capable of several, say of n, values, as a single-valued function of position upon an n-sheeted surface. When, however, we seek to apply Cauchy's contour integral theorem to integrals of algebraic functions

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considered on this surface, we are at once met by the fact that it is in general possible to draw closed curves upon the surface which are not capable of being continuously deformed to evanescence, and do not form the complete boundary of any portion of the surface. Such a circumstance arises also, evidently, for many surfaces; as for instance for the surface of an anchor ring. The question arises for such a surface, what is the least number of irreducible closed curves by means of which all others can be represented. In the case of a surface utilised in Riemann's manner for the representation of an algebraic function, the number so arising has the greatest importance for the theory, and is the most fundamental of the characters used to discriminate between algebraic functions of different individualities. When we pass from a surface of two dimensions to a closed space of n dimensions, and therein consider closed spaces of r dimensions, there is a similar question. Let two such closed spaces of order r be regarded as equivalent when either can be continuously deformed into the other within the given space of n dimensions; there will be a least number, k, of closed spaces of order r in terms of which every other such space can be represented in the form

$$P \equiv m_1 P_1 + \ldots + m_k P_k,$$

wherein $m_1, \ldots, m_k$ are integers. And there will be such a number k for each value of r which is less than n. These so-called numbers of Betti are in fact equal in pairs, the number k for any r being equal to the number k for $r' = n - r$. This theorem requires, evidently enough, much greater precision in defining the meaning of equivalence than we can attempt here. For instance in the closed three-dimensional space interior to an anchor ring, every closed curve is clearly representable in terms of one such curve (unless itself deformable to evanescence); and every closed surface in this three-dimensional space, if not itself deformable to evanescence, is deformable to one surface, whose shape is that of an anchor ring interior to the given anchor ring, so that the two numbers of Betti are each equal to unity. It is obvious that two spaces which are capable of being put into point to point correspondence with one another must have the same numbers of Betti. Conversely, however, it was shown by Poincaré that the equalities of these numbers are not the only descriptive similarities necessary in order that two spaces should be capable of such correspondence.

The theory just referred to is suggested by the discussion of Riemann's surface. Riemann's own theory of the functions arising for such a surface was based upon a theorem of existence of potential functions, for which the evidence was, in the light of subsequent scrutiny, undoubtedly insufficient. The theorem in question, for which the physical suggestion is extremely cogent, has thence become the centre of a wide literature. To this also Poincaré contributed, with an extensive paper expounding a method of his own, in addition to which he wrote long papers dealing in general with the differential equations of mathematical physics.

In this survey we have left aside many of Poincaré's discoveries, for instance, his brilliant additions to Laguerre's theory of the class of integral functions, or to Weierstrass's theory of monogenic functions. We have expounded instead some matters wherein is well seen the great generality and abstractness of much of his work. If his writings had been limited to his contributions to theory of functions, they would have left an enduring mark. We pass, however, now to consider in a few lines his extensive publications in the field of Astronomy and Dynamics.

As has been said, Poincaré was Professor of Astronomy from 1896, and, in addition to pure mathematics, he was probably interested from the first also in physical questions. As early as 1881, while yet Ingénieur des Mines, in 'Liouville's Journal' (vol. 7, p. 376), in a "Mémoire sur les Courbes Définies par une Équation Différentielle," we find the words:— "Prenons pour exemple le problème des trois corps; ne peut on pas se demander si l'un des corps restera toujours dans une certaine région du ciel." These words would seem to give the key to Poincaré's work in Astronomy and Dynamics; to ascertain whether the theory leads us to expect stability of motion and periodical recurrence of position may be said to have been his constant preoccupation. The publication of G. W. Hill's 'Researches in the Lunar Theory,' in America, in 1877–8, seems to have greatly impressed him. In vol. 1 of the Bulletin Astronomique (1884) he published a paper, "Sur Certaines Solutions Particulières du Problème des Trois Corps," which generalised Hill's idea of a periodic orbit for the Moon. And in the Preface to vol. 1 (1892) of his 'Méthodes Nouvelles de la Mécanique Céleste,' speaking of Hill's contributions to the theory, he says: "Dans cette œuvre . . . il est permis d'apercevoir le germe de la plupart des progrès que la science a fait depuis." Many of the leading ideas of his theory of orbits were expounded in his essay "Sur le Problème des Trois Corps et les Équations de la Dynamique," which obtained the prize offered by the King of Sweden. This was finished in 1888, and published in revised form in 1890. In addition to this are to be mentioned the 'Méthodes Nouvelles,' already referred to (vol. 1, 1892; vol. 2, 1894; vol. 3, 1899), and the Sorbonne Lectures on Celestial Mechanics (vol. 1, 1905; vol. 2, 1907–9).

The dominating idea of the work is the possibility of the existence of solutions of theoretical exactness and of periodic character. In the case of the Earth and Sun and Moon, regarding the Sun as moving with constant angular velocity in a circle about the Earth and the Moon as moving in the same plane, G. W. Hill obtained, by actual computation, an orbit of the Moon relatively to the uniformly rotating line joining the Earth to the Sun, which is both re-entrant and symmetrical. Poincaré obtains a generalisation of this for any dynamical system in which the differential equations have an appropriate form, of wide generality, by reasoning which is quite general and quite simple; but this reasoning requires an appreciation of Cauchy's theorems of existence for the solutions of differential equations; and it is a characteristic property of the series which express the periodic solutions that they converge. From the periodic orbit of the Moon Hill obtained, by variation of the equations, an equation for the motion of the Moon's perigee. In order to calculate the frequency of its oscillations without solving the equation, he introduced the use of determinants of indefinitely great order. That Poincaré should investigate the convergence of the method, and so set up a new engine of analysis, as an incident to his astronomical work, is characteristic of him. He further considers in much detail the general method of variation and the quantities which generalise the frequency considered by Hill, and their expansion as power series, as part of his theory of characteristic exponents. In another direction, also, he adopts the idea, suggested by Hill, of making the periodic solution the centre of the theory, by considering solutions which coincide with the periodic solutions after an infinite time, or did so coincide an infinite time before the present. These are the so-called asymptotic solutions. Both the periodic solutions and the asymptotic solutions are particular solutions of the equations, not containing the full number of arbitrary constants. Whereas the former converge, the latter, when expanded in terms of the small quantities, do not; they are, however, definitely and formally shown to be capable of use for approximations, in the manner of Stirling's series for the gamma function. The interplay between the original equations and the equations deduced by variation is again exemplified in Poincaré's consideration of integral invariants. In the motion of an incompressible fluid the integral which expresses the volume of any portion of the fluid is unaltered by the motion, if always taken over the same particles of the fluid. He obtains other integrals having the same property, and considers their relations in many aspects. That a quantity should contain in its expression a term of the form $t\cos(mt+h)$ is rendered by him as a statement that the quantity, though not remaining for all time of limited magnitude, does yet return infinitely often to within arbitrary nearness of its original value. This becomes a text for the consideration of dynamical systems with such a property—stable *à la Poisson*. In particular, a proof is given, as illustrating the theory of integral invariants, that for incompressible fluid in a closed vessel, if we consider the particles occupying at any instant a particular small volume, these particles (speaking generally) return infinitely often to this volume. This theory of integral invariants reappears in Poincaré's recent paper (Journ. de Physique, January, 1912) written in support of Planck's Theory of Quanta.

But it is impossible not to consider the relation of Poincaré's periodic solutions with the expansions used by practical astronomers, and a large part of his writing deals with this matter. Series had gradually been introduced containing only sines and cosines—that is terms A $\cos(nt+h)$, but no terms such as $t\cos(nt+h)$, or such as t^k, in which the time occurs outside the periodic functions—the evident intention being to obtain series which might serve to express the circumstances for all time. Apparently the possibility of such series may have been recognised by d'Alembert (*cf.* E. W. Brown, 'Lunar Theory,' p. 239). Poincaré attributes the series to Newcomb ('Smithsonian Contributions to Knowledge,' December, 1874), who used them for the motion of the planets, and after him to Lindstedt. For the case of the Moon Delaunay's series are to be referred to (1860). Poincaré investigates Lindstedt's series again, and extends their scope; but he proves that they are not as a rule convergent. His method of proof is extremely simple, if not wholly convincing for all possible cases. It may be said to be part of his theory of periodic solutions. It is related also to his general theorem as to the existence of uniform integrals of the astronomical equations. He proves, however, that the Lindstedt series are asymptotic, in the sense in which Stirling's series for the gamma function are asymptotic; they give a rule for writing down a finite number of terms approximating very closely to the functions sought, but the approximation cannot be made arbitrarily close. To the consideration of these series and the related investigations of Delaunay, of Bohlin, and of Gyldén, a large part of the second volume of the 'Méthodes Nouvelles' is devoted. In the Sorbonne lectures a different plan of exposition is followed. Lagrange's method of successive approximation is first used to obtain expansions wherein the time occurs explicitly outside the periodic functions. A proof is then given that the terms in which the time enters in this way may be absorbed; for if they be omitted they can be re-found from the terms which remain, by a change of variables and subsequent expansion ('Leçons de Mécanique Céleste,' vol. 1, 1905, pp. 172, 198, 268). There is a further theorem of great importance which, like the proof of the divergence of Lindstedt's series, occurs in Poincaré's prize essay. The problem of three bodies has the classical integrals, which belong to any dynamical system, known as those of energy and momentum. It was proved by Bruns that, apart from and independent of these, the problem allows no other algebraic integral. To this Poincaré adds the theorem that the problem possesses no single-valued integral. The statement is for certain restricted values of the parameters; upon these restrictions we need not now enter.

This account leaves aside many matters dealing with the theory of orbits to which Poincaré devotes attention, and it does not represent his whole contribution to Astronomy. For very soon after the publication of the paper dealing with periodic orbits (Bull. Astr., vol. 1, 1884) Poincaré was publishing investigations issuing in 1885 in a great paper (Acta Math., vol. 7) dealing with the forms of rotating masses of fluid and their stability. As his investigations in regard to periodic orbits begin with acknowledgments to G. W. Hill, so this paper begins by quoting the results announced by Thomson and Tait in the 'Natural Philosophy,' Without entering into precise mathematics it would seem to be impossible to give here any competent account of Poincaré's work; the questions of stability involved are still matter of controversy. Poincaré considers a series of possible shapes of relative equilibrium, bestowing especial care upon the critical values of the parameters. He is thus led to consider the possibility of Jacobi's ellipsoid of unequal axes changing gradually into a shape which may be likened to a pear spinning about a line at right angles to its long axis. The suggestion is that the thinner portion (the stalk end) may gradually become detached. The reader will find it interesting to turn to the remarks made by Sir George Darwin in presenting to Poincaré the gold medal of the Royal Astronomical Society, in 1909. It was a subject upon which no one could speak with greater authority. Evidently Poincaré's investigation was to him a revelation of intellectual mastery for which he had the profoundest respect. Besides this work there remains, however, also Poincaré's work in regard to tides. It may be sufficient, perhaps, to refer to Sir George Darwin's brief indication of his concurrence, in this matter, with what is undoubtedly a very common feeling in regard to much of Poincaré's applied mathematics, namely, that the great generality of his methods is apt to militate against any quite immediate application. Undoubtedly he has no scruple in bringing the most advanced and the most modern theories of pure mathematics into service; and where such theory is not already in existence he invents it.

When we turn from Poincaré's astronomical work to his work in Physics, we enter upon ground which has already been much trodden. It may be sufficient to call attention to the large number of volumes of Sorbonne lectures, edited by his pupils, dealing freshly with the whole field of recent discovery and discussion in electricity, optics, thermodynamics, beside those dealing with matters already referred to.

But in addition to all this mathematical and physical work, Poincaré was also a prolific writer on general questions of philosophic interest. How far his contribution to these matters was new, and how far he stated in a brilliant way the critical conclusions which are common to many to-day, it must be for others to decide. At least, while taking up the humblest and simplest attitude in face of the immensity of the universe, he preached in no uncertain way the dignity of the pursuit of truth. "Thought is the lightning flash between two infinities of blackness. But it is the lightning which matters."

His writings on such matters are accessible to all and of general interest. It is unnecessary to expound them here.

H. F. B.